# Parity Dominating Sets in Grid Graphs

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**Abstract**

A positive even closed (open) dominating set of a graph $G$ is a subset $D$ of the vertices such that each vertex in $G$ has a positive even number of vertices of $D$ in its closed (open) neighborhood. In this paper we find all positive even open dominating sets in the $m \times n$ grid graph $G_{m,n}$ and, with the help of a computer, find all $m, n$ such that $G_{m,n}$ has a positive even closed dominating set. We also find all dominating sets $D$ such that each vertex of $G_{m,n}$ has precisely two vertices of $D$ in its closed (open) neighborhood. A brief survey of some previous work on parity domination is also presented.

## 1 Introduction

A subset $D$ of the vertex set $V$ of a graph $G$ is an *even open dominating set* if $|N(v)| \cap D|$ is even for each $v \in V$, where $N(v) = \{u \in V : uv \text{ is an edge of G}\}$ is the open neighborhood of $v$. $D$ is an *even closed dominating set* (sometimes just called an even dominating set) if $|N[v] \cap D|$ is even for each $v \in V$, where $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of $v$. The notions of odd open dominating set and odd closed dominating set (sometimes just called odd dominating set) are defined analogously.

A fundamental result first proved by Sutner is that every graph contains an odd dominating set, see for example [2, 14], which is not the case of the other three combinations of even/odd and open/closed defined above. Parity domination has been studied in a number of recent papers, for example [1, 2, 3, 7, 9, 10, 13].

Note that by our definitions, a nonempty even open or even closed dominating set $D$ might not be a dominating set, because $N(v) \cap D$ might be empty for some $v$. For example, the subset $D$ in the $5 \times 2$ grid grid graph corresponding to the positions of the 1's in the $5 \times 2$ matrix in Figure 1(a) is an even open dominating set that is not a dominating set (the vertex in row 3, column 1 is not dominated), while the set $D$ corresponding to the positions of the 1's in Figure 1(b) is an even closed dominating set that is not a dominating set of the the $5 \times 3$ grid grid graph (the vertices in row 3, column 1 and in row 3 column 3 are not dominated).

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \qquad\qquad \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

(a) (b)

Figure 1.

In [9, 10, 11, 12], Goldwasser and Klostermeyer used algebraic methods to find all pairs $(m, n)$ such that there exists an odd/even open/closed dominating set in the $m \times n$ grid graph $G_{m,n}$, and they determined how many of such sets $G_{m,n}$ contains.

A *positive even closed dominating set* is an even closed dominating set $D$ where $|N[v] \cap D| > 0$ for each vertex $v$ (so it is an even dominating set that is also a dominating set), with an analogous definition for *positive even open dominating set*. In this paper, we find all positive even open dominating sets in $G_{m,n}$ and, with the help of a computer, find all $m, n$ such that $G_{m,n}$ has a positive even closed dominating set. The algebraic methods used in [9, 10, 11, 12] do not suffice to solve these problems, because they do not distinguish an empty neighborhood from one which is non-empty and even.

A subset $D$ of the vertex set $V$ of a graph $G$ is a *perfect open dominating set* if $|N(v) \cap D| = 1$ for each $v \in V$ and is a *perfect closed dominating set* if $N[v] \cap D| = 1$ for each $v \in V$. Note that a perfect closed dominating set in the $n$-cube is a perfect binary Hamming code of length $n$ and minimum distance three. The sets are often called *total perfect codes* and *perfect codes*, respectively, in the literature, though we use our terminology for consistency and clarity sake within this paper.

A subset $D$ of the vertex set $V$ of a graph $G$ is a *doubly perfect open dominating set* if $|N(v) \cap D| = 2$ for each $v \in V$ and is a *doubly perfect* *closed dominating set* if $|N[v] \cap D| = 2$ for each $v \in V$. In this paper, we find all doubly perfect open and closed dominating sets in $G_{m,n}$ for all $m$ and $n$.

In $G_{m,n}$, as in any bipartite graph, open domination problems are generally easier to analyze than closed domination problems, because the selection of vertices to put in the dominating set operates independently within each part of the vertex bipartition.

## 2 Background

We say that a binary $m \times n$ matrix $D$ is an odd (even) open dominating set matrix for $G_{m,n}$ if the positions of the 1's in $D$ correspond to an odd (even) open dominating set in $G_{m,n}$. Denote the rows in such a matrix by $r_1, r_2, \ldots, r_m$. For example, the $2 \times 4$ matrix with $r_1 = 1001$ and $r_2 = 1001$ is an odd open dominating set matrix. Throughout the paper, let 0 denote the all-zeroes vector (when clear from the context).

A basic observation is that once $r_1$ is fixed, the remaining rows of an even open dominating set matrix can be computed by the following recurrence:

$$r_{i,j} = r_{i-1,j-1} + r_{i-1,j+1} + r_{i-2,j} \ \ (mod\ 2) \tag{1}$$

where undefined entries are taken to be zero. In the case of an odd open dominating set matrix, simply modify the equation as follows:

$$r_{i,j} = 1 + r_{i-1,j-1} + r_{i-1,j+1} + r_{i-2,j} \ \ (mod\ 2) \tag{2}$$

Then $r_1, r_2, \ldots, r_m$ are the rows of an $m \times n$ even/odd open dominating set matrix if and only if $r_{m+1} = 0$ using Equation (1)/(2).

Of course, odd/even closed dominating set matrices can be defined in an analogous manner by adding an $r_{i-1,j}$ term to the right hand side of Equations (1) and (2).

Let $P_n$ be the adjacency matrix for the path on $n$ vertices with the vertices in the usual order and let $B_n = P_n + I_n$ ($I_n$ is the $n \times n$ identity matrix). Let $f_i$ be the $i^{th}$ Fibonacci polynomial defined over $GF(2)$ by

$$f_n = xf_{n-1} + f_{n-2} \quad n \geq 2,\ f_0 = 0, f_1 = 1$$

(so $f_2 = x, f_3 = x^2 + 1, f_4 = x^3, f_5 = x^4 + x^2 + 1$).

In [9, 12] it was noted that if the first row of an even open dominating set matrix with $n$ columns is the binary $n$-vector $w$, then the $i^{th}$ row is $r_i = f_i(P_n)w$, while the $i^{th}$ row of an even closed dominating set matrix with first row $w$ is $y_i = f_i(B_n)w$. It was shown in [12] that $f_{n+1}$ is the characteristic polynomial of $P_n$, so the smallest integer $i$ such that $r_i = f_i(P_n)w = 0$ for all $n$-vectors $w$ is $i = n + 1$. Hence the following.

**Theorem 1** *[12] For each binary n-vector w, there exists a unique* $n \times n$ *even open dominating set matrix with first row w.*

Goldwasser and Klostermeyer also determined the number of even/odd dominating sets contained in each grid graph.

**Theorem 2** *Let* $m, n$ *be positive integers and let* $d+1$ *be the gcd of* $m+1$ *and* $n+1$*. Then*
*(i) The number of even open dominating sets of* $G_{m,n}$ *is* $2^d$*.*
*(ii) The number of odd open dominating sets of* $G_{m,n}$ *is* $2^d$ *if there does not exist a positive integer t such that* $\frac{m+1}{2^t}$ *and* $\frac{n+1}{2^t}$ *are both odd integers (which is the case if m or n is even) and 0 if there does exist such a positive integer t.*

Goldwasser, Klostermeyer, and Trapp [9] determined the number of even closed dominating sets in $G_{m,n}$.

**Theorem 3** *[9] The number of even dominating sets in* $G_{m,n}$ *is* $2^d$ *where d is the degree of the greatest common divisor of* $f_{m+1}(x)$ *and* $f_{n+1}(x+1)$ *where* $f_0, f_1, \ldots$ *is the sequence of Fibonacci polynomials over* $GF(2)$*.*

Since the symmetric difference of an even and odd dominating set is an odd dominating set, from Sutner's theorem it follows that $G_{m,n}$ has the same number of odd closed dominating sets as even closed dominating sets.

It follows from Theorem 3 that if $f_{n+1}(x+1)$ divides $f_{m+1}(x)$ then $y_{m+1} = f_{m+1}(B_n)w = 0$ for each binary $n$-vector $w$, which would mean that for each $w$ there exists an $m \times n$ even closed dominating set matrix with first row $w$. For each $n$ does there exist an $m$ such that $f_{n+1}(x+1)$ divides $f_{m+1}(x)$? This question was answered in [11].

**Theorem 4** *[11] For each positive integer* $n \neq 5$ *there exists a positive integer* $m < 3 \cdot 2^{\frac{n}{2}}$ *such that* $f_{n+1}(x+1)$ *divides* $f_{m+1}(x)$*. Hence in any* $t \times n$ *even closed dominating set matrix with* $n \neq 5$ *and* $t \geq 3 \cdot 2^{\frac{n}{2}}$ *at least one of the first* $3 \cdot 2^{\frac{n}{2}}$ *rows is all zeroes.*

It is a simple exercise to find all perfect closed dominating sets in $G_{m,n}$. We believe this was first done in [17].

**Theorem 5** *The grid graph* $G_{m,n}$ $(m \geq n)$ *has a perfect closed dominating set if and only if either:*
*(1)* $n = 1$
*(2)* $n = 2$ *and m is odd*
*(3)* $m = n = 4$*.*

Goldwasser and Klostermeyer found all perfect open dominating sets in $G_{m,n}$, solving a problem stated in [6].

**Theorem 6** *[16] (a) $G_{m,1}$ contains a perfect open dominating set if and only if $m \not\equiv 1 (\text{mod } 4)$.*
*(b) Suppose $m, n$ are greater than 1. $G_{m,n}$ contains a perfect open dominating set if and only if one of $m, n$ (say $m$) is even, and the other, $n$, is congruent to -3, -1, or 1 $(\text{mod } m + 1)$.*

# 3 Positive Even Open Dominating Sets

An $n \times n$ (0-1)-matrix with $n \geq 2$ is a *bulls-eye* matrix if all entries in rows 1 and $n$ and columns 1 and $n$ are ones, all other entries in rows 2 and $n-1$ and columns 2 and $n-1$ are zeroes, all other entries in rows 3 and $n-2$ and columns 3 and $n-2$ are ones (so there are alternating rings of ones and zeroes), and so on. We display bulls-eye matrices for $n = 4, 5, 6, 7$ in Figure 2.

| | | | |
|---|---|---|---|
| 1111 | 11111 | 111111 | 1111111 |
| 1001 | 10001 | 100001 | 1000001 |
| 1001 | 10101 | 101101 | 1011101 |
| 1111 | 10001 | 101101 | 1010101 |
| | 11111 | 100001 | 1011101 |
| | | 111111 | 1000001 |
| | | | 1000001 |

Figure 2.

It is easy to check that if $n$ is even then an $n \times n$ bulls-eye matrix is a doubly perfect open dominating set matrix, while if $n$ is odd then all but one open neighborhood contains precisely two ones: if $n \equiv 3(\text{mod } 4)$ then center entry has four ones in its open neighborhood, while if $n \equiv 1(\text{ mod } 4)$, the center entry has an empty open neighborhood. If $m = t(n+1) - 1$ for some positive integer $t$, we can construct an $m \times n$ even open dominating set matrix by concatenating $t$ $n \times n$ bulls-eye matrices, each separated by a rows of zeroes. Such a matrix, which we call a *concatenated bulls-eye matrix*, is a positive even open dominating set matrix if $n \not\equiv 1(\text{mod } 4)$ and is a doubly perfect open dominating set matrix if $n$ is even.

We think of an $m \times n$ positive even open dominating set matrix an an $m \times n$ rectangle partitioned into $mn$ squares colored alternately white and black like a (possibly non-square) chessboard (so the colors correspond to the vertex partition in the bipartite graph), each square with the entry 0 or 1. We assume that $m \geq n$ and that the (1, 1) square is white. We refer to the squares in positions (1, $r$), (2, $r-1$), $\ldots$, ($r$, 1) as the $r$-*diagonal* for $r \in \{1, 2, \ldots, n\}$. If $r$ is odd, all the squares in the $r^{th}$ diagonal are white, while if $r$ is even, all the squares in the $r^{th}$ diagonal are black. Since we are considering open neighborhoods, in constructing a positive even open dominating set matrix $A$, the entries in the white and black diagonals operate independently.

Clearly the 2-diagonal of $A$ must be 1 1 (so that the (1, 1) square has a positive even closed neighborhood). If $n \geq 4$, then the entries in the 4-diagonal, in order starting with (1, 4), must be 1001 or 0110. If it is 1001, then the (3, 4) and (4, 3) entries must be 1's, so if $n \geq 6$ the 6-diagonal must be 101101. If the 4-diagonal is 0110, then the (1,6) and (2, 5) entries must be 1's, so the 6-diagonal must be 110011. In general, if we have filled in the entries for all $r$-diagonals with $r$ even and $r \leq 2k \leq n+2$, for some $k$, then there will be precisely two ways to fill in the $(2k+2)$-diagonal so that all the squares in the $(2k+1)$-diagonal have even (possibly empty) open neighborhoods and these two ways are 0-1 complements of one another. However, if the $2k$-diagonal has two consecutive entries that are 0's, or has first or last entry 0, then to avoid an empty open neighborhood at least one of the two possibilities is eliminated.

Due to these considerations, it is easy to show that $A$ cannot be a positive even open dominating set matrix if either

(i) the smallest value of $k$ such that the $(1, 2k)$ entry is equal to 0 is an odd number, or
(ii) the $(1, 2t)$ and $(1, 2s)$ entries are both 0 for some $t$ and $s$ with $t \neq s$.

So the only possibilities are that all entries in black squares in the first row are 1's, or precisely one of them is a 0 and this must be the $(1, 4j)$ entry for some positive integer $j$.

In Figure 3(a), we show the way to fill in the entries along the first eight black diagonals with the (1, 12) entry equal to 0 and all other entries in black squares in the first row equal in 1. Note that if the entries in the first four black diagonals are as shown in Figure 3(a), but if the entries in the fifth black diagonal were the 0-1 complement of the ones show in Figure 3(a), then the (5, 5) square would have an empty open neighborhood. Taking the 0-1 complement of the seventh or eighth black diagonal also produces an empty open neighborhood.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | 0 | - | 1 | - | 1 | - | 1 |
| 1 | - | 0 | - | 0 | - | 0 | - | 0 | - | 1 | - | 1 | - | 0 | - | - | - |
| - | 0 | - | 1 | - | 1 | - | 1 | - | 0 | - | 0 | - | 0 | - | - | - | - |
| 1 | - | 1 | - | 0 | - | 0 | - | 1 | - | 1 | - | 1 | - | - | - | - | - |
| - | 0 | - | 0 | - | 1 | - | 0 | - | 0 | - | 0 | - | - | - | - | - | - |
| 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | - | - | - | - | - | - |
| - | 0 | - | 0 | - | 1 | - | 0 | - | 0 | - | - | - | - | - | - | - | - |
| 1 | - | 1 | - | 0 | - | 0 | - | 1 | - | - | - | - | - | - | - | - | - |
| - | 0 | - | 1 | - | 1 | - | 1 | - | - | - | - | - | - | - | - | - | - |
| 1 | - | 0 | - | 0 | - | 0 | - | - | - | - | - | - | - | - | - | - | - |
| - | 1 | - | 1 | - | 1 | - | - | - | - | - | - | - | - | - | - | - | - |
| 0 | - | 0 | - | 0 | - | - | - | - | - | - | - | - | - | - | - | - | - |
| - | 1 | - | 1 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 1 | - | 0 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| - | 0 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 1 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |

Figure 3(a). Partial Even Open Dominating Set Matrix (Black Diagonal Entries)

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | - | 1 | - | 0 | - | 1 | - | 1 | - | 1 | - | 0 | - | 1 |
| - | 0 | - | 1 | - | 1 | - | 0 | - | 0 | - | 1 | - | 1 | - |
| 1 | - | 0 | - | 0 | - | 0 | - | 1 | - | 0 | - | 0 | - | - |
| - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | - | - |
| 0 | - | 0 | - | 0 | - | 0 | - | 1 | - | 0 | - | - | - | - |
| - | 1 | - | 1 | - | 1 | - | 0 | - | 0 | - | - | - | - | - |
| 1 | - | 0 | - | 0 | - | 1 | - | 1 | - | - | - | - | - | - |
| - | 0 | - | 1 | - | 0 | - | 0 | - | - | - | - | - | - | - |
| 1 | - | 1 | - | 1 | - | 1 | - | - | - | - | - | - | - | - |
| - | 0 | - | 1 | - | 0 | - | - | - | - | - | - | - | - | - |
| 1 | - | 0 | - | 0 | - | - | - | - | - | - | - | - | - | - |
| - | 1 | - | 1 | - | - | - | - | - | - | - | - | - | - | - |
| 0 | - | 0 | - | - | - | - | - | - | - | - | - | - | - | - |
| - | 1 | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 1 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |

Figure 3(b). Partial Even Open Dominating Set Matrix (White Diagonal Entries)

Now consider the white diagonals. Figure 3(b) shows the entries in the first eight white diagonals if $A = [a_{ij}]$ is an $m \times n$ positive even closed dominating set matrix with $m \geq n \geq 15$ and $a_{1,5} = a_{1,3} = 0$ but the other entries in white squares in the first row equal to 1. Note that if the first five white diagonals are as in Figure 3b, then the sixth white diagonal must also be as in the figure (so $a_{1,11}$ cannot be equal to 0). Furthermore, there is a unique way to choose the entries for any subsequent white diagonals (because each has two consecutive 0 entries): the entry in the first row must be 1. Thus $A$ cannot be a positive even open dominating set matrix if either

(iii) more than two white entries in the first row are 0's, or
(iv) $a_{1,2k-1} = a_{1,2j-1} = 0$ and $k - j$ is odd, for some $k, j$.

So the only possibilities not yet eliminated for the entries in the white squares in the first row are that at most one square is 0 or precisely two are equal to 0 and these are in columns $2j - 1$ and $2k - 1$ where $k - j$ is an even positive integer.

We can use this "diagonals" argument to characterize positive even open dominating set matrices. The results are stated mainly in terms of the first row of such matrices, which of course determines all subsequent rows. As noted in [6] and [12], an $n \times n$ even dominating set matrix with a single 1 in the first row has the form shown in Figure 4. Note that the matrix is symmetric about both main diagonals.

```
0010000
0101000
1010100
0101010
0010101
0001010
0000100
```

Figure 4.

Any even open dominating set matrix $A$ can be expressed as the sum of $j$ such matrices, where $j$ is the number of 1's in the first row of $A$. It follows that any $n \times n$ even open dominating set matrix is symmetric about both main diagonals. In particular, the number of rows in the first row, last row, first column, and last columns must all be the same.

The following results characterize positive even open dominating set matrices.

**Theorem 7** *If $A$ is an $m \times n$ doubly perfect open dominating set matrix where $m \geq n$, then $n$ is even and $A$ is a concatenation of bulls-eye matrices (so $m \equiv -1 (\mathrm{mod}\ n+1)$).*

In place of "$A$ is a concatenation of bulls-eye matrices" in Theorem 7, we could have said "the first row of $A$ is all 1's."

**Theorem 8** *Suppose $A$ is an $m \times n$ positive even open dominating set matrix with $m \geq n$ that is not doubly perfect. Then either:*

*1. $m > n, n \equiv 3 (\mathrm{mod}\ 4)$ and $A$ is a concatenation of $n \times n$ bulls-eye matrices, or*

*2. $m = n$ and the first row of $A$ has at most two 0's, with the following possibilities:*
*(a) $n \equiv 3 (\mathrm{mod}\ 4)$ with the first row having either*
*(i) no 0's*
*(ii) one 0 in column $j$ where $j$ is odd*
*(iii) two 0's, in columns $j, k$ where $j$ and $k$ are odd and $j \equiv k (\mathrm{mod}\ 4)$.*
*(b) $n$ is even, with $I = \{i : 1 \leq i \leq n$ and $i$ is a multiple of 4$\}, K = \{k : 1 \leq k \leq n$ and n+1-k is a multiple of 4$\}$ and with the first row having either*
*(i) one 0 in column $j$, where $j$ is in either $I$ or $K$*
*(ii) two 0's, in columns $j$ and $r$, where one of $j, r$ is in $I$ and the other is in $K$.*

*Proof of Theorems 7 and 8:* Assume $A$ is an $m \times n$ positive even open dominating set matrix with $m \geq n$. By Theorem 1, if $m > n$ then the $(n+1)^{st}$ row of $A$ is all 0's. First assume that $m = n$.

**Case 1** $n = 4p + 1$ for some positive integer $p$.

With the (1,1) square white, the "diagonals argument" for the black squares shows that at most one black entry in the first row is 0, and if there is one, it must be in column number $4j$, for some positive integer $j$. But after a left-right reflection, this would be column number $(4p+1)+1-4j = 4(p-j)+2$, an impossibility since this is not a multiple of four. So there cannot be a black 0 in the first row. But if all black entries in the first row are 1's, then (as in the $5 \times 5$ bulls-eye matrix in Figure 2) the center entry has an empty neighborhood. Hence if $n \equiv 1(\bmod 4)$, then there are no $n \times n$ positive even open dominating set matrices (nor $m \times n$ if $m \geq n$).

**Case 2** $n = 4p - 1$ for some positive integer $p$.

Now it seems to be possible to have a black 0 in column $4j$ of the first row, because after a left-right reflection, this would be column $4(p-j)$. If $n = 4p - 1$, then the $(2p, 2p)$ entry of the $n \times n$ bulls-eye matrix has all 1's in its open neighborhood. Changing the entry in column $4j$ to a 0 has the effect of changing all entries in a diamond-shaped pattern, like the positions of the 1's in the matrix in Figure 4 ($(2, 4j-1)$ and $(2, 4j+1)$ in the second row and so on). So now the $(2p, 2p)$ entry will have an empty open neighborhood. Hence all black entries in the first row must be 1's.

By prohibitions (iii) and (iv) in the "white diagonals argument", the only possibilities for $n = 4p - 1$ are those listed in (i), (ii), and (iii) in 2(a) of Theorem 8. We now verify that each of these does indeed yield a positive even open dominating set matrix.

Possibility (a)(i) gives a bulls-eye matrix with all open neighborhoods containing two 1's, except the center square has four 1's in its open neighborhood.

In possibility (a)(ii), precisely one white entry in the first row is 0. For example, if $n = 11$, then the white diagonals would be as for the first six white diagonals in Figure 3(b). The rest of the white entries in this $11 \times 11$ matrix would then be determined by a reflection over the slope 1 main diagonal. It is not hard to show that all open neighborhoods of white entries have precisely two 1's. As in (a)(i), the center entry has an open neighborhood consisting of four black 1's but all other open neighborhoods (both black and white) contain two 1's.

Figure 3(b) shows the white diagonals in possibility (a)(iii) with $n = 15$ and $a_{1,5} = a_{1,3} = 0$. The rest of the white diagonal entries are determined by refection over the slope 1 main diagonal. It is not hard to show that there will be four white open neighborhoods with four 1's, while all other white open neighborhoods have two 1's. In fact, if $a_{l,j} = a_{1,k} = 0$ where $j$ and $k$ are odd with $j < k$ then the $(\frac{k-j}{2}, \frac{j+k}{2})$ entry has an open neighborhood with four 1's as do the three other corresponding entries after reflections about the main diagonals. Hence we do get a positive even open dominating set matrix with a total of five open neighborhoods with four 1's.

**Case 3** $n$ is even.

By the "diagonals argument", there can be at most one black 0 in the first row, and if there is one, its column number is in $I$. Since $n$ is even, by symmetry, the same argument shows there can be at most one white 0 in the first two, and if there is one its column number is in $K$. If there is a black 0 in the first row in column number $4j$ it is not hard to show that the $(2j, 2j)$ entry has an open neighborhood of four black 1's (as in Figure 3(a)), as dos the $(n+1-2j, n+1-2j)$ entry, while all other black neighborhoods in the first $n$ rows contain precisely two black 1's. So if $n$ is even, possibility (b)(i) yields a positive even open dominating set matrix with precisely two open neighborhoods having four 1's, while possibility (b)(ii) yields a positive even open dominating set matrix with precisely four open neighborhoods having four 1's.

We have shown that if an $n \times n$ matrix has a 0 in the first row, then it cannot be a doubly perfect open dominating set matrix, completing the proof of Theorem 7.

If $A$ is any $m \times n$ even open dominating set matrix where $m > n$, then, because of the symmetry about both main diagonals (of the first $n$ rows), the $n^{th}$ row is the reversal of the first row. Since the $(n+1)^{st}$ row is all zeroes, the $(n+2)^{nd}$ row is the same as the $n^{th}$ row. If the first row has a 0, then the $(n+1)^{st}$ row has an entry with an empty open neighborhood. Hence any $m \times n$ even open dominating set matrix with $m > n$ must have all 1's in the first row, completing the proof of Theorem 8. □

By examining the various possibilities in Theorem 8, it is easy to count the number of $n \times n$ positive open dominating set matrices: if $n = 4p$ or $n = 4p + 2$ there are $(p + 1)^2$ of them, if $n = 4p - 1$ there are $p^2 + p + 1$ of them, and if $n = 4p + 1$ there are none. Each has at most two 0's in each of row 1, row $n$, column 1, and column $n$ and none has more than five open neighborhoods containing four 1's.

We remark that each doubly perfect open dominating set matrix $A$ can be expressed as the sum of two perfect open dominating set matrices $B$ and $C$. In each ring of 1's in each bulls-eye in $A$, choose two consecutive 1's for $B$, then two consecutive 1's for $C$, and so on. It is easy to see how to make these choices so that $B$ and $C$ are perfect.

## 4 Positive Even Closed Dominating Sets

As discussed in Section 2, given any $n$-vector $w$, we can use the closed version of Equation (1) to generate a sequence $r_1, r_2, \ldots$ of $n$-vectors with $r_1 = w$, and there exists an $m \times n$ even closed dominating set matrix with first row $w$ if and only if $r_{m+1} = 0$. By Theorems 3 and 4, if $n \neq 5$, then for each first row $w$, there exists an integer $m < 3 \cdot 2^{\frac{n}{2}}$ such that $r_{m+1} = 0$ (for $n = 5, r_{24} = 0$ for each first row $w$). If $r_{i+1} = 0$ for some $i$, then clearly $r_{i+2} = r_i$, and in fact $r_{i+2+j} = r_{i-j}$ for $j = 1, 2, \ldots, i-1$. Then $r_{2i+2} = 0$ and $r_{2i+3} = r_{2i+1} = r_1$, so if $m$ is the smallest integer such that $r_{m+1} = 0$, then the sequence $r_1, r_2, \ldots$ is periodic with period either $m+1$ or $2m+2$. Hence if there exists an $m \times n$ positive even closed dominating set matrix $A$ with $m^{th}$ row all 1's, then there exists a positive even closed dominating set matrix of size $(2m+1) \times n$ and if $A$ has first and $m^{th}$ row all 1's then there exists one of size $[t(m+1)-1] * n$ for all positive integers $t$.

So we want to know for which values of $n$ does there exist an $m \times n$ positive even closed dominating set matrix with first and last row all 1's. Certainly for $n = 1$ and $n = 2$ (the $1 \times 2$ and $2 \times 1$ matrices with both entries equal to 1). Certainly for $n = 4, 5, 8$ as shown in Figure 5(a)(b)(c). It turns out that these are the only such values of $n$.

```
1 1 1 1
0 1 1 0
0 1 1 0
1 1 1 1
  (a)
```

```
1 1 1 1 1
0 1 1 1 0
0 1 0 1 0
1 0 1 0 1
1 1 1 1 1
1 1 0 1 1
1 1 1 1 1
1 0 1 0 1
0 1 0 1 0
0 1 1 1 0
1 1 1 1 1
   (b)
```

```
1 1 1 1 1 1 1 1
0 1 1 1 1 1 1 0
0 1 0 0 0 0 1 0
1 0 0 1 1 0 0 1
1 0 1 0 0 1 0 1
0 0 1 0 0 1 0 0
1 1 0 1 1 0 1 1
0 0 1 0 0 1 0 0
1 0 1 0 0 1 0 1
1 0 0 1 1 0 0 1
0 1 0 0 0 0 1 0
0 1 1 1 1 1 1 0
1 1 1 1 1 1 1 1
      (c)
```

```
1 1 1 1 1 1 1 1 1 1 ...
0 1 1 1 1 1 1 1 1
0 1 0 0 0 0 0 0
1 0 0 1 1 1 1
1 0 1 0 1 1
0 0 1 1 1
1 1 1 1
0 1 1
0 0
0
      (d)
```

Figure 5.

**Lemma 9** *If $A$ is an $m \times n$ positive even closed dominating set matrix with first and last rows all 1's, then $n \in \{1, 2, 4, 5, 8\}$.*

*Proof:* It is easy to check that if $n \in \{3, 6, 7, 9\}$ and you generate an even closed dominating set matrix row by row starting with an all 1's first row, then you get an empty closed neighborhood before you get another row of all 1's. If $n \geq 10$, then the entries $(i, j)$ for the generated matrix with $i + j \leq 11$ as as shown in Figure 5(d), so the $(9, 1)$ entry has a closed neighborhood containing no 1's. □

**Theorem 10** *Let $n$ be a positive integer not in $\{1, 2, 4, 5, 8\}$. If $m_n \geq 6 \cdot 2^{\frac{n}{2}}$, then $G_{m,n}$ does not have a positive even closed dominating set.*

*Proof:* By Theorem 4, if $n \neq 5$, then in any $t \times n$ even closed dominating set matrix $A$ with $t \geq 3 \cdot 2^{\frac{n}{2}}$, one of the first $3 \cdot 2^{\frac{n}{2}}$ rows is all 0's. Let $k$

be minimal such that $r_k = 0$. Then $r_{2k} = 0$ as well, and $r_{k-1} = r_{k+1}$ and $r_{2k-1} = r_{2k+1}$. Unless both $r_1$ and $r_{k-1}$ are all 1's, then $A$ has an entry with closed neighborhood having no 1's. But by Lemma 9, $r_1$ and $r_{k-1}$ cannot both be all 1's. □

Using a simple computer-aided technique, we have verified the following.

**Theorem 11** *If $m$ and $n$ are integers greater than fifteen, then $G_{m,n}$ does not have a positive even closed dominating set.*

*Verification:* If $A = [a_{i,j}]$ is a (not necessarily positive) even closed dominating set matrix with $a_{i,j}$ predetermined for $j = 1, 2, \ldots, k$, then by the closed version of Equation (1), then entries $a_{i,j}$ are determined for all $i, j$ such that $i + j \leq k + 1$ (forming a triangle in the upper left corner of $A$). Using a computer, we computed this triangular submatrix for all $2^{32}$ possible choices for $a_{1,1}, a_{1,2}, \ldots, a_{1,32}$ and discovered that each one has an entry whose closed neighborhood contains no 1's. (This computation took 30 hours on a 2.4 GHz Intel Celeron). That means that there are only finitely many ordered pairs $(m, n)$ such that $n \notin \{1, 2, 4, 5, 8\}$ and there exists an $m \times n$ positive even closed dominating set matrix. Another simple exhaustive computer search found them all, and there are none where both $m$ and $n$ are greater than 15. □

If $n \in \{1, 2, 4, 5, 8\}$, then we can string together positive even closed dominating set matrices with $n$ columns and first and last row all 1's (each successive one followed by a row of 0's) to get bigger positive even closed dominating set matrices with $n$ columns. Along with the results of our computer searches, this gives us all $(m, n)$ such that $G_{m,n}$ has a positive even closed dominating set:

**Theorem 12** *If $m \geq n$, then $G_{m,n}$ has a positive even closed dominating set if and only if either*
*(1)* $n = 1$ *and* $m \equiv -1 (\text{mod } 3)$
*(2)* $n = 2$ *and $m$ is odd and greater than 2*
*(3)* $n = 4$ *and* $m \equiv -1 (\text{mod } 5)$
*(4)* $n = 5$ *and* $m \equiv -1 (\text{mod } 12)$
*(5)* $n = 8$ *and* $m \equiv -1 (\text{mod } 14)$
*(6)* $(m, n)$ *is one of* $(5, 3)$, $(5, 5)$, $(7, 5)$, $(8, 6)$, $(11, 7)$, $(14, 9)$, $(29, 9)$, $(11, 11)$, $(15, 11)$, $(17, 13)$, $(16, 14)$, $(23, 15)$.

We note that the middle row of the $11 \times 7$ and $29 \times 9$ matrices referred to in (6) of Theorem 12 is all 0's.

It would be of interest to find an algebraic or combinatorial proof of Theorem 11.

Recall that $D$ is a doubly perfect closed dominating set of a graph $G$ with vertex set $V$ if $|N[v] \cap D| = 2$ for each $v \in V$. In Figure 6, we show some doubly perfect dominating set matrices.

```
1    11    110    0110
1    00    001    1001
0    11    101    1001
1    00    100    0110
1    11    011
```

Figure 6. Four Matrices

It is easy to see that there exists an $m \times 1$ doubly perfect closed dominating set matrix if and only of $m \equiv -1 (\text{mod } 3)$ and there exists one of size $m \times 2$ if and only if $m$ is odd (and these are all unique).

**Theorem 13** *If $G_{m,n}$ has a doubly perfect dominating set where $m$ and $n$ are greater than 2, then either $m = n = 4$ or $m = 5$ and $n = 3$ (or vice versa).*

The proof of Theorem 13 is easy and will be omitted. Note that the existence of a doubly perfect dominating set in $G_{m,n}$ is equivalent to the tiling of an $m \times n$ rectangle with $1 \times 2$ tiles which do not share an edge such that any untiled square is adjacent to precisely two tiles.

It is easy to check that each $m \times n$ doubly perfect dominating set matrix is the sum of two perfect closed dominating set matrices except if $m = 5$ and $n = 3$ (or vice versa).

# 5 Open Problems

Goldwasser, Klostermeyer and Ware [11] left the following question.

**Question 1** *[11] If $G(n)$ is the fraction of the integers $t \in \{1, 2, \ldots, n\}$ such that $G_{t,t}$ has a non-empty even closed dominating set, what is $\lim_{n\to\infty} G(n)$?*

It is reported in [11] that $G(10,000) = 0.423$.

Since every graph has an odd dominating set, Caro, Klostermeyer, and Goldwasser considered the cardinality of the smallest odd dominating set in $G_{m,n}$, denoted $\gamma_{odd}(G_{m,n})$ [3, 4]. The size of the smallest dominating set of $G_{m,n}$ is denoted $\gamma(G_{m,n})$. The following results were obtained, but finding better bounds remains a challenging problem.

**Fact 14** *$\gamma_{odd}(G_{m,n}) \leq \frac{20}{7}\gamma(G_{m,n})$ where $G_{m,n}$ is a sufficiently large grid graph.*

An infinite family of grids was shown in [3] to have $\frac{\gamma_{odd}(G_{m,n})}{\gamma(G_{m,n})} \approx \frac{20}{9}$.

A *non-zero (mod k) dominating set* of a graph $G$ is a subset, $D$, of the vertices such that each vertex in $G$ has a non-zero (mod $k$) number vertices in $D$ in its closed neighborhood. An odd dominating set is a non-zero (mod 2) dominating set. This natural generalization of parity domination was introduced in [5]. Caro and Jacobson proved that every tree has a non-zero (mod $k$) dominating set, for all $k \geq 2$ [5]. Yuster generalized this result for graphs containing at most one cycle [18]. Dvorak, Cerny, Jelinek, Podbrdsky, Komlos and Mares used the probabilistic method (the Local Lemma to be precise) to show that if $\Delta(G) \leq c^k$, where $c$ is a constant dependent on $\frac{\Delta(G)}{\delta(G)}$, then $G$ has a non-zero (mod $k$) dominating set [8]. As such, all regular graphs and all grid graphs contain non-zero (mod $k$) dominating sets for all $k \geq 2$ (see also [15]).

To determine whether every graph has a non-zero (mod $k$) dominating set is a significant open problem.

**Conjecture 2** *[5] For all $k \geq 2$, every graph contains a non-zero (mod k) dominating set.*

Of course, only odd $k$ are of interest, since it is known that every graph contains a non-zero (mod 2) dominating set.

# References


[1] A. Amin, L. Clark, and P. Slater (1998), Parity Dimension for Graphs, *Disc. Math.*, vol. 187, pp. 1-17

[2] Y. Caro (1996), Simple Proofs to Three Parity Theorems, *Ars Comb.*, 42, pp. 175-180

[3] Y. Caro and W. Klostermeyer (2003), The Odd Domination Number of a Graph, *J. Comb. Math Comb. Comput.*, vol. 44 (2003), pp. 65-84

[4] Y. Caro, W. Klostermeyer, and J. Goldwasser (2001), Odd and Residue Domination Numbers of a Graph, *Discuss. Math. Graph Theory*, vol. 21, pp. 119-136

[5] Y. Caro and M. Jacobson (2003), On non-zero(mod k) dominating sets, *Discuss. Math. Graph Theory*, vol. 23, pp. 189-199

[6] R. Cowen, S. Hechler, J. Kennedy, and A. Steinberg (2004), Odd Neighborbood Transversals for Grid Graphs, manuscript

[7] Y. Dodis and P. Winkler (2001), "Universal Configurations in Light-Flipping Games," *Proc.* $12^{th}$ *SIAM Symp. Disc. Algorithms*, pp. 926-927

[8] Z. Dvorak (2003), personal communication

[9] J. Goldwasser, W. Klostermeyer, and G. Trapp, (1997), Characterizing Switch-Setting Problems, *Linear and Multilinear Algebra*, vol. 43, pp. 121-135

[10] J. Goldwasser and W. Klostermeyer (1997), Maximization Versions of "Lights Out" Games in Grids and Graphs, *Cong. Num.*, vol. 126, pp. 99-111

[11] J. Goldwasser, W. Klostermeyer, and H. Ware (2002), Fibonacci Polynomials and Parity Domination in Grid Graphs, *Graphs and Comb.*, vol. 18 (2002), pp. 271-283

[12] J. Goldwasser and W. Klostermeyer (2005), Odd and Even Dominating Sets with Open Neighborhoods, to appear in *Ars Comb.*

[13] M. Halldorsson, J. Kratochvil, and J. Telle (1999), Mod-2 Independence and Domination in Graphs, *Proc. Workshop on Graph-Theoretic Concepts in Computer Sci. '99*, Bergen, Norway

[14] T. Haynes, S. Hedetniemi, and P. Slater (1998), **Fundamentals of Domination in Graphs**, Marcel Dekker, New York

[15] C. Jerozal and W. Klostermeyer (2003), The Number of Non-Zero (mod $k$) Dominating Sets of Some Graphs, *International Mathematical Journal*, vol. 4 (2003), pp. 431-438

[16] W. Klostermeyer and J. Goldwasser, Total Perfect Codes in Grid Graphs (2005), to appear in *Bull. ICA*

[17] D. Van Wieren, M. Livingstone, and Q. Stout (1993), "Perfect Dominating Sets on Cube Connected Cycles," *Cong. Num.*, vol. 97, pp, 51-70

[18] R. Yuster (2004), Nowhere 0 mod $p$ Dominating Sets in Multigraphs, manuscript