WHAT IS...COP NUMBER?

ANTHONY BONATO

The cop number is a simple notion originating from a game played on a graph. Despite this apparent simplicity, consideration of the cop number leads to many questions in structural, probabilistic, and algorithmic graph theory. In the game of *Cops and Robbers*, there are two players, a set of *cops* and a single *robber*. We are given an undirected graph G with loops on each vertex. Players occupy vertices of G, and move along edges to neighboring vertices or remain on their current vertex. The cops move first by occupying a set of vertices; one cop can occupy only a single vertex, although more than one can occupy the same vertex. The robber then chooses a vertex to occupy, and the players move at alternate ticks of the clock. The game is played with perfect information, so the players see each others' moves. The cops win if they can *capture* the robber by moving to a vertex occupied him; otherwise, the robber wins. While many variations are possible (such as players moving at different speeds or playing with imperfect information) we focus on the game as described here. We consider only finite graphs, although the cop number is also studied in the infinite case. Cops and Robbers has found application to multiple-agent moving-target search in artificial intelligence, and variants of the game have been recently considered in fields such as robotics and mathematical counter-terrorism.

The *cop number of a graph* G, written $c(G)$, is the minimum number of cops needed to win in G. Placing a cop on each vertex clearly guarantees a win for the cops, so the cop number is well defined. If G is disconnected, then $c(G)$ is the sum of the cop numbers of the connected components; hence, we consider only connected graphs. References for the results discussed here may be found in [1].

Cop-win graphs are those where only one cop is needed to win, and they form the first and simplest case to analyze. If one vertex is adjacent to all others as in cliques or wheels, then of course the graph is cop-win. *Trees* are graphs with no cycles. Trees are cop-win, since the cop chases the robber along the unique path connecting them until the robber reaches a vertex with degree one. A degree one vertex u is a *corner* with a special property: there is vertex v such that v is adjacent to all the vertices adjacent to u (including u itself). By considering the second to last move of the cop before the cop captures the robber, it is evident that a cop-win graph must have at least one corner. Deleting this corner gives rise to another cop-win graph: whenever the robber moves to the corner u, the cop plays as if the cop were on v. This observation along with an induction proves that a graph is cop-win if and only if we may iteratively delete corners and end up with a single vertex.

A *planar graph* is one that can be drawn in the plane without edge crossings. The famous Four Color Theorem states that, for any planar graph, we need at most four colors to assign colors to the vertices in such a way that adjacent vertices receive different colors. Aigner and Fromme [2] introduced the cop number in 1984, and proved that a planar graph has cop number at most 3. For example, the dodecahedron is a planar graph with cop number 3. One of the main tools used in their proof were *isometric paths*: A path is isometric if distances between vertices in the path are the same as in the graph. They showed that one cop can *guard* an isometric path, in the sense that we can move a cop along vertices of the path in such a way that if the robber moved onto the path, the robber would be captured. To guard an isometric path, the cop exploits a *retraction* (a graph homomorphism which is the identity on its image) onto the path: the cop simply captures the image of the robber on the path. Despite the characterization described earlier for cop-win graphs, there is no known characterization of cop-win planar graphs.

An equally enticing and challenging aspect is that existing graph parameters appear in only a few bounds for cop numbers. A *dominating set* S has the property that all vertices not in S are adjacent to some vertex of S. The *domination number* of G is the minimum order of a dominating set in G. The cop number is bounded above by the domination number, as the cops simply occupy a minimum order dominating set on their first move and catch the robber in the next round. Unfortunately, this bound is far from tight as the reader can check in the case of paths. The minimum order of a cycle in G is called its *girth*. If G has girth at least 5, then the cop number is bounded below by the minimum degree of G. The *genus* of a graph G, written g, is the smallest k such that G can be drawn on a sphere with k handles so that distinct edges do not intersect except at common vertices. Schroeder proved that $c(G) \leq \left\lfloor \frac{3}{2}g \right\rfloor + 3$, and conjectured that $c(G) \leq g + 3$.

How large can the cop number be as a function of the order of the graph? Graphs arising from finite geometry provide some insight into this question. Consider a projective plane P of order q, and its incidence graph $G(P)$. The graph $G(P)$ has vertices the points and lines of P, and so has $2q^2 + 2q + 2$ vertices. No two points (or lines) are adjacent, and a point is adjacent with a line if it is on that line. For example, the Fano plane has incidence graph isomorphic to the Heawood graph, which has cop number 3. See Figure 1. As the girth of

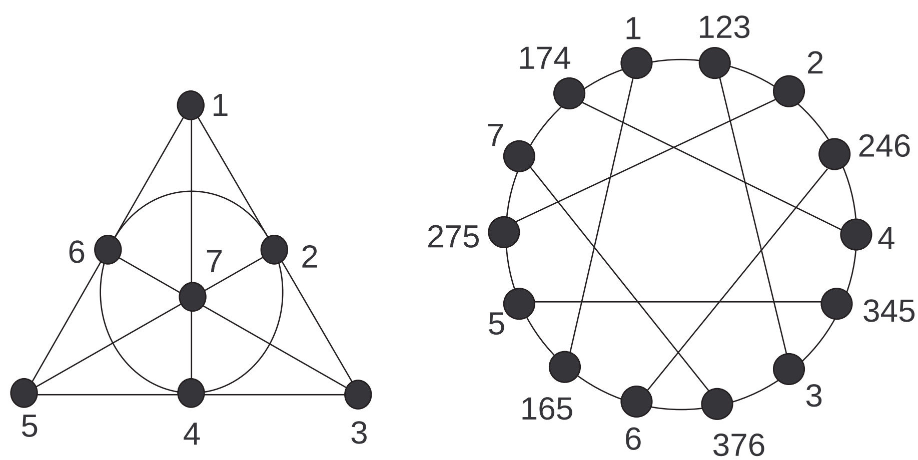

FIGURE 1. The Fano plane and its incidence graph. Points and lines of the plane are labelled on the graph.

$G(P)$ is 6 and each vertex has degree $q + 1$, it is not hard to see that the cop number is at least $q + 1$. Hence, the cop number of a graph with n vertices can be as large as a constant multiple of $\sqrt{n}$.

Let $c(n)$ be the maximum value of $c(G)$, where G is a connected graph of order n. Meyniel's conjecture states that $c(n) = O(\sqrt{n})$. In other words, for n sufficiently large, the cop number is at most a constant multiple of $\sqrt{n}$ (as is the case for incidence graphs of projective planes). Frankl communicated the conjecture in his 1987 paper, where he used a greedy argument with isometric paths and the Moore bound to prove that $c(n) = O\left(n\frac{\log\log n}{\log n}\right)$. Meyniel's conjecture was largely forgotten until recently, and is now gathering much research attention. The best known upper bound is

$$c(n) = O\left(\frac{n}{2^{(1+o(1))\sqrt{\log_2 n}}}\right), \tag{1}$$

which was proven independently by three groups of researchers using the probabilistic method. The $n^{1-o(1)}$ bound in (1) is far from the conjecture, and even proving that $c(n) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$ remains open. At the present time, the jury is still out on whether Meyniel's conjecture holds. Partial evidence in favor of the conjecture comes from its having been proved for binomial random graphs $G(n, p)$ for a wide range of $p = p(n)$.

We finish by highlighting some algorithmic aspects of the cop number. If k is a fixed integer and G is given as input, then one can determine whether $c(G) \leq k$ by doing a polynomial-time computation with running time $O(n^{2k+3})$. Unfortunately, because this bound is exponential in k, it is therefore impractical for large k. If k is not fixed (and so may be a function of n), then determining if $c(G) \leq k$ is **NP**-hard. We do not know, however, if this problem is in **NP**. It is conjectured that computing the cop number when k is not fixed is **EXPTIME**-complete, which would imply that it is among the hardest problems solvable in exponential time.

References

[1] A. Bonato, R.J. Nowakowski, *The Game of Cops and Robbers on Graphs*, American Mathematical Society, Providence, Rhode Island, 2011.

[2] M. Aigner, M. Fromme, A game of cops and robbers, *Discrete Applied Mathematics* **8** (1984) 1–11.

Department of Mathematics, Ryerson University, Toronto, ON, Canada, M5B 2K3

E-mail address: abonato@ryerson.ca