CHAPTER 1

System of First Order Differential Equations

In this chapter, we will discuss system of first order differential equations. There are many applications that involving find several unknown functions simultaneously . Those unknown functions are related by a set of equations that involving the unknown functions and their first derivatives. For example, in Chapter Two, we studied the epidemic of contagious diseases. Now if

- $S(t)$ denotes number of people that is susceptible to the disease but not infected yet.
- $I(t)$ denotes number of people actually infected.
- $R(t)$ denotes the number of people have recovered.

If we assume

- The fraction of the susceptible who becomes infected per unit time is proportional to the number infected, b is the proportional number.
- A fixed fraction rS of the infected population recovers per unit time, $0 \leq r \leq 1$.
- A fixed fraction of the recovers g become susceptible and infected, $0 \leq g \leq 1$. proportional function.

The system of differential equations model this phenomena are

$$\begin{aligned} S' &= -bIS + gR \\ I' &= bIS - rI \\ R' &= rI - gR \end{aligned}$$

The numbers of unknown function in a system of differential equations can be arbitrarily large, but we will concentrate ourselves on 2 to 3 unknown functions.

1. Principle of superposition

Let $a_{ij}(t)$, $b_j(t)$ $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$ be known function, and $x_i t$, $i = 1, 2, \cdots, n$ be unknown functions, the linear first order system of differential equation for $x_i(t)$ is the following,

$$\begin{aligned}
x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t)\\
x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t)\\
x_3'(t) &= a_{31}(t)x_1(t) + a_{32}(t)x_2(t) + \cdots + a_{3n}(t)x_n(t) + b_3(t)\\
&\vdots\\
x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_1(t)
\end{aligned}$$

Let $\boldsymbol{x}(t)$ be the column vector of unknown functions x_it, $i = 1, 2, \cdots, n$, $\boldsymbol{A}(t) = (a_{ij}(t)$, and $\boldsymbol{b}(t)$ be the column vector of known functions b_it, $i = 1, 2, \cdots, n$, we can write the first order system of equations as

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t) \tag{1}$$

- When $n = 2$, the linear first order system of equations for two unknown functions in matrix form is,

$$\begin{bmatrix} \mathrm{x}_1'(\mathrm{t}) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \mathrm{a}_{11}(\mathrm{t}) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{b}_1(\mathrm{t}) \\ b_2(t) \end{bmatrix}$$

- When $n = 3$, the linear first order system of equations for three unknown functions in matrix form is,

$$\begin{bmatrix} \mathrm{x}_1'(\mathrm{t}) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} \mathrm{a}_{11}(\mathrm{t}) & a_{12}(t) & a_{13} \\ a_{21}(t) & a_{22}(t) & a_{23} \\ a_{31}(t) & a_{32}(t) & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \\ x_3 t \end{bmatrix} + \begin{bmatrix} \mathrm{b}_1(\mathrm{t}) \\ b_2(t) \\ b_3(t) \end{bmatrix}$$

A **solution** of equation (1) on the open interval I is a column vector function $\boldsymbol{x}(t)$ whose derivative (as a vector-values function) equals $\boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)$. The following theorem gives existence and uniqueness of solutions,

THEOREM 1.1. *If the vector-valued functions $\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$ are continuous over an open interval I contains t_0, then the initial value problem*

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \end{cases}$$

has an unique vector-values solution $\boldsymbol{x}(t)$ that is defined on entire interval I for any given initial value $\boldsymbol{x}_0$.

When $\boldsymbol{b}(t) \equiv \boldsymbol{0}$, the linear first order system of equations becomes

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t),$$

which is called a **homogeneous equation**.

As in the case of one equation, we want to find out the general solutions for the linear first order system of equations. To this end, we first have the following results for the homogeneous equation, THEOREM 1.2. **Principle of Superposition** *Let* $\boldsymbol{x}_1(t),\ bx_2(t), \cdots, \boldsymbol{x}_n(t)$ *be* n *solutions of the homogeneous linear equation*

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$$

on the open interval I. *If* $c_1,\ c_2,\ \cdots,\ c_n$ *are* n *constants, then the linear combination*

$$c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + c_3\boldsymbol{x}_3(t) + \cdots + c_n\boldsymbol{x}_n(t)$$

is also a solution on I.

EXAMPLE 1.1. *Let*

$$\boldsymbol{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(t)$$

, $\boldsymbol{x}_1(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{t}} \\ 0 \end{bmatrix}$ *and* $\boldsymbol{x}_2(t) = \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$ *are two solutions, as*

$$bx_1'(t) = \begin{bmatrix} (\mathrm{e}^{\mathrm{t}})' \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{t}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{t}} \\ 0 \end{bmatrix}$$

and

$$bx_2'(t) = \begin{bmatrix} 0 \\ (e^{-2t})' \end{bmatrix} = \begin{bmatrix} 0 \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

By the Principle of Superposition, for any two constants c_1 *and* c_2

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) = c_1 \begin{bmatrix} \mathrm{e}^{\mathrm{t}} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} \mathrm{c_1 e^t} \\ c_2 e^{-2t} \end{bmatrix}$$

is also solution. We shall see that it is actually the general solution.

The next theorem gives the general solution of linear system of equations,

THEOREM 1.3.
- *Let* $\boldsymbol{x}_1(t),\ \boldsymbol{x}_2(t),\ \cdots,\ bx_n(t)$ *be* n *linearly independent (as vectors) solution of the homogeneous system*

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t),$$

then for any solution $\boldsymbol{x}_c(t)$ *there exists* n *constants* $c_1,\ c_2,\ \cdots,\ c_n$ *such that*

$$\boldsymbol{x}_c(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots + c_n\boldsymbol{x}_n(t).$$

We call $\boldsymbol{x}_c(t)$ *the general solution of the homogeneous system.*

If $x_p(t)$ is a particular solution of the nonhomogeneous system,

$$\boldsymbol{x}(t) = \boldsymbol{B}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

and $x_c(t)$ is the general solution to the associate homogeneous system,

$$\boldsymbol{x}(t) = \boldsymbol{B}(t)\boldsymbol{x}(t)$$

then $\boldsymbol{x}(t) = \boldsymbol{x}_c(t) + \boldsymbol{x}_p(t)$ is the general solution.

EXAMPLE 1.2. *Let*

$$\boldsymbol{x}'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -4t^2 + 5t \\ -6t^2 + 7t + 1 \end{bmatrix} \boldsymbol{x}(t)$$

, $\boldsymbol{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $\boldsymbol{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ are two linearly independent solutions. and $\boldsymbol{x}_p(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix}$ is a particular solution. By Theorem 1.3,

(2) $\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \boldsymbol{x}_p(t) = \begin{bmatrix} 3c_1e^{2t} + c_2e^{-5t} + t^2 \\ 2c_1e^{2t} + 3c_2e^{-5t} + t \end{bmatrix}$

is the general solution. Now suppose we want to find a particular solution that satisfies the initial condition $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then let $t = 0$ in (2), we have

$$\boldsymbol{x}(0) = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

which can be written in matrix form,

$$\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

Solve this equation, we get $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So the particular solution is $\boldsymbol{x}(t) = \begin{bmatrix} 3e^{2t} - e^{-5t} + t^2 \\ 2e^{2t} - 3e^{-5t} + t \end{bmatrix}$.

From the above example, we can summarize the general steps in find a solution to initial value problem,

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \end{cases}$$

- **Step One:** Find the general solution $\boldsymbol{x}_c = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots + c_n\boldsymbol{x}_n(t)$, where $\boldsymbol{x}_1(t),\ \boldsymbol{x}_2(t),\ \cdots,\ \boldsymbol{x}_n(t)$ are a set of linearly independent solutions, to the associate homogeneous system, $\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$.
- **Step Two:** Find a particular solution $\boldsymbol{x}_p(t)$to the nonhomogeneous system, $\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)$.
- **Step Three:** Set $\boldsymbol{x}(t) = \boldsymbol{x}_c(t) + \boldsymbol{x}_p(t)$ and use the equation $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, to determine $c_1,\ c_2,\ \cdots,\ c_n$.

2. Homogeneous System

We will use a powerful method called eigenvalue method to solve the homogeneous system

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

where $\boldsymbol{A}$ is a matrix with constant entry. We will present this method for $\boldsymbol{A}$ is either a 2×2 or 3×3 cases. The method can be used for $\boldsymbol{A}$ is an $n\times n$ matrix. The idea is to find solutions of form

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t}, \tag{3}$$

a straight line that passing origin in the direction $\boldsymbol{v}$. Now taking derivative on $\boldsymbol{x}(t)$, we have

$$\boldsymbol{x}'(t) = \lambda\boldsymbol{v}e^{\lambda t} \tag{4}$$

put (3) and (2.2) into the homogeneous equation, we get

$$\boldsymbol{x}'(t) = \lambda\boldsymbol{v}e^{\lambda t} = \boldsymbol{A}\boldsymbol{v}e^{\lambda t}$$

So

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v},$$

which indicates that λ must be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{v}$ is an associate eigenvector.

2.1. $\boldsymbol{A}$ is a 2×2 matrix.

Suppose

$$\boldsymbol{A} = \begin{bmatrix} \mathrm{a}_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then the characteristic polynomial $p(\lambda)$ of $\boldsymbol{A}$ is

$$p(\lambda) = |\boldsymbol{A}-\lambda I| = (a_{11}-\lambda)*(a_{22}-\lambda)-a_{12}a_{21} = \lambda^2-(a_{11}+a_{22})+(a_{11}a_{22}-a_{12}a_{22}.$$

So $p(\lambda)$ is a quadratic polynomial of λ. From Algebra, we know that $p(\lambda) = 0$ has either 2 distinct real solutions, or a double solution, or 2 conjugate complex solutions. The following theorem summarize the solution to the homogeneous system, THEOREM 2.1. *Let $p(\lambda)$ be the characteristic polynomial of $\boldsymbol{A}$, for $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$,*

Case 1: *$p(\lambda) = 0$ has two distinct real solutions λ_1 and λ_2.*

Suppose $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \end{bmatrix}$ are associate eigenvector (i.e, $\boldsymbol{A}\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1$ and $\boldsymbol{A}\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2$) Then the general solution is

$$\boldsymbol{x}_c(t) = c_1\boldsymbol{v}_1 e^{\lambda_1 t} + c_2\boldsymbol{v}_2 e^{\lambda_2 t}$$

And

$$\Phi(t) = \begin{bmatrix} \mathrm{v}_{11}\mathrm{e}^{\lambda_1 \mathrm{t}} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{bmatrix}$$

is called the **fundamental matrix***(A* **fundamental matrix** *is a square matrix whose columns are linearly independent solutions of the homogeneous system).*

Case 2: *$p(\lambda) = 0$ has a double solutions λ_0.*

In this case $p(\lambda) = (\lambda - \lambda_0)^2$ and λ_0 is a zero of $p(\lambda)$ with multiplicity 2.

(1) λ_0 **has two linearly independent eigenvectors:**

Suppose $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \end{bmatrix}$ are associate linearly independent eigenvectors. Then the general solution is

$$\boldsymbol{x}_c(t) = (c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2)e^{\lambda_0 t}$$

And

$$\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} \mathrm{v}_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

(2) λ_0 **has only one associate eigenvector:**

Suppose $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \end{bmatrix}$ is the only associated eigenvector and $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \end{bmatrix}$ is a solution of

$$(\lambda_0 I - \boldsymbol{A})\boldsymbol{v}_2 = \boldsymbol{v}_1.$$

Then the general solution is,

$$\boldsymbol{x}_c(t) = (c_1\boldsymbol{v}_1 + c_2(t\boldsymbol{v}_1 + \boldsymbol{v}_2)e^{\lambda_0 t}$$

And

$$\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} \mathrm{v}_{11} & (v_{11}t + v_{12}) \\ v_{21} & (v_{21}t + v_{22}) \end{bmatrix}$$

is the fundamental solution matrix.

Case 3: $p(\lambda) = 0$ *has two conjugate complex solutions* $a+bi$ *and* $a-bi$. *Suppose* $\boldsymbol{v} = \begin{bmatrix} \mathrm{v}_{11} + \mathrm{iv}_{12} \\ v_{21} + iv_{22} \end{bmatrix}$ *is the associate complex eigenvector with respect to* $a+bi$, *then the general solution is,* $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \end{bmatrix}$ *and* $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \end{bmatrix}$

$$\boldsymbol{x}_c(t) = [c_1(\boldsymbol{v}_1 \cos(bt) - \boldsymbol{v}_2 \sin(bt))c_2(\boldsymbol{v}_2 \cos(bt) + \boldsymbol{v}_1 \sin(bt))]e^{at}.$$

And

$$\Phi(t) = e^{at} \begin{bmatrix} \mathrm{v}_{11} \cos(\mathrm{bt}) - \mathrm{v}_{12} \sin(\mathrm{bt}) & v_{12} \cos(bt) + v_{11} \sin(bt) \\ v_{21} \cos(bt) - v_{22} \sin(bt) & v_{22} \cos(bt) + v_{21} \sin(bt) \end{bmatrix}$$

is the fundamental matrix.

From Theorem 2.1, let $\Phi(t)$ be the fundamental matrix, the general solution is given by $\boldsymbol{x}_c(t) = \Phi(t)\boldsymbol{c}$, with $\boldsymbol{c} = \begin{bmatrix} \mathrm{c}_1 \\ c_2 \end{bmatrix}$ and the solution that satisfies a given initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \Phi(t)\Phi(t_0)^{-t}\boldsymbol{x}_0$$

EXAMPLE 2.1. **Two distinct eigenvalues case** *Find the general solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & -3 \\ -1 & -5 \end{bmatrix} \boldsymbol{x}(t)$

Solution Using Mathcad , functions **eigenvals()** and **eigenvecs()**

> In Mathcad , eigenvecs(M) Returns a matrix containing the eigenvectors. The nth column of the matrix returned is an eigenvector corresponding to the nth eigenvalue returned by eigenvals.

we find,$\lambda_1 = -\frac{3}{2} + \frac{1}{2}\sqrt{61}$ and $\lambda_2 = -\frac{3}{2} - \frac{1}{2}\sqrt{61}$ with associated eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} -7-\sqrt{61} \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -7+\sqrt{61} \\ 2 \end{bmatrix}$ respectively. So the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} (-7-\sqrt{61})\mathrm{e}^{(-\frac{3}{2}+\frac{1}{2}\sqrt{61})\mathrm{t}} & (-7+\sqrt{61})e^{(-\frac{3}{2}-\frac{1}{2}\sqrt{61})t} \\ 2e^{(-\frac{3}{2}+\frac{1}{2}\sqrt{61})t} & 2e^{(-\frac{3}{2}-\frac{1}{2}\sqrt{61})t} \end{bmatrix}$$

and the general solution is, for $\boldsymbol{c} = \begin{bmatrix} \mathrm{c}_1 \\ c_2 \end{bmatrix}$,

$$\boldsymbol{x}_c(t) = \Phi(t)\boldsymbol{c}$$

⊣

EXAMPLE 2.2. **One double eigenvalues with two linearly independent eigenvectors** *Find the general solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}(t)$.

Solution The eigenvalue is $\lambda_0 = 2$ and associated eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the general solution is $\boldsymbol{x}_c = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$ ⊣

EXAMPLE 2.3. **One double eigenvalues with only one eigenvector** *Find the solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & -3 \\ 3 & 8 \end{bmatrix} \boldsymbol{x}(t)$ *and* $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Solution Using Mathcad , functions **eigenvals()** and **eigenvecs()** we can find a double eigenvalue $\lambda_0 = 5$ and eigenvector $\begin{bmatrix} -.707 \\ .707 \end{bmatrix}$. Notice, the symbolic operator $\dot{\to}$(bring up by either [Shift][Ctrl][.] or [Ctrl][.]) will not work with eigenvecs() this time, but since multiply an eigenvector by a nonzero constant still get an eigenvector, we can choose $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

To find $\boldsymbol{w}$ that satisfies $(\boldsymbol{A} - \lambda_0 \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}_1$ λ_0 we will solve $(\boldsymbol{A} - \lambda_0 \boldsymbol{I}) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. That is,

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

One solution is $w_1 = 1$ and $w_2 = 0$

So the fundamental matrix is

$$\Phi(t) = e^{5t} \begin{bmatrix} -3 & -3t + 1 \\ 3 & 3t \end{bmatrix}$$

and the general solution is, $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$,

$$\boldsymbol{x}_c(t) = \Phi(t)\boldsymbol{c}$$

Now, $\Phi(0) = e^{5(0)} \begin{bmatrix} -3 & -3(0) + 1 \\ 3 & 3(0) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & 0 \end{bmatrix}$ and $\Phi(0)^{-1} = \frac{1}{3} \begin{bmatrix} 0 & -1 \\ -3 & -3 \end{bmatrix}$.

Hence, the particular solution is $\boldsymbol{x}(t) = \Phi(t)\Phi(0)^{-1}\boldsymbol{x}_0 = e^{5t} \begin{bmatrix} -3t + 4 \\ 3t - 3 \end{bmatrix}$ ⊣

Example 2.4. **Two conjugate complex eigenvalues case** *Find the general solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \boldsymbol{x}(t)$

Solution Using Mathcad , functions **eigenvals()** and **eigenvecs()** we find two conjugate complex eigenvalues, $\lambda_1 = 2 + i\sqrt{3}$ and $\lambda_2 = 2 - i\sqrt{3}$ with associated eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} \sqrt{3} \\ -i \end{bmatrix}$ with respect to λ_1. Compare this with the Theorem 2.1, we have $a = 2, b = \sqrt{3}, v_{11} = \sqrt{3}, v_{21} = 0, v_{12} = 0$, and $v_{22} = -1$.

So the fundamental matrix is

$$\Phi(t) = e^{2t} \begin{bmatrix} \sqrt{3}\cos(\mathrm{bt}) & -\sin(bt) \\ \sqrt{3}\sin(bt) & -\cos(bt) \end{bmatrix}$$

and the general solution is, $\boldsymbol{c} = \begin{bmatrix} \mathrm{c}_1 \\ c_2 \end{bmatrix}$,

$$\begin{aligned} \boldsymbol{x}_c(t) &= \Phi(t)\boldsymbol{c} = e^{2t} \begin{bmatrix} \sqrt{3}\cos(\sqrt{3}\mathrm{t}) & -\sin(\sqrt{3}t) \\ \sqrt{3}\sin(\sqrt{3}t) & -\cos(\sqrt{3}t) \end{bmatrix} \begin{bmatrix} \mathrm{c}_1 \\ c_2 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} \sqrt{3}\mathrm{c}_1\cos(\sqrt{3}\mathrm{t}) - \mathrm{c}_2\sin(\sqrt{3}\mathrm{t}) \\ \sqrt{3}c_1\sin(\sqrt{3}t) - c_2\cos(\sqrt{3}t) \end{bmatrix} \end{aligned}$$

Suppose we want to find a solution such that $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$$\begin{aligned} \boldsymbol{x}(t) &= \Phi(t)\Phi(0)^{-1}\boldsymbol{x}(0) \\ &= e^{2t} \begin{bmatrix} \sqrt{3}\cos(\sqrt{3}\mathrm{t}) & -\sin(\sqrt{3}t) \\ \sqrt{3}\sin(\sqrt{3}t) & -\cos(\sqrt{3}t) \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} \sqrt{3}\cos(\sqrt{3}\mathrm{t}) & -\sin(\sqrt{3}t) \\ \sqrt{3}\sin(\sqrt{3}t) & -\cos(\sqrt{3}t) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -2 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} \cos(\sqrt{3}\mathrm{t}) + +2\sin(\sqrt{3}\mathrm{t}) \\ -\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t) \end{bmatrix} \end{aligned}$$

⊣

2.2. *A* is a 3×3 **matrix.** Suppose

$$\boldsymbol{A} = \begin{bmatrix} \mathrm{a}_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the characteristic polynomial $p(\lambda)$ of $\boldsymbol{A}$ given by

$$p(\lambda) = |\boldsymbol{A} - \lambda I|,$$

is a cubic polynomial of λ. From Algebra, we know that $p(\lambda) = 0$ has either 3 distinct real solutions, or 2 distinct solutions and one is a double solution, or one real solution and 2 conjugate complex solutions, or a triple solution. The following theorem summarize the solution to the homogeneous system,

THEOREM 2.2. *Let $p(\lambda)$ be the characteristic polynomial of $\boldsymbol{A}$, for $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$,*

Case 1: *$p(\lambda) = 0$ has three distinct real solutions λ_1, λ_2, and λ_3.* *Suppose* $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$, *and* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ *are associate eigenvector (i.e,* $\boldsymbol{A}\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1$, $\boldsymbol{A}\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2$, *and* $\boldsymbol{A}\boldsymbol{v}_3 = \lambda_3\boldsymbol{v}_3$*) Then the general solution is*

$$\boldsymbol{x}_c(t) = c_1\boldsymbol{v}_1e^{\lambda_1 t} + c_2\boldsymbol{v}_2e^{\lambda_2 t} + c_3\boldsymbol{v}_3e^{\lambda_3 t}$$

And the **fundamental matrix** *is*

$$\Phi(t) = \begin{bmatrix} \mathrm{v}_{11}\mathrm{e}^{\lambda_1 \mathrm{t}} & v_{12}e^{\lambda_2 t} & v_{13}e^{\lambda_3 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & v_{23}e^{\lambda_3 t} \\ v_{31}e^{\lambda_1 t} & v_{32}e^{\lambda_2 t} & v_{33}e^{\lambda_3 t} \end{bmatrix}.$$

Case 2: *$p(\lambda) = 0$ has a double solutions λ_0.* *So $p(\lambda) = (\lambda - \lambda_0)^2(\lambda - \lambda_1)$, and λ_0 has multiplicity 2. Let* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ *is the eigenvector associated with λ_1.*

[1] λ_0 **has two linearly independent eigenvectors:** *Suppose* $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \end{bmatrix}$ *and* $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \end{bmatrix}$ *are associate linearly independent eigenvectors. Then the general solution is*

$$\boldsymbol{x}_c(t) = (c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2)e^{\lambda_0 t} + c_3\boldsymbol{v}_3e^{\lambda_1 t}$$

And

$$\Phi(t) = \begin{bmatrix} \mathrm{v}_{11}\mathrm{e}^{\lambda_0 \mathrm{t}} & v_{12}e^{\lambda_0 t} & v_{13}e^{\lambda_1 t} \\ v_{21}e^{\lambda_0 t} & v_{22}e^{\lambda_0 t} & v_{23}e^{\lambda_1 t} \\ v_{31}e^{\lambda_0 t} & v_{32}e^{\lambda_0 t} & v_{33}e^{\lambda_1 t} \end{bmatrix}$$

[2] λ_0 **has one eigenvector:**

Suppose $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ *is the associated eigenvector with respect to* λ_0 *and* $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ *is a solution of*

$$(\lambda_0 I - \boldsymbol{A})\boldsymbol{v}_2 = \boldsymbol{v}_1.$$

Then the general solution is,

$$\boldsymbol{x}_c(t) = (c_1\boldsymbol{v}_1 + c_2(t\boldsymbol{v}_1 + \boldsymbol{v}_2))e^{\lambda_0 t} + c_3\boldsymbol{v}_3 e^{\lambda_1}$$

And

$$\Phi(t) = \begin{bmatrix} \mathrm{v}_{11}\mathrm{e}^{\lambda_0 \mathrm{t}} & (v_{11}t + v_{12})e^{\lambda_0 t} & v_{13}e^{\lambda_1} \\ v_{21}e^{\lambda_0 t} & (v_{21}t + v_{22})e^{\lambda_0 t} & v_{23}e^{\lambda_1} \\ v_{31}e^{\lambda_0 t} & (v_{31}t + v_{32})e^{\lambda_0 t} & v_{33}e^{\lambda_1} \end{bmatrix}$$

is the fundamental solution matrix.

Case 3: $p(\lambda) = 0$ *has two conjugate complex solutions* $a \pm bi$ *and a real solution* λ_1.

Suppose $\boldsymbol{v} = \begin{bmatrix} \mathrm{v}_{11} + \mathrm{iv}_{12} \\ v_{21} + iv_{22} \\ v_{31} + iv_{32} \end{bmatrix}$ *is the associate complex eigenvector with respect to* $a + bi$*, then the general solution is, let* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{13} \\ v_{23} \\ V_{33} \end{bmatrix}$*, are associated eigenvectors with respect to* λ_1*,*

$$\boldsymbol{x}_c(t) = [c_1(\boldsymbol{v}_1\cos(bt) - \boldsymbol{v}_2\sin(bt))c_2(\boldsymbol{v}_2\cos(bt) + \boldsymbol{v}_1\sin(bt))]e^{at} + c_3\boldsymbol{v}_3 e^{\lambda_1}.$$

And

$$\Phi(t) = e^{at}\begin{bmatrix} \mathrm{v}_{11}\cos(\mathrm{bt}) - \mathrm{v}_{12}\sin(\mathrm{bt}) & v_{12}\cos(bt) + v_{11}\sin(bt) & v_{13}e^{\lambda_1} \\ v_{21}\cos(bt) - v_{22}\sin(bt) & v_{22}\cos(bt) + v_{21}\sin(bt) & v_{23}e^{\lambda_1} \\ v_{31}\cos(bt) - v_{32}\sin(bt) & v_{32}\cos(bt) + v_{31}\sin(bt) & v_{33}e^{\lambda_1} \end{bmatrix}$$

is the fundamental matrix.

Case 4: $p(\lambda) = 0$ *has solution* λ_0 *with multiplicity 3.*
In this case, $p(\lambda) = (\lambda - \lambda_0)^3$.

[1] λ_0 **has three linearly independent eigenvectors.**

Let $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ V_{32} \end{bmatrix}$, *and* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{13} \\ v_{23} \\ V_{33} \end{bmatrix}$ *be the three linearly independent eigenvectors. Then the general* *solution is* $\boldsymbol{x}_c(t) = (c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3)e^{\lambda_0 t}$ *and fundamental matrix is* $\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} \mathrm{v}_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & V_{23} \\ v_{31} & & \\ v_{32} & & \\ V_{33} & & \end{bmatrix}$

[2] λ_0 **has two linearly independent eigenvectors.**

Suppose $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ V_{32} \end{bmatrix}$ *are the linearly independent eigenvectors. Let* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{13} \\ v_{23} \\ V_{33} \end{bmatrix}$, *then only one of the two equations,* $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_1$ *or* $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2$ *can has a solution that is linearly independent with* $\boldsymbol{v}_1, \boldsymbol{v}_2$.

Suppose $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2$ *generates such a solution. Then the general solution is* $\boldsymbol{x}_c(t) = [c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3(t\boldsymbol{v}_2 + \boldsymbol{v}_3)]e^{\lambda_0 t}$ *and fundamental matrix is* $\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} \mathrm{v}_{11} & v_{12} & tv_{12} + v_{13} \\ v_{21} & v_{22} & tv_{22} + V_{23} \\ v_{31} & v_{32} & tv_{32} + V_{33} \end{bmatrix}$

[3] λ_0 **has only one eigenvector.**

Let $\boldsymbol{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ *be the linearly independent eigenvectors. Let* $\boldsymbol{v}_2 = \begin{bmatrix} \mathrm{v}_{12} \\ v_{22} \\ V_{32} \end{bmatrix}$ *and* $\boldsymbol{v}_3 = \begin{bmatrix} \mathrm{v}_{13} \\ v_{23} \\ V_{33} \end{bmatrix}$ *be two vectors that satisfies*

$$(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \textit{and} \quad (\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2.$$

Then the general solution is $\boldsymbol{x}_c(t) = [c_1\boldsymbol{v}_1 + c_2(t\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_3(t^2\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{v}_3)]e^{\lambda_0 t}$ *and fundamental matrix is* $\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} \mathrm{v}_{11} & tv_{11} + v_{12} & t^2v_{11} + tv_{12} + v_{13} \\ v_{21} & tv_{21} + v_{22} & t^2v_{21} + tv_{22} + V_{23} \\ v_{31} & tv_{31} + v_{32} & t^2v_{31} + tv_{32} + V_{33} \end{bmatrix}$

REMARK 2.1. *Suppose* $\boldsymbol{A}$ *is an* $n \times n$ *matrix, for the homogeneous system* $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, *three general case would happen*

Case 1: $\boldsymbol{A}$ *has* n *distinct eigenvalues* $\lambda_i, \quad i = 1, 2, \cdots, n$ *with linearly independent eigenvectors* $\boldsymbol{v}_i, \quad i = 1, 2, \cdots, n$ *then the general solution will be* $\boldsymbol{x}_c(t) = c_1\boldsymbol{v}_1 e^{\lambda_1} + c_2\boldsymbol{v}_2 e^{\lambda_2} + \cdots + c_n\boldsymbol{v}_n e^{\lambda_n}$

Case2: *$\boldsymbol{A}$ has $m < n$ distinct eigenvalues, in this case some eigenvalues would have multiplicity greater than 1.*

Suppose λ_r has multiplicity r. Depending on how many linearly independent eigenvectors are associated with λ_r the situation could be very complex. Let p be the number of linearly eigenvectors associated with λ_r, then $d = r - p$ is called the **deficit** *of λ_r. The simply cases are either $d = 0$ or $d = r - 1$. When $0 < d < r - 1$ the situation could be very complex.*

Suppose $d = r - 1$ and $\boldsymbol{v}_1$ is the only eigenvector associate with λ_r, then one will have to solve $r - 1$ equations $(\boldsymbol{A} - \lambda_r)^i \boldsymbol{v}_{i+1} = \boldsymbol{v}_i, \quad i = 1, 2, \cdots, r-1$. And the general solution would contains terms like $[c_1\boldsymbol{v}_1 + c_2(\boldsymbol{v}_1 t + \boldsymbol{v}_2) + c_3(\boldsymbol{v}_1 t^2 + \boldsymbol{v}_2 t + \boldsymbol{v}_3) + \cdots + c_r(\boldsymbol{v}_1^r + \boldsymbol{v}_2 t^{r-1} + \cdots + \boldsymbol{v}_r)]e^{\lambda_r}$.

Case 3: *A complex root $a + bi$ with associated eigenvector $\boldsymbol{v}_a + i\boldsymbol{v}_b$, then the general solution contains term, $[c_1(\boldsymbol{v}_a \cos(bt) - \boldsymbol{v}_b \sin(bt)) + c_2(\boldsymbol{v}_a \sin(bt) + \boldsymbol{v}_b \cos(bt))]e^{at}$.*

REMARK 2.2. *Suppose $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \boldsymbol{x}_3(t), \cdots, \boldsymbol{x}_n(t)$ are n linearly independent solution for $n \times n$ homogeneous system, $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, the* **fundamental matrix** *$\Phi(t)$ is a matrix whose columns are $\boldsymbol{x}_i(t), i = 1, 2, \cdots, n$.*

EXAMPLE 2.5. *(***Two distinct eigenvalues***) Find the general solution to*

$$\begin{aligned} x_1' &= 3x_1 + 4x_2 - 2x_3 \\ x_2' &= 2x_1 + x_2 - 4x_3 \\ x_3' &= x_1 + 2x_2 \end{aligned}$$

Solution Let $\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix}$ The equations can be written in matrix form $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$.

Using Mathcad , functions **eigenvals()** and **eigenvecs()** we find,$\lambda_1 = 2$ and $\lambda_2 = 1$ with associated eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ respectively. Since λ_1 has multiplicity 2 as 1 appeared twice in the result of eigenvals() function, we need to solve the equation $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2$.

To use Mathcad ,

(1) you first compute $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_3$ using the following sequences of key stroke,
  [*] type ([Ctrl][M], set the rows and columns in the matrix definition popup menu, input the data for $\boldsymbol{A}$,
  [*] type -[Ctrl][M] set the row and column number and input data for $\lambda_1 \boldsymbol{I}$,
  [*] type)[Ctrl][M], now set 1 as column number, enter a, b, c in the place holders,
  [*] type [Ctrl][.] to compute symbolically and you get.

(2) Using the **Given** **Find** block to find a solution. Type **Given** in a blank space, type a+2b-c[Ctrl]= 1 and 2a-4c[Ctrl]=0 in two rows, then type key word **Find** following by typing (a,b)[Ctrl][.] you will get the solution in terms of c.

$$\left[\begin{pmatrix} 3 & 4 & -2 \\ 2 & 1 & -4 \\ 1 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} 2 \cdot a + 4 \cdot b - 2 \cdot c \\ 2 \cdot a - 4 \cdot c \\ a + 2 \cdot b - c \end{pmatrix}$$

Given

$$a + 2 \cdot b - c = 1$$

$$a - 4 \cdot c = 0$$

$$\text{find}(a, b) \rightarrow \begin{pmatrix} 4 \cdot c \\ \frac{-3}{2} \cdot c + \frac{1}{2} \end{pmatrix}$$

Set $c = 1$, we get $\boldsymbol{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

So the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -4\mathrm{e}^{2\mathrm{t}} & e^t & (t+4)e^t \\ e^{2t} & 0 & -e^t \\ -2e^{2t} & e^t & (t+1)e^t \end{bmatrix}$$

and the general solution is,

$$\boldsymbol{x}_c(t) = c_1 \boldsymbol{v}_1 e^{2t} + c_2 \boldsymbol{v}_2 e^t + c_3(t\boldsymbol{v}_2 + \boldsymbol{v}_3)e^t$$

⊣

Example 2.6. *(**One eigenvalue with deficit 1**) Find the solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -4 & 0 & -1 \end{bmatrix} \boldsymbol{x}(t)$ *and* $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$

Solution Using Mathcad , functions **eigenvals()** (Notice the **eigenvecs()** will not find a good result in this case due to the rounding error.) we find, $\lambda_0 = 1$ is the only eigenvalue. To find the associate eigenvectors we compute (Using $(\boldsymbol{A} - \lambda_0 \boldsymbol{I})\boldsymbol{v} = 0$)

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I})\boldsymbol{v} = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{v}_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} 2\mathrm{v}_1 + \mathrm{v}_3 \\ 2v_1 + v_3 \\ -4v_1 - 2v_3 \end{bmatrix} = 0$$

We have only $2v_1 + v_3 = 0$ for three variables v_1, v_2, v_3, this indicates that v_2 can be any value, and set $v_1 = 1$ find $v_3 = -2$, So $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ are two eigenvectors.

To find the generalize eigenvector associated with λ_0 we will have to solve two equations

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I}) \begin{bmatrix} \mathrm{w}_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix},$$

and

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I}) \begin{bmatrix} \mathrm{w}_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

From (2.2),

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{w}_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix},$$

we get two inconsistent equations $2w_1 + w_3 = 1$ and $2w_1 + w_3 = 0$. So now solution can be found in this case.

From (2.2),

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{w}_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

we get one equation $2w_1 + w_3 = 1$ choose $w_3 = 1$ we get $w_1 = 0$, since w_2 can be anything, we set $w_2 = 1$. So $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and we can verify that $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are linearly independent.

So the fundamental matrix is

$$\Phi(t) = e^t \begin{bmatrix} 1 & 1 & t \\ 0 & 1 & t+1 \\ -2 & -2 & -2t+1 \end{bmatrix}$$

and the general solution is,

$$\boldsymbol{x}_c(t) = [c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + c_3(t\boldsymbol{v}_2 + \boldsymbol{v}_3)]e^t$$

Now, $\Phi(0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & 1 \end{bmatrix}$ and $\Phi(0)^{-1} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$.

Hence, the particular solution is $\boldsymbol{x}(t) = [\boldsymbol{v}_1 + 3\boldsymbol{v}_2]e^t$ ⊣

EXAMPLE 2.7. (**One eigenvalue with deficit 2**) *Find the general solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 2 & -2 & 3 \end{bmatrix} \boldsymbol{x}(t)$.

Solution Using Mathcad , functions **eigenvals()** (Notice the **eigenvecs()** will not find a good result in this case due to the rounding error.) we find, $\lambda_0 = 3$ is the only eigenvalue. To find the associate eigenvectors we compute (Using $(\boldsymbol{A} - \lambda_0 \boldsymbol{I})\boldsymbol{v} = 0$)

$$(\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{v}_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} -\mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3 \\ v_1 + v_2 + v_3 \\ 2v_1 - 2v_2 \end{bmatrix} = 0$$

We have only one eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

To find the generalize eigenvector associated with λ_0 we will have to solve two equations

$$(\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1,$$

and

$$(\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2,$$

From (2.2),

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

we have two equations

$$\begin{cases} b - a & = \ 1 \\ a + b + c = 1 \end{cases}$$

Choosing $a = 0$, we get $b = 1, c = 0$. Hence $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

From (2.2),

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

we have two equations

$$\begin{cases} b - a & = \ 0 \\ a + b + c = 1 \end{cases}$$

Choosing $a = 0$, we get $b = 0, c = 1$. So $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and we can verify that $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are linearly independent.

So the fundamental matrix is

$$\Phi(t) = e^t \begin{bmatrix} 1 & t & t^2 \\ 1 & 1+t & t^2+t \\ -2 & -2t & -2t^2+1 \end{bmatrix}$$

and the general solution is,

$$\boldsymbol{x}_c(t) = [c_1\boldsymbol{v}_1 + c_2(t\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_3(t^2\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{v}_3)]e^{3t}$$

⊣

Example 2.8. *(***Two conjugate complex eigenvalues case***)* *Find the general solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x}(t)$

Solution Using Mathcad , functions **eigenvals()** and **eigenvecs()** we find two conjugate complex eigenvalues and one real eigenvalue, $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2} + i\frac{1}{2}\sqrt{19}$, and $\lambda_3 = \frac{1}{2} - i\frac{1}{2}\sqrt{19}$ with associated eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 + \mathrm{i}\sqrt{19} \\ -2 + i\sqrt{19} \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + i \begin{bmatrix} \sqrt{19} \\ \sqrt{19} \\ 0 \end{bmatrix}$ with respect to λ_3. Compare this with the Theorem 1.3, we have $a = \frac{1}{2}, b = \frac{1}{2}\sqrt{19}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ (real part of $\boldsymbol{v}$), and $\boldsymbol{v}_3 = \begin{bmatrix} \sqrt{19} \\ \sqrt{19} \\ 0 \end{bmatrix}$ (imaginary part of $\boldsymbol{v}$).

The general solution is,

$$\boldsymbol{x}_c(t) = c_1\boldsymbol{v}_1e^t + c_2(\boldsymbol{v}_2\cos(\frac{1}{2}\sqrt{19}) - \boldsymbol{v}_3\sin(\frac{1}{2}\sqrt{19}))e^{\frac{1}{2}t} + c_3(\boldsymbol{v}_2\sin(\frac{1}{2}\sqrt{19}) + \boldsymbol{v}_3\cos(\frac{1}{2}\sqrt{19}))e^{\frac{1}{2}t}$$

⊣

3. Nonhomogeneous System of Equations

To find solutions to the initial value problem of nonhomogeneous equations $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ we follow the steps below,

(1) Find the general solution $\boldsymbol{x}_c(t) = \Phi(t)\boldsymbol{c}$ to homogeneous equation $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, where $\Phi(t)$ is the fundamental matrix.
(2) Find a particular solution $\boldsymbol{x}_p$ to $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$
(3) The general solution to the nonhomogeneous equation $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$ is $\boldsymbol{x}(t) = \boldsymbol{x}_c(t) + \boldsymbol{x}_p(t)$. Using $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ to determine the coefficient vector $\boldsymbol{c}$.

The following theorem gives one way to find a particular solution based on the fundamental matrix,

Theorem 3.1. *Let $\Phi(t)$ be a fundamental matrix of $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, a particular solution to $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$ is given by*

$$\boldsymbol{x}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{b}(t)\,dt.$$

Example 3.1. *Find the general solution to*

$$\begin{aligned} x_1' &= 3x_1 + 4x_2 - 2x_3 + t^2 \\ x_2' &= 2x_1 + x_2 - 4x_3 \\ x_3' &= x_1 + 2x_2 - t^2 \end{aligned}$$

Solution Let $\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix}$, and $\boldsymbol{b}(t) = \begin{bmatrix} \mathrm{t}^2 \\ 0 \\ -t^2 \end{bmatrix}$. The equations can be written in matrix form $\boldsymbol{x}'(t) =$

$\boldsymbol{Ax}(t)+\boldsymbol{b}(t)$. From Example 2.5, we know that the fundamental matrix to $\boldsymbol{x}(t) = \boldsymbol{Ax}(t)$ is

$$\Phi(t) = \begin{bmatrix} -4\mathrm{e}^{2\mathrm{t}} & e^t & (t+4)e^t \\ e^{2t} & 0 & -e^t \\ -2e^{2t} & e^t & (t+1)e^t \end{bmatrix}$$

To find a particular solution, we first compute $\Phi^{-1}(t)\boldsymbol{b}(t) = \begin{bmatrix} -\frac{2}{5}\mathrm{t}^2\mathrm{e}^{-2\mathrm{t}} \\ -(\frac{2}{5}t^3 + \frac{11}{5}t^2)e^{-t} \\ \frac{2}{5}t^2e^{-t} \end{bmatrix}$

then we compute

$$\begin{aligned} \Phi(t)\int \Phi^{-1}(t)\boldsymbol{b}(t)\,dt &= \Phi(t)\begin{bmatrix} \int -\frac{2}{5}\mathrm{t}^2\mathrm{e}^{-2\mathrm{t}}\,\mathrm{dt} \\ \int -(\frac{2}{5}t^3 + \frac{11}{5}t^2)e^{-t}\,dt \\ \int \frac{2}{5}t^2e^{-t}\,dt \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5}\mathrm{t}^2 + 2\mathrm{t} + \frac{16}{5} \\ -\frac{1}{5}t^2 - \frac{3}{5}t - \frac{7}{10} \\ \frac{9}{5}t^2 + \frac{24}{5}t + \frac{29}{5} \end{bmatrix}. \end{aligned}$$

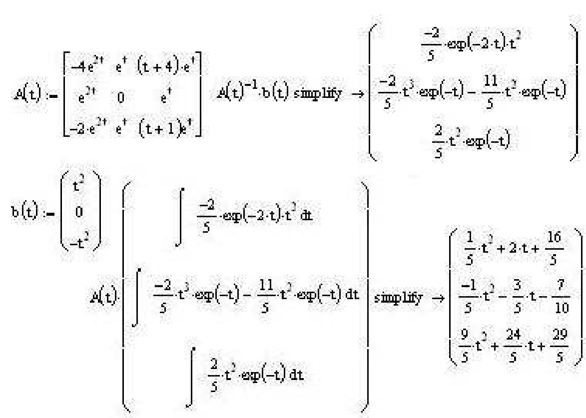
And so the general solution is,

$$\begin{aligned} \boldsymbol{x}(t) &= c_1\begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^t \\ &+ c_3\left(t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}\right)e^t + \begin{bmatrix} \frac{1}{5}\mathrm{t}^2 + 2\mathrm{t} + \frac{16}{5} \\ -\frac{1}{5}t^2 - \frac{3}{5}t - \frac{7}{10} \\ \frac{9}{5}t^2 + \frac{24}{5}t + \frac{29}{5} \end{bmatrix} \end{aligned}$$

The following is a screen shot that shows how to carry out the computation in Mathcad ,

To use Mathcad ,

(1) Define fundamental matrix $A(t)$ and $\boldsymbol{b}(t)$ in the same line (not as shown in graph), and compute in the next line $A^{-1}b(t)$
(2) type A(t)*[Ctrl][M] choose column as 1, at each place holder, type [Ctrl][I] to get the indefinite integral,
(3) and put the corresponding entry of $A^{-1}b(t)$ in the integrant position.
(4) press [Shift][Ctrl][.] type key work **simplify**



THEOREM 3.2. *If* $\Phi(t)$ *is the fundamental matrix for* $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, *, and* $\boldsymbol{x}_p(t) = \int \Phi^{-1}(t)\boldsymbol{b}(t)\,dt$, *then* $\boldsymbol{x}(t) = \Phi(t)\Phi^{-1}(t_0)(\boldsymbol{x}_0 - \boldsymbol{x}_p(t_0)) + \boldsymbol{x}_p(t)$ *is the solution to the nonhomogeneous initial value problem,*

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

EXAMPLE 3.2. *Find the solution to* $\boldsymbol{x}'(t) = \begin{bmatrix} 2 & -3 \\ 3 & 8 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} \mathrm{e}^{2\mathrm{t}} \\ -2e^{2t} \end{bmatrix}$ *and* $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solution From Example 2.3 is the fundamental matrix is

$$\Phi(t) = e^{5t} \begin{bmatrix} -3 & -3t+1 \\ 3 & 3t \end{bmatrix}$$

and $\Phi(0)^{-1} = \frac{1}{3} \begin{bmatrix} 0 & -1 \\ -3 & -3 \end{bmatrix}$.

Now $\boldsymbol{b}(t) = \begin{bmatrix} \mathrm{e}^{2\mathrm{t}} \\ -2e^{2t} \end{bmatrix}$, using the formula $\boldsymbol{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\boldsymbol{b}(t)\,dt$ and Mathcad , we have $\boldsymbol{x}_p(t) = \begin{bmatrix} 0 \\ \frac{1}{3}e^{2t} \end{bmatrix}$

Therefore,

$$\Phi(0)^{-1}(\boldsymbol{x}(0) - \boldsymbol{x}_p(0)) = \frac{1}{3} \begin{bmatrix} 0 & -1 \\ -3 & -3 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \right) = \begin{bmatrix} -\frac{2}{3} \\ -8 \end{bmatrix}.$$

and the solution is

$$\boldsymbol{x}(t) = \Phi(t)\Phi(0)^{-1}(\boldsymbol{x}(0) - \boldsymbol{x}_p(0)) + \boldsymbol{x}_p(t) = e^{5t} \begin{bmatrix} 24\mathrm{t} - 6 \\ -24t - 2 + \frac{1}{3}e^{-3t} \end{bmatrix}$$

⊣

4. Higher order differential equations

One can transform equations that involving higher order derivatives of unknown functions to system of first order equations. For example, suppose $x(t)$ is an unknown scalar function that satisfies

$$mx''(t) + cx'(t) + kx(t) = f(t)$$

an equation can be used to model a spring system with external force $f(t)$ or an RCL electronic circuit with an energy source $f(t)$.

Now if we set $x_1(t) = x(t)$ and $x_2(t) = x'(t)$ we then get an system of first order equations

$$x_1'(t) = x_2(t) \tag{5}$$

$$x_2'(t) = -\frac{c}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{f(t)}{m} \tag{6}$$

In general, if we have an differential equation that involving nth order derivative $x^{(n)}(t)$ of unknown function $x(t)$,

$$x^{(n)} = a_0x(t) + a_1x'(t) + \cdots + a_{n-1}x^{(n-1)} + f(t),$$

we can transform it into an system of first order equations of n unknown functions $x_1(t) = x(t),\ x_2(t) = x'(t),\ x_3(t) = x^{(2)}(t),\ \cdots,\ x_n(t) = x^{(n-1)}(t)$, and using the eigenvalue method for system of differential equation to solve the higher order equation.

Example 4.1. *Transform the differential equation* $x^{(3)} + 3x^{(2)} - 7x'(t) - 9x = \sin(t)$ *into system of first order equations.*

Solution Here the highest order of derivative is third derivative $x^{(3)}$ of $x(t)$. So we transfer it into system of 3 equations.

Let $x_1(t) = x(t),\ x_2(t) = x'(t),\ x_3(t) = x''(t)$, we have

$$x_1'(t) = x_2(t) \tag{7}$$

$$x_2'(t) = x_3(t) \tag{8}$$

$$x_3'(t) = -3x_3(t) + 7x_2(t) + 9x_1(t) - \sin(t) \tag{9}$$

Let $\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{x_1(t)} \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 7 & -3 \end{bmatrix}$, and $\boldsymbol{b}(t) = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$ we can write the system of equation in matrix form $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$. ⊣

Example 4.2. *Find the general solution for the 3rd order differential equation* $x^{(3)} + 3x^{(2)} - 7x'(t) - 9x = \sin(t)$.

Solution From previous example, Example 4.1, Let $\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{x_1(t)} \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 7 & -3 \end{bmatrix}$, and $\boldsymbol{b}(t) = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$ we can write the system of equation in matrix form $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$. Using Mathcad we find the eigenvalues are $\lambda_1 = -1, \lambda_2 = -1 + \sqrt{10}, \lambda_3 = -1-\sqrt{10}$ with associate eigenvectors, $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 + \sqrt{10} \\ 11 - 2\sqrt{10} \end{bmatrix}$, and $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -1-\sqrt{10} \\ 11+2\sqrt{10} \end{bmatrix}$ respectively (after multiply the results of Mathcad by some constants). So the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-\mathrm{t}} & e^{(-1+\sqrt{10})t} & e^{(1+\sqrt{10})t} \\ -e^{-t} & (-1+\sqrt{10})e^{(-1+\sqrt{10})t} & -(1+\sqrt{10})e^{(1+\sqrt{10})t} \\ e^{-t} & (11-2\sqrt{10})e^{(-1+\sqrt{10})t} & (11+2\sqrt{10})e^{(1+\sqrt{10})t} \end{bmatrix}$$

From $\Phi(t)$ we find a particular solution

$$\boldsymbol{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\boldsymbol{b}(t)\,dt = \begin{bmatrix} \frac{3}{52}\cos(\mathrm{t}) - \frac{5}{39}\sin(\mathrm{t}) \\ \frac{1}{26}\cos(t) - \frac{35}{156}\sin(t) \\ -\frac{3}{52}\cos(t) - \frac{8}{39}\sin(t) \end{bmatrix}$$

Hence the general solution to the system is

$$\begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \\ x_3(t) \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{c}_1\mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2\mathrm{e}^{(-1+\sqrt{10})\mathrm{t}} + \mathrm{c}_3\mathrm{e}^{(1+\sqrt{10})\mathrm{t}} + \frac{3}{52}\cos(\mathrm{t}) - \frac{5}{39}\sin(\mathrm{t}) \\ -c_1e^{-t} + c_2(-1+\sqrt{10}e^{(-1+\sqrt{10})t} - c_3(1+\sqrt{10})e^{(1+\sqrt{10})t} + \frac{1}{26}\cos(t) - \frac{35}{156}\sin(t) \\ c_1e^{-t} + c_2(11-2\sqrt{10})e^{(-1+\sqrt{10})t} + c_3(11+2\sqrt{10})e^{(1+\sqrt{10})t} - \frac{3}{52}\cos(t) - \frac{8}{39}\sin(t) \end{bmatrix}$$

and $x_1(t) = c_1e^{-1t} + c_2e^{(-1+\sqrt{10})t} + c_3e^{(1+\sqrt{10})t} + \frac{3}{52}\cos(t) - \frac{5}{39}\sin(t)$ is the general solution to the third order ordinary differential equation

$$x^{(3)} + 3x^{(2)} - 7x'(t) - 9x = \sin(t)$$

. ⊣

EXAMPLE 4.3. *Find the solution to the initial value problem*

$$x'' - 10x' + 9x = te^t, \quad x(0) = 1, x'(0) = -1.$$

Solution Since the given equation is of second order, we will have two unknowns $x_1(t) = x(t), x_2(t) = x'(t)$ to transform the equation into a system of first order equations,

$$\begin{aligned} x_1'(t) &= x_2 \\ x_2'(t) &= 10x_2 - 9x_1 + te^t, \end{aligned}$$

and the initial conditions are $x_1(0) = x(0) = 1 \;\; x_2(0) = x'(0) = -1$.

Now let $\boldsymbol{x}(t) = \begin{bmatrix} \mathrm{x}_1(\mathrm{t}) \\ x_2(t) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -9 & 10 \end{bmatrix}$, and $\boldsymbol{b}(t) = \begin{bmatrix} 0 \\ te^t \end{bmatrix}$.
We have the matrix version of this equation, $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$

Using Mathcad , we find the eigenvalues $\lambda_1 = 1$, $\lambda_1 = 9$, and associate eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$. And fundamental matrix $\Phi(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{t}} & e^{9t} \\ e^t & 9e^{9t} \end{bmatrix}$.

From $\Phi(t)$ we find a particular solution

$$\boldsymbol{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\boldsymbol{b}(t)\, dt = \begin{bmatrix} -\frac{1}{512}(32\mathrm{t}^2 + 8\mathrm{t} + 1)\mathrm{e}^{\mathrm{t}} \\ -\frac{1}{512}(32t^2 + 72t + 9)e^t \end{bmatrix}$$

The solution with initial values $\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is given by

$$\begin{aligned} \boldsymbol{x}(t) &= \Phi(t)\Phi^{-1}(0)(\boldsymbol{x}_0 - \boldsymbol{b}(0)) + \boldsymbol{b}(t) \\ &= \begin{bmatrix} \left(-\frac{1}{16}\mathrm{t}^2 - \frac{1}{64}\mathrm{t} + \frac{639}{512}\right)\mathrm{e}^{\mathrm{t}} - \frac{127}{512}\mathrm{e}^{9\mathrm{t}} \\ \left(-\frac{1}{16}t^2 - \frac{9}{64}t + \frac{631}{512}\right)e^t - \frac{1143}{512}e^{9t} \end{bmatrix}. \end{aligned}$$

Hence the solution to the initial value problem of the second order differential equation is $x(t) = x_1(t) = \left(-\frac{1}{16}t^2 - \frac{1}{64}t + \frac{639}{512}\right)e^t - \frac{127}{512}e^{9t}$. ⊣

Project

At beginning you should enter: Project title, your name, ss#, and due date in the following format

Project One: Define and Graph Functions

John Doe
SS# 000-00-0000

Due: Mon. Nov. 23rd, 2003

You should format the text region so that the color of text is different than math expression. You can choose color for text from **Format–>Style** select normal and click **modify**, then change the settings for font. You can do this for headings etc.

(1) **Solutions To System of Equations**
Finding solution to linear system using Mathcad and study the long time dynamic behavior of the solutions.
- Find general solution to

$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

Plot several solutions with different initial values in
[-] xt-plane, yt-plane
xy-plane. Here you will need to define range variable $t = 0, 0.1 \cdots 7$ and set $X := x(t)$, $Y := y(t)$. The graph in xy-plane is called the trajectory. If this models the movement of a satellite, what is its trajectory.

- Find general solution to

$$\begin{cases} x' = -8y \\ y' = 18x \end{cases}$$

Plot several solutions with different initial values in
[-] xt-plane, yt-plane
xy-plane. Here you will need to define range variable $t = 0, 0.1 \cdots 7$ and set $X := x(t)$, $Y := y(t)$. The graph in xy-plane is called the trajectory. If this models the movement of a satellite, what is its trajectory.

- Find general solution to

$$\begin{cases} x' = 2x - y \\ y' = y - 3x \end{cases}$$

Plot several solutions with different initial values in
[-] xt-plane, yt-plane
xy-plane. Here you will need to define range variable $t = 0, 0.1 \cdots 7$ and set $X := x(t)$, $Y := y(t)$. The graph in xy-plane is called the trajectory. If this models the movement of a system of two species, what is you conclusion about interdependency of these species? Can you find initial value such that $x(t) = 0$ (distinct) for some t? what about $y(t)$.

(2) **Solution of Higher order equation** In general $mx'' + cx' + kx = f(t)$ models a object with mass m attached to a spring with constant k and damping force that is proportional to the velocity x', $c \geq 0$, $k > 0$. Suppose $m = 1$ and $f(t) = Ae^{-at}\sin(bt)$, that is the external force is oscillatory ($b > 0$) and diminishing ($a > 0$) Find solutions and graph the solutions.
   - $c = b = 0$ Find general solution and graph some particular solutions.
   - $c = 20, k = 10, a = 0, b = \frac{1}{4}, A = 1$
   - $c = 2, k = 3, A = 100, a = 2, b = \sqrt{2}$