Markov versus Gaussian white noise in propagation of wave arising in Mathematical Neurosciences

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Abstract: In the presence of external rectangular signal over a relatively large period of time, Saddle node bifurcation on a limit cycle (SNLC) systems readily exhibits slow wave propagation. We are concerned with the interplay between Markov noise and SNLC media vs. white noise and SNLC media, is that a wave propagation through network of chain of SNLC's through a circular process. The purpose of these investigations is to analyze the potentially constructive role of noise on the circular process in some canonical network arises in mathematical neurosciences. We observed that for relatively large variance of noise the propagation does not sustain through the network.

1 Introduction

In this paper, we study a model that arises in mathematical neurosciences by considering a network of $N$ excitatory neurons of Type $I$ [3], connected through a circular ring using the method derived in [2]. In this network additive noise does not adequately describe the systems behavior. Thus we consider a phase lock loop that arises in mathematical neurosciences as a canonical model of Type $I$ excitatory neurons. We also consider some external signal as a part of synaptic input function. Voltage Controlled Oscillator Network (see, [1]) provides a base for modelling this type of network.

2 VCO as SNLC bifurcation

We consider the following VCO network arises in mathematical neuroscience as a canonical model of excitable neural tissue of Type $I$, see [3]. A network of $N$ such electronic circuits that are connected as described in Figure 1, where $\theta_i$

![VCO as SNLC bifurcation - \( \dot{\theta} \)-network](image)

is the phase of VCO$_i$ output, $\omega_i$ is the center frequency of VCO$_i$, and $\mu_i$ is the sensitivity of VCO$_i$, and we consider

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\[ \dot{\theta}_i = \omega_i + \mu_i \left( f(\theta_i) + V(\theta_{i-1}, \dot{\theta}_{i-1}) \right) \]

Typically \( f(\theta) = \cos \theta \), \( \omega \approx 1 \). VCO is the model of SNLC's and traditionally, \( V(\theta, 0) = K \cos \theta \). Hoppensteadt (see, [4]) proposed \( V(\theta, 0) = V(\theta) = K \dot{\theta} \) and described as "\( \dot{\theta} \)-network".

### 3 \( \dot{\theta} \)-network

A mathematical model for a phase lock loop is posed in terms of voltage put out by the VCO having fixed wave form, say \( \cos \theta \), and a variable phase \( \dot{\theta}(t) \). The circuits are designed so modelling is in terms of phase variable. We consider a network of \( N \) neurons using VCON's described by a phase variables \( \dot{\theta}_j(t), j = 1, ..., N \) and the voltage output of each VCO is \( \cos \theta_j(t) \). Thus using methodology of canonical models for neural network [3] and [1], we write

\[
\tau_j \dot{\theta}_j = \omega_j + \cos \theta_j + \mu \sum_{k=1}^{N} C_{j,k}(y(t/c)) \dot{\theta}_k(t) 
\]

where \( \tau_j \in \mathbb{R} \), the time constant, \( y(t) \) is a Markov process taking values in a set \( Y \) in which it is ergodic with ergodic distribution \( \rho(dy) \), which is the noise process. We will call \( y(t) \) as a Markov noise. \( C_{j,k}(y) \) is an \( N \times N \) connection matrix and \( \mu \) is the strength of connection. \( \omega_j \) is a parameter, which characterizes the dynamics of the system. \( \epsilon \ll 1 \) is a small parameter. It measures the ratio of time scales between noise (fast) and the system response. \( \dot{\theta}_j(t) \in \mathbb{R} \), defines phase variable of the \( j \)-th neuron.

Using our \( \dot{\theta} \) network model (1), we consider the connection in the network is defined by the matrix \( C \) having dimensions \( N \times N \). Following this, using synaptic input function \( F(t) \), we write a neural Networks in terms of their phases supposing that the output of a synapse is proportional to \( F \); that is neurotransmitter is released in proportion to the number of action potential arriving per unit time (see, [1], [4]). We have

\[
\begin{align*}
\tau_j \dot{\theta}_j(t) &= \omega_j + \cos \theta_j(t) + F(t) + \mu \sum_{k=1}^{N} C_{j,k} \dot{\theta}_k(t) \\
\dot{\theta}_j(0) &= 0
\end{align*}
\]

where \( \dot{\theta}_j(t) \in \mathbb{R} \) is the phases, \( \tau_j \), is time constants, \( \mu \) is the strength of connection and matrix \( C \) is an \( N \times N \) matrix of circular connection defined by

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Using the matrix \( C \) we can write (2) in matrix form,

\[(\tau - \mu C) \dot{\theta}(t) = \omega + F(t) + \cos \theta(t).\]

Now if we use Markov chain (and Gaussian white) noise in the connection then the matrix \( C \) can be written as

\[
C(y(t/c)) = \begin{pmatrix}
0 & y(t/c) & 0 & \cdots & 0 \\
0 & 0 & y(t/c) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & y(t/c) \\
y(t/c) & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

and then using \( C(y(t/c)) \) (4), we obtain,

\[(\tau - \mu C(y(t/c)) \dot{\theta}(t) = \omega + F(t) + \cos \theta(t).\]

If the matrix \( \tau - \mu C(y) \) is invertible, then we have

\[\dot{\theta} = (\tau - \mu C(y(t/c))^{-1}(\omega + F(t) + \cos \theta(t))\]

We considered \( y(t) \) is a Markov processes in a set \( Y = \{y_1, y_2\} \), in which it is ergodic and using two-state transition probability matrix, one can show it has an ergodic distribution defined by \( \rho = (\rho_1, \rho_2) \), where \( \rho_1 \) and \( \rho_2 \) can be defined in terms of transition probability.
4 Compare wave propagation using Markov and Gaussian white noise

We have observed that the SNLC system (1) in the presence of noise (Markov vs. Gaussian white) does not sustain wave propagation, see Figure 4 and Figure 5 whereas solution of the system (1) without noise has shown in Figure 3. Moreover, Figure 3 describes the switching behavior from state 0 to 1 and vise versa keeping ε (update) fixed but varying a and b (as we know as a and b are small differential system (1) generates larger variance, for more detail see, [6]).

We have also observed that if the synaptic input function F(t) is large wave propagation can be chaotic. Another observation in the speed of propagation changes if we change w in the small neighborhood of 1.

5 Concluding remarks

Experimentation indicates that as step size decreases, the lack of independence in the samples from a random generator typically degrades the computation before rounding errors becomes significant.

Future work will address the error convergence of this network as well as continuous analogue namely, Fredholm Volterra integro-differential equations modelling neural networks which also include the effects of memory which is subject to noise occurring on different time scale than the system response.

References

Figure 4: Compare the effect of wave propagation using Markov and Gaussian white noise of $\delta$-network model (1). Left column is the contour plot of $\cos \theta(t)$ using Markov noise choosing $a = 0.4, b = 0.3; a = 0.3, b = 0.2; a = 0.2, b = 0.1; a = 0.1, b = 0.05$ (with increasing variance of stochastic differential system). Right column is the contour plot of $\cos \theta(t)$ using Gaussian white noise with increasing variance choosing $0.01, 0.1, 1.0, 10.0$. In terms of comparison, both Markov and Gaussian white (noise) columns of graph shows that as variance increases wave propagation break down.

Figure 5: Comparison of wave propagation in $\delta$-network model (1) using Markov noise(Markov Chain) vs Gaussian white noise(GWN). Left column from top to bottom have shown the effect of Markov noise vs. considering four different pair $a = 0.9, b = 0.8; a = 0.4, b = 0.3; a = 0.04, b = 0.03; a = 0.004, b = 0.003$ and its contour plot of $\cos \theta(t)$. Right column from top to bottom have shown the effect of Gaussian white noise using same variance as it was computed using model (1).