An Introduction to Symplectic Maps and Generalizations of the Toda Lattice

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Abstract

We construct particular Hamiltonian systems associated to graphs embedded in Euclidean $n-$dimensional space and apply the dual transformation to this system. This result extends the one-dimensional dual transformation associated to the Toda lattice. We also discuss some applications to solid state lattice theory.

1 Introduction

Imagine N particles and $N-1$ springs such that only nearest neighbor interactions occur. Let each spring force be governed by an exponential, and consequently non-linear, potential and suppose that the particles an springs are equidistantly placed on $\mathbb{R}$:

N particles and $N-1$ springs

The integer N may be fixed as finite or infinite, or N could be chosen to be an element from a finite cyclic group to invoke periodicity in the lattice. Then , for a given lattice particle, m_j, we have that: $j+N \equiv j$ for $j \in \mathbb{Z}^+ \cup \{0\}$. The system we describe is called the Toda lattice [5] where we refer to "lattice" in the sense of solid state physics. We can form a Hamiltonian or Energy function to describe the dynamical behavior of this system if, say, a force is applied to one end of the lattice. However, we typically want to write the equations of motion for the lattice in the Lax form which is well-known from scattering theory. The Lax form easily shows that the Toda lattice is a completely integrable system which means that we can deduce the constants of motion.

Of all the possible solutions, we often choose to focus on soliton behavior in the lattice. Toda originally designs this lattice and a type of canonical transformations, or symplectic mapping, known as the dual transformation, to describe phenomena in solid state lattice theory. In the theory of vibrations of both linear and on-linear imperfect crystals, different systems can arise with the same frequency spectra [6]. The systems studied so far are one-dimensional and we are curious as whether different systems occur with the same spectra in n-dimensional space with $n \geq 1$ and in spaces with non-zero curvature.

Kukuhata [3] shows the dual transformation can also be applied to another integrable system, the $(1+1)-$dimensional $O(3)$ non-linear sigma model. In this application of the dual transformation, a new integrable model is created from an already known integrable model exactly analogous to Toda's original set-up. In addition, this new system has a distinct topology from the old system. Thus, in terms of searching for new, integrable models, the dual transformation has proved itself quite useful. What we find particularly interesting with regard to integrable models is the following: one needs to realize that the dual transformation is a point transformation. This is one of the simplest canonical transformations that can be constructed. Thus, use of the dual transformation on the Toda lattice, as well as on the sigma models, allows one to conjecture that other canonical transformations or symplectic mappings applied to all known integrable models can yield new integrable models.

In this paper we study the dual transformation applied to N particles and $N-1$ springs that are no longer constrained to the real line. For simplicity, assume the Hamiltonian system formed to be embedded in Euclidean $n-$space with the standard metric, i.e. the metric tensor $g_{ij} = \delta_{ij}$ where δ_{ij} is the Kronecker tensor for $i, j = 1, 2, ..., n$. Also assume that $N < \infty$ and that all N particles are connected to at least one spring. We plan to remove these restrictions in the next paper.

Section 2 of the paper briefly reviews the classical Toda lattice and the dual transformation associated to it. We provide the review as an outline to Section 3 as it extends the classical case to higher dimensions. First, we have to describe what it means to extend to higher dimensions. Our interpretation of one possible extension is to create graphs that can act as dynamical systems. We decide to take a graph (a finite one, for simplicity), position masses at the vertices of the graph, and exchange the edges of the graphs with springs. Again, for simplicity, we assume that the springs behave uniformly, although not necessarily linearly. We then re-express the Hamiltonian with respect to the graphs rather than focusing on the underlying manifold. We focus on graphs which may be neatly embedded into Euclidean $n-$dimensional space for the purpose of this paper. Hence Section 3 gives information regarding how graphs behave when "equipped" with a symplectic $2-$form.

2 Review of Toda's Lattice and the Dual Transformation

Toda originally constructed a linear one-dimensional lattice of particles an springs that obeyed Hooke's law and applied a canonical transformation to that system [5]. This mapping was termed the "dual transformation." The mapping was later applied to springs with non-linear potentials [6]. For the systems considered, the terminology "dual" was typically used to emphasize the essential ingredient in the symplectic mapping: particles "became" springs and springs "became" particles in the new, or dual, system.

We first review Toda's dual transformation as a way of introducing the generalized systems of Section 3.

First fix $N < \infty$ (for simplicity) and consider a chain of N particles evenly distributed on the real line in which a spring lies between each pair of particles. (The system is considered classical mechanical and not quantum mechanical which is a restriction to a simple illustration of the overall methodology and which is to be removed in later notes.) The masses of the particles are denoted by m_j for $j = 1, 2, ...N$. Suppose that the potential energy of the spring between the $j-1^{st}$ and j^{th} particle is described by the function $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$ written $\phi_j(q_j - q_{j-1})$ where q_j is the displacement of the j^{th} particle from the origin. The difference of displacements reflects the spring's physical expansion or compression. If the system is uniformly harmonic, then for all $j = 1, 2, ..., N$, $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$ is known to be

$$\phi_j(q_j - q_{j-1}) = \frac{k_j}{2}(q_j - q_{j-1})^2 \text{ where } k_j \text{ is the } j^{th} \text{ spring constant.}$$

If the system is uniformly anharmonic, we recall that Hooke's law is not obeyed. For example, suppose we want to consider the nonlinear potential as originally devised by Toda [7]. The $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$ in this case is defined as

$$\phi_j(q_j - q_{j-1}) = \frac{a}{b}e^{-b(q_j - q_{j-1} - \sigma)} + a(q_j - q_{j-1}) + c$$

where a, b, σ are fixed parameters and c is a constant for all $j = 1, 2, ..., N$.

Next we introduce the symplectic manifold. If $\mathbb{R}^N$ is a manifold described by the system of coordinates $\{q_j | j = 1, 2, ..., N\}$, set the phasespace equal to $\mathcal{M} = \mathbb{R}^N$ equipped with the following $2-$form $\omega = \sum_{j=1}^{N} dq \wedge dp_j$ where p_j is the momentum conjugate to the coordinate q_j for $j = 1, 2, ...N$. In other words, if $f, g \in C^\infty(\mathcal{M})$, the Poisson bracket is defined by

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right).$$

The conjugate momentum, p_j, satisfies the following relations:

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \text{ and } \{q_i, p_j\} = \delta_{ij} \text{ for all } i, j = 1, 2, ...N.$$

Therefore we can define a Hamiltonian, or energy function, $H : \mathcal{M} \to \mathbb{R}$ on the system by setting $H = T + U$ where T is the kinetic energy and U the potential energy such that

$$T = \sum_{j=1}^{N} \frac{p_j^2}{2m_j} \text{ and } U = \sum_{j=1}^{N} \phi_j(q_j - q_{j-1}) + f_{q_N}$$

with f a force acting on the N^{th} particle from the right. We assume a "fixed end condition" for the "1^{st} particle":$q_1 \equiv 0$. With respect to the dynamics of the system, a force is applied at time t_0 and the system evolves for $t \geq t_0$. In this paper, however, we stop short of studying the evolution of the system and focus instead on the introductory aspects of the dynamics. Specifically, what is investigated is a particular symplectic mapping, the dual transformation, which creates "equivalent" Hamiltonians, i.e. new dynamical systems "equivalent" to the original systems.

For notational convenience, the $2N-$tuples $(q_1, ..., q_N, p_1, ..., p_N)$ are abbreviated (q, p). Hence all $2-$tuples in this section are really $2N-$tuples. The reader can expect similar abbreviations throughout the paper.

We now use a two-step process to explicitly construct the dual transformation. This process utilizes an intermediate mapping which maps the canonical coordinates (q, p) to new coordinates (r, s). Then these (r, s) coordinates are mapped to (Q, P). The composite mapping, $\psi : (q, p) \to (Q, P)$ is the symplectic mapping of interest. We note that Toda does not prove that this mapping is symplectic. We provide an explicit proof to illustrate the ideas involved.

Recall that to show a finite dimensional mapping is symplectic, it suffices to show that the matrix representation of ψ, Ψ, satisfies

$$\Psi J \Psi^t = J \text{ where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the rank N identity matrix and t is matrix transpose. It is equally valid to check that the map is symplectic is to notice if the transformed coordinates are canonical such that

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \text{ and } \{Q_i, P_j\} = \delta_{ij} \text{ for all } i, j = 1, 2, ...N.$$

We shall investigate both approaches. (See Abraham and Marsden [1] or Goldstein [2] for more information on symplectic mappings).

Define the relative displacement between the $(j-1)^{st}$ and j^{th} particle by

$$r_j = q_j - q_{j-1}$$

which is the "generalized coordinate" for $j = 1, 2, ..., N$. Define a momentum conjugate to r_j to be

$$s_j = \frac{\partial L}{\partial \dot{r}_j}$$

where the Lagrangian $L = T - U$ and where $j = 1, 2, ..., N$. Next, introduce the new momentum P_j and a new coordinate Q_j conjugate to P_j by

$$r_j = \frac{P_j}{a} \text{ and } s_j = -aQ_j$$

for $j = 1, 2, ..., N$ and a an arbitrarily fixed constant.

Proposition 1 *The map $\psi : \mathbb{R}^N \to \mathbb{R}^N$ defined by*

$$(q, p) \to (Q, P)$$

is a symplectic map.

We recall that a conjugate momentum formed from a Lagrangian vector field is symplectic. Thus, if the Poisson bracket is defined in terms of the new canonical coordinates (r, s), the map ψ is symplectic. We would like to give an in-depth proof to illustrate the action.

Proof. First, we demonstrate that the map ψ is symplectic by inspecting the matrix representation of ψ and then by checking that the Poisson bracket action on the transformed coordinates (Q, P).

Suppose $0 < j \leq N$. We have that $r_j = q_j - q_{j-1}$. Hence

$$P_j = a(q_j - q_{j-1}).$$

Now we investigate Q_j as a function of p_j. The Lagrangian is a function of r and $\dot{r}$, namely $L = T(\dot{r}_1, \dot{r}_2, ..., \dot{r}_N) - U(r_1, r_2, ..., r_N)$. Therefore

$$s_j = \frac{\partial L}{\partial \dot{r}_j} = \frac{\partial T}{\partial \dot{r}_j}.$$

Since $r_j = q_j - q_{j-1}$ with $q_0 \equiv 0$, we get by inversion

$$q_j = \sum_{l=1}^{j} r_l.$$

Thus, since $p_j = m_j \dot{q}_j$

$$\begin{aligned} s_j &= \frac{\partial T}{\partial \dot{r}_j} \\ &= \frac{\partial}{\partial \dot{r}_j}\left[\sum_{i=1}^{N} \frac{p_i^2}{2m_i}\right] \\ &= \frac{\partial}{\partial \dot{r}_j}\left[\sum_{i=1}^{N} \frac{m_j \dot{q}_j^2}{2}\right] \\ &= \frac{\partial}{\partial \dot{r}_j}\left[\sum_{i=1}^{N} \frac{m_j}{2}\left(\sum_{l=1}^{j} \dot{r}_l\right)^2\right] \\ &= \sum_{i=j}^{N} m_i(\dot{r}_1 + \dot{r}_2 + ... + \dot{r}_N). \end{aligned}$$

This means that

$$s_j - s_{j+1} = m_i(\dot{r}_1 + \dot{r}_2 + ... + \dot{r}_j) = m_j \dot{q}_j = p_j.$$

Since $m_{N+1} \equiv 0$, we have that $s_N = p_N$. Or, in other words,

$$s_j = \sum_{i=j}^{N} p_i.$$

Hence,

$$Q_j = -\frac{1}{a}\sum_{i=j}^{N} p_i.$$

Therefore the matrix representation Ψ of ψ is

$$\Psi = \left[\begin{array}{cc} 0 & \Psi_\mu \\ \Psi_\lambda & 0 \end{array}\right]$$

where Ψ_λ is a bidiagonal matrix of rank N and Ψ_μ is an upper-triangular matrix of rank N where for all $i, j = 1, 2, ..., N$

$$(\Psi_\lambda)_{ij} = \begin{cases} -\frac{1}{a} & \text{for } j \geq i, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and } (\Psi_\mu)_{ij} = \begin{cases} a & \text{for } j = i, \\ -a & \text{for } j = i-1, \\ 0 & \text{otherwise} \end{cases} .$$

In other words, $\nu = [QP]^t := [Q_1 Q_2 ... Q_N P_1 P_2 ... P_N]^t$ and $\eta = [qp]^t := [q_1 q_2 ... q_N p_1 p_2 ... p_N]^t$. By direct computation, we find that

$$\Psi J \Psi^t = J$$

where Ψ^t is the transpose of Ψ with real entries.

Next we demonstrate the Poisson bracket. Let

$$D = (d_{i,j}) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $P_i = a(q_i - q_{i-1})$ implies

$$P_j = a \sum_{j=1}^{N} d_{i,j} q_j.$$

Thus

$$\frac{\partial P_i}{\partial q_k} = a \sum_{j=1}^{N} d_{i,j} \delta_{j,k} = a d_{i,k}.$$

Also, $Q_j - Q_{j-1} = \frac{-1}{a} p_j$ implies that

$$p_j = -a \sum_{j=1}^{N} d_{j,i} Q_i.$$

Inversion gives

$$Q_i = -\frac{1}{a} \sum_{j=1}^{N} d_{j,i} p_j.$$

Therefore

$$\frac{\partial Q_i}{\partial p_k} = -\frac{1}{a} \sum_{j=1}^{N} d_{j,i} \delta_{j,k} = -\frac{1}{a} d_{i,k} = \begin{cases} 0 & \text{if } k < i \\ -\frac{1}{a} & \text{if } k \geq i. \end{cases}$$

Thus the Poisson bracket is verified by:

$$\begin{aligned}
\{Q_j, P_k\} &= \sum_{j=1}^{N} \left[\left(\frac{\partial Q_j}{\partial q_i}\right)\left(\frac{\partial P_k}{\partial p_i}\right) - \left(\frac{\partial P_k}{\partial q_i}\right)\left(\frac{\partial Q_j}{\partial p_i}\right) \right] \\
&= -\sum_{j=1}^{N} \left(\frac{\partial P_k}{\partial q_i}\right)\left(\frac{\partial Q_j}{\partial p_i}\right) \\
&= -\sum_{j=1}^{N} \left(\frac{\partial P_k}{\partial q_i}\right)\left(\frac{\partial Q_i}{\partial p_j}\right)^t \\
&= \sum_{j=1}^{N} d_{k,i} d_{j,i} = \delta_{k,j}
\end{aligned}$$

since $DD^{-1} = I$. Also, $Q = Q(p)$ and $P = P(q)$ implies

$$\{Q_j, Q_k\} = \{P_j, P_k\} = 0.$$

Hence (Q, P) are canonical coordinates and the dual transformation, ψ, is a symplectic map. ■

Note that if $a = 1$, the map $\psi : (p, q) \to (P, Q)$ corresponds to a rotation by $-\frac{\pi}{2}$. The Hamiltonian of the dual system for a harmonic potential is

$$H_{dual} = \sum_{j=1}^{N} \left[\phi\left(\frac{P_j}{a}\right) + f\frac{P_j}{a} \right] + \sum_{j=1}^{N-1} \left[\frac{a^2}{2m_j}(Q_j - Q_{j-1})^2 \right] + \frac{a^2}{2m_n} Q_N^2.$$

From this we can see that the Hamiltonians H and H_{dual} are equivalent under the symplectic map. Of particular interest for vibrations in crystals is the fact that the frequency spectra of the normal mode of vibration are equivalent. This topic will be explored in a subsequent paper.

3 The Dual Transformation of Higher Dimensional Toda-type Lattices

In this section we shall define what we mean by Hamiltonians on graphs and symplectic maps on graphs. First, fix a graph $\Gamma = \Gamma(V, E)$ where V is the vertex set of Γ and E is the edge set of Γ. In general, we want to restrict to graphs which can be embedded into a vector space F. For simplicity, we consider the special case $F = \mathbb{R}^n$ with the standard distance $g_{ij} = \delta_{ij}$ for $i, j = 1, ..., n$. Also for simplicity we shall assume Γ is connected. If not, we would need to consider the connected components of Γ. We also assume that Γ is finite. In later notes, we shall remove these conditions.

Set $\Gamma = \Gamma(V, E)$ and suppose $y = y(v, t)$ is a mapping of the vertices of Γ and continuous time such that $y \in L^2(V, \mathbb{R})$. We employ the simplest possible difference scheme to illustrate the general technique.

Let the difference operator be defined by $d : L^2(V, \mathbb{R}) \to L^2(E, \mathbb{R})$ such that

$$dy(e, t) := y(e_+, t) \to y(e_-, t)$$

where $e \in E$ such that e_+ and e_- are the initial and terminal vertices of e, respectfully. We can also denote an edge e by $e = [e_+, e_-]$.

Note: The graphs need not have directed edges. However, a natural direction occurs via geodesics which we soon construct.

Now given Γ, we place exactly one particle at each vertex $v \in V$. Thus the dynamical system has exactly the cardinality of V, or $|V|$, particles. We place at least one spring at each edge $e \in E$ so that the system has *at least* the cardinality of E, or $|E|$ springs. We can now imagine the graph Γ as a dynamical system with particles attached by springs embedded in $n-$dimensional Euclidean $n-$space where $n \geq 1$. As the system evolves, the edges are naturally flexible as a result of the spring potential action.

We now investigate the canonical coordinates and conjugate momenta for the dynamical system of Γ. We would like to define the "canonical coordinate" for a system system of particles placed at vertices of Γ to be the map $q \in L^2(V, \mathbb{R})$ such that

$$q(v, t) = q_v(t) = \begin{bmatrix} q_1(v, t) & q_2(v, t) & ... & q_n(v, t) \end{bmatrix}^t.$$

Thus, in actuality, the map $q_v(t)$ restricted to a vertex v is a vector in $\mathbb{R}^n$. This definition differs from the standard definition of a canonical coordinate which is that of an element from the set

$$\{q_i(t) | i = 1, ..., n \cdot |V|\}.$$

This idea assumes that Γ is identified to a manifold, or more generally, a vector space of dimension $n \cdot |V|$. In our definition, q is expected to carry more information.

For example, if Γ is a Cayley graph, then q can be re-interpreted as a "time-dependent" representation of the group G associated to the graph Γ. (In this situation, q maps not to $\mathbb{R}$ but to another vector space). The map q is not a vector, but possibly a block matrix. Thus we want to make the following definition of a "canonical coordinate of Γ" in a way which encompasses both the standard definition and the desired for Γ.

Definition 1 *If a particle placed at a vertex $v \in V(\Gamma)$, its "canonical coordinate" is defined to be*

$$q_v(t) = q(v,t) \in L^2(V, \mathbb{R}).$$

The collection of maps $\{q_v | v \in V\}$ is said to be the collection of canonical coordinates for Γ.

The map q_v is always a map in $\mathbb{R}^n$ and a particle at vertex v will remain there constrained by nearest neighbor springs unless the system breaks apart. In general, we will now use the symbol Γ to refer to both the dynamical system and the graph Γ. The graph Γ is also free to evolve in time and, usually, at least one vertex is physically fixed to a point in space and hence can never move. This vertex is the analogue of the fixed end point of the previous section.

Definition 2 *The "conjugate momentum" of a particle situated at $v \in V$ is defined to be*

$$p_v(t) = p(v,t) \in L^2(V, \mathbb{R})$$

where m_v is the mass of the particle at $v \in V$ and $p_v(t) = m_v \dot{q}_v$. The collection of maps $\{p_v | v \in V\}$ is said to be the collection of conjugate momenta for Γ.

Combined, the points $\{q_v, p_v | v \in V\}$ are defined to be the canonical coordinated for the phasespace of Γ. We will again appeal to the shorthand notation (q, p) when specifying these coordinates.

We now investigate the symplectic manifold associated to the graph Γ. We have a natural symplectic manifold for Γ if the the graph can be identified to a manifold of dimension $n \cdot |V|$. Specifically, the natural phasespace is the object

$$(\mathcal{M}, \omega) := (\mathbb{R}^{2n\cdot|V|}, \sum_{i=1}^{n\cdot|V|} dq_i \wedge dp_i)$$

with the standard Poisson bracket. However, we find this standard phasespace cumbersome and want to define a new symplectic manifold for Γ as

$$(\mathcal{M}, \omega) := (\mathcal{M}, \sum_{v \in V} dq_v \wedge dp_v)$$

with Poisson bracket

$$\{f, g\} = \sum_{v \in V} \left(\frac{\partial f}{\partial q_v} \frac{\partial g}{\partial p_v} - \frac{\partial f}{\partial p_v} \frac{\partial g}{\partial q_v} \right)$$

for $f, g \in C^\infty(\mathcal{M})$ where $\mathcal{M} = \mathbb{R}^{2n|V|}$. In order to avoid confusion, we shall call the structure $(\mathcal{M}, \omega)$ the "graph-theoretic phasespace" $\mathcal{M}$ with the "graph-theoretic 2-form"ω. Together, $(\mathcal{M}, \omega)$ will be the called the "symplectic manifold for the graph Γ." Again we emphasize that our definitions are for computational convenience and for those graphs which may not embed naturally into manifolds.

Note: In an unnatural situation, we set $\mathcal{M}$ equal to some affine vector space, or even perhaps something more ad-hoc.

For $(\mathcal{M}, \omega)$, the canonical coordinates (p, q) have the Poisson action

$$\{p_v, p_{v'}\} = \{q_v, q_{v'}\} = 0 \text{ and } \{q_v, p_{v'}\} = \delta_{v,v'} \text{ for all } v, v' \text{ in } V.$$

Define the Hamiltonian, or energy function, $H : \mathcal{M} \to \mathbb{C}$ with $(\mathcal{M}, \omega)$ the above symplectic manifold for the dynamical system Γ. As before, $H = T + U$ where T is the kinetic energy and U is the potential energy such that

$$T = \frac{1}{2} \sum_{v \in V} m_v p_v^2 \text{ and } U = \sum_{e \in E} \phi(dq(e)) + f_{qv_{\alpha_0}}$$

where $p_v^2 = \sum_{i,j=1}^{n} \delta_{ij} p_i(v) p_j(v)$, $\phi(\cdot)$ is the spring potential energy between the particle at e_- and e_+ and f is a force which acts on the fixed vertex v_{α_0}.

3.1 A symplectic map for Γ

A symplectic map applied to the canonical coordinates (p, q) may be defined in the following way: if $e \in E$ has initial vertex e_- and terminal vertex e_+, define the relative displacement between them, r_e, as the "generalized coordinate" for $e \in E$ where

$$r_e = dq(e).$$

Define the "conjugate momentum to r_e" as

$$s_e = \frac{\partial L}{\partial \dot{r}_e}$$

with the Lagrangian $L = T - U$. By generalized coordinate and momentum we are implying that the Poisson bracket is

$$\{r_e, r_{e'}\} = \{s_e, s_{e'}\} = 0 \text{ and } \{r_e, s_{e'}\} = \delta_{e,e'} \text{ for all } e, e' \text{ in } E.$$

Next, introduce a new momentum P_v and a new coordinate Q_v conjugate to P_v such that

$$r_e = \frac{P_{e_+}}{a} \text{ and } s_e = -aQ_{e_+} \text{ where } a \text{ is a fixed scalar.}$$

If Γ happens to be a Cayley graph, notate the edge $e = [g, gs]$ where $g \in G$ is a fixed group and $s \in S$ where S is a generating set for G. The element g

corresponds to e_- and gs corresponds to e_+ of e. Then, fro a given $e \in E$, the new momenta and coordinates (P, Q) are

$$r_{[g,gs]} = \frac{P_{gs}}{a} \text{ and } s_{[g,gs]} = -aQ_{gs}.$$

The pulling back from the vertex to an edge is not necessarily well-defined for an arbitrary graph Γ, so for simplicity we shall restrict *all* calculations in this paper to trees embedded in Γ or graphs which are already trees. We remove these restrictions in the next paper.

The map ψ is defined to be the composite of the two mappings as in Section 1. Thus an intermediate mapping is constructed which maps the coordinates (q, p) to new coordinates (r, s), and then to new coordinates (Q, P).

Proposition 2 *If $\Gamma = \Gamma(V, E)$ is a tree, the map $\psi : \mathbb{R}^{2n|V|} \to \mathbb{R}^{2n|V|}$ defined by*

$$(q, p) \to (Q, P)$$

is a symplectic map.

Note: If we work with a graph that is not a tree, the situation may not be as dire. Thus is due to the fact that any connected graph contains a maximal tree. See Serre [4] for a proof of this. If this is the case, then we can construct a subgraph which contains all of the vertices and some of the edges from our original graph. Thus, use of a maximal tree as an approximation of a particular graph may possibly yield useful results regarding the graph dynamics.

The proof of the proposition above follows from a sequence of lemmas which serve to classify the types of trees.We reserve this sequence for the following paper.

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