

Generalized Kloosterman Sums over Rings of Order 2^r

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Abstract

After presenting several applications of Kloosterman sums in analytic number theory, generalized Kloosterman sums attached to Gauss sums over the ring of 2^r elements are explicitly evaluated and are shown to be sines and cosines.

1 Introduction

In the last century Kloosterman sums have played an increasingly important role in the analytic theory of numbers. First evaluated by Salié [10] in 1929, Kloosterman sums are a type of exponential sum which are finite analogues of Bessel functions. They appear in the Hardy-Ramanujan-Rademacher formula for the partition function $p(n)$ where $p(n)$ is the number of ways of writing n as a sum of integers. More general Kloosterman sums of order n are associated with metaplectic forms on $GL(2)$ in Kazhdan and Patterson [6]. Gelfand, Graev, and Piatetski-Shapiro [4] gives interpretations of Kloosterman sums involving representations of p -adic $GL(2)$.

We should note that Kloosterman sums were first introduced by Kloosterman in the context of estimating the number of representations of an integer by a quadratic form, i.e. the number of (x, y) in $\mathbb{Z} \times \mathbb{Z}$ such that $ax^2 + bxy + cy^2 = n$ for fixed n and a, b, c integers.

Besides appearing in the Fourier coefficients of modular forms, estimates of these sums play an important role in giving partial results to the now proved Ramanujan conjecture (see Selberg [11]). Weil [12] found the estimate of $2p^{\frac{1}{2}}$ for the sums mod p using the Riemann hypothesis for zeta functions for curves over finite fields. Deligne gives an efficient way to estimate sums over finite fields but sums over finite rings which are not fields are less well-understood.

Although Kloosterman sums of odd prime powers have been studied extensively, many authors avoid the case where $p = 2$. This case is important as Kloosterman

sums for fields of characteristic 2 have been useful in error-correcting codes [1] and Kloosterman sums over the ring of integers modulo p^r have been helpful in studying the spectra of Euclidean graphs (see [3], [8] and [9]). After stating some well-known facts about Kloosterman sums, we evaluate several generalized Kloosterman sums.

We begin by introducing Kloosterman sums $K(a, b)$ and defining the generalized Kloosterman sum $K'(a, b)$. After quickly simplifying $K'(a, b)$ and reviewing Salié's results, we evaluate $K'(a, b)$.

2 Definitions

Here \mathbb{Z}_{2^r} is the quotient ring of integers \mathbb{Z} modulo the ideal of multiples of 2^r .

DEFINITION: For $v \in \mathbb{Z}_{2^r}^*$ the *Gauss Sum* is defined by

$$G_v^{(r)} = \sum_{y \in \mathbb{Z}_{2^r}} e^{\frac{2\pi i v y^2}{2^r}}.$$

DEFINITION: If κ is a character of the multiplicative group $\mathbb{Z}_{2^r}^*$, then define the *Kloosterman sum*

$$K^{(r)}(\kappa \mid a, b) = \sum_{v \in \mathbb{Z}_{2^r}^*} \kappa(v) e^{\frac{2\pi i (av + b\bar{v})}{2^r}}$$

where \bar{v} is the multiplicative inverse of $v \pmod{2^r}$ such that $v\bar{v} \equiv 1 \pmod{2^r}$.

We define the *generalized Kloosterman sum*

$$K'(a, b) = \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} [G_v^{(r)}]^n e^{\frac{2\pi i (av + b\bar{v})}{2^r}}.$$

This sum is generalized in that it contains a Gauss sum of order n inside of it. This sum is encountered in the evaluation of eigenvalues of the adjacency matrix associated to Euclidean graphs over rings modulo 2^r .

2.1 Simplification of $K'(a, b)$

For $r \geq 3$,

$$\begin{aligned} K'(a, b) &= \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} [G_v^{(r)}]^n e^{\frac{2\pi i (av + b\bar{v})}{2^r}} \\ &= 2^{\frac{nr}{2}} \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} (1 + i^v)^n \left(\frac{2}{v}\right)^{rn} e^{\frac{2\pi i (av + b\bar{v})}{2^r}}. \end{aligned}$$

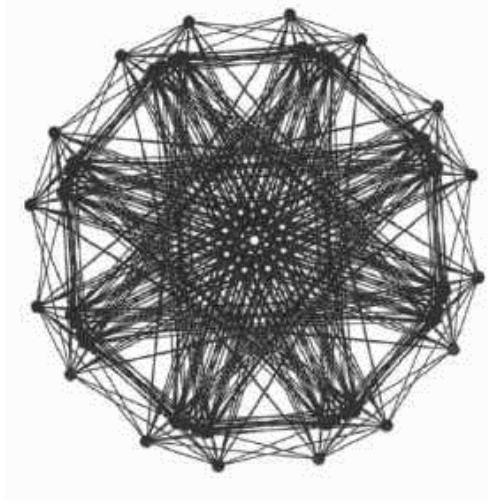


Figure 1: A Euclidean graph modulo 8 created using Mathematica

We now reduce $K'(a, b)$ to the cases where $n \equiv 0, 1, 2,$ or $3 \pmod 4$. For Euclidean graphs, n corresponds to the dimension of the vertices of the graphs. In Figure 1, the Euclidean graph has dimension $n = 2$. Thus there are 64 vertices as each vertex corresponds to $[k, k]$ where k is an integer mod 8.

$$\text{Let } K'(a, b; \chi_1) = \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} e^{\frac{2\pi i(av+b\bar{v})}{2^r}}, K'(a, b; \chi_2) = \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} i^v e^{\frac{2\pi i(av+b\bar{v})}{2^r}},$$

$$K'(a, b; \chi_3) = \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} \left(\frac{2}{v}\right) e^{\frac{2\pi i(av+b\bar{v})}{2^r}} \text{ and } K'(a, b; \chi_4) = \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} i^v \left(\frac{2}{v}\right) e^{\frac{2\pi i(av+b\bar{v})}{2^r}}.$$

Then:

for $n \equiv 0 \pmod 4$,

$$\begin{aligned} K'(a, b) &= 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n}{4}} \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} e^{\frac{2\pi i(av+b\bar{v})}{2^r}} \\ &= 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n}{4}} K'(a, b; \chi_1); \end{aligned}$$

for $n \equiv 1 \pmod{4}$,

$$\begin{aligned} K'(a, b) &= 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-1}{4}} \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} \left(\frac{2}{v}\right)^r e^{\frac{2\pi i(av+b\bar{v})}{2^r}} + \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} i^v \left(\frac{2}{v}\right)^{rn} e^{\frac{2\pi i(av+b\bar{v})}{2^r}} \\ &= \begin{cases} 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-1}{4}} \{K'(a, b; \chi_1) + K'(a, b; \chi_2)\} & \text{for } r \text{ even} \\ 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-1}{4}} \{K'(a, b; \chi_3) + K'(a, b; \chi_4)\} & \text{for } r \text{ odd.} \end{cases} \end{aligned}$$

for $n \equiv 2 \pmod{4}$,

$$\begin{aligned} K'(a, b) &= 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n-2}{4}} \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} i^v e^{\frac{2\pi i(av+b\bar{v})}{2^r}} \\ &= 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n-2}{4}} K'(a, b; \chi_2); \end{aligned}$$

for $n \equiv 3 \pmod{4}$,

$$\begin{aligned} K'(a, b) &= 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-3}{4}} \left\{ \sum_{\substack{2|v \\ v \in \mathbb{Z}_{2^r}^*}} \left[-\left(\frac{2}{v}\right)^r e^{\frac{2\pi i(av+b\bar{v})}{2^r}} + i^v \left(\frac{2}{v}\right)^{rn} e^{\frac{2\pi i(av+b\bar{v})}{2^r}} \right] \right\} \\ &= \begin{cases} 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-3}{4}} \{-K'(a, b; \chi_1) + K'(a, b; \chi_2)\} & \text{for } r \text{ even} \\ 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-3}{4}} \{-K'(a, b; \chi_3) + K'(a, b; \chi_4)\} & \text{for } r \text{ odd.} \end{cases} \end{aligned}$$

3 Kloosterman Sums

Salié ([10], pp.91-100) explicitly calculated the Kloosterman sum $S(a, b; 2^r)$, which we denote as $K'(a, b; \chi_1)$, on $\mathbb{Z}/2^r\mathbb{Z}$ with $r > 1$ where

$$S(a, b; 2^r) = \sum_{\substack{2|v \\ 0 < v < 2^r}} e^{\frac{2\pi i(av+b\bar{v})}{2^r}}.$$

Katz [5] interprets Salié's work as a method of stationary phase and uses the Taylor expansion of f to evaluate sums of $e^{\frac{2\pi i f(x)}{p^r}}$ over $x \pmod{p^r}$. This rewrites the sums as a sum over x with $f'(x)$ congruent to $0 \pmod{p^j}$ for $0 \leq j \leq r$ and an inner sum which is a Gauss sum if $2^{r-j} \geq r$ and p odd.

We use the notation $K'_{2^r}(a, b; \chi_1)$ below in cases where a specific r is chosen.

4 Evaluation of $K'(a, b; \chi_1)$

4.1 Properties of $K'(a, b; \chi_1)$

Let $\varepsilon = e^{\frac{2\pi i}{2^r}}$.

We first look at the symmetry property of $K'(a, b; \chi_1)$. If at least one of a or b is odd we see below that $K'(a, b; \chi_1) = K'(1, ab; \chi_1)$ by the change of variables $w = av$, $\bar{w} = \bar{a}\bar{v}$ and $\bar{v} = a\bar{w}$ where a is odd. We also notice that $K'(a, b; \chi_1) = K'(-a, -b; \chi_1)$. Now we derive some formulas for various a and b .

For $r = 1$,

$$K'(a, b; \chi_1) = (-1)^{a+b}.$$

For $r \geq 2$ the numbers v where $2 \nmid v$ have inverses \bar{v} such that $v\bar{v} \equiv 1 \pmod{2^r}$. We can represent v as $v = w + 2^{r-1}s$ where $2 \nmid w$, $w = 1, \dots, 2^{r-1} - 1$, and $s = 0$ or 1 . It is easy to verify that $\bar{v} = \bar{w} - 2^{r-1}s\bar{w}^2 \pmod{2^r}$ for $r \geq 2$. If either a or b is odd, substituting gives

$$K'(a, b; \chi_1) = \sum_{\substack{2 \nmid w \\ 0 < w < 2^{r-1}}} \varepsilon^{aw+b\bar{w}} [1 + (-1)^{a-b\bar{w}^2}].$$

Since w is odd, w^2 is odd which gives:

Lemma 1 $K'(a, b; \chi_1)$ is zero iff $a \not\equiv b \pmod{2}$ for $r \geq 2$.

Proof. See Salié [10], p.98.

Theorem 2 If $\bar{a}b \not\equiv 1 \pmod{8}$, then $K'(a, b; \chi_1)$ is zero for $r \geq 6$.

Proof. See Salié [10], p.98.

By Lemma 1 and Thm. 2, we may assume both a and b odd. Also, since any $\bar{a}b \equiv 1 \pmod{8}$ can be written as $\bar{a}b \equiv c^2 \pmod{2^k}$ for $k \geq 3$, we can assume that $\bar{a}b \equiv c^2 \pmod{2^r}$.

4.2 Evaluation of $K'(a, b; \chi_1)$

Theorem 3 Given a and b odd such that $\bar{a}b \equiv c^2 \pmod{2^r}$,

$$K'(a, b; \chi_1) = 2^{\frac{r+3}{2}} \begin{cases} \cos \left[\frac{4\pi ac}{2^r} + (-1)^{\frac{ac-1}{2}} \frac{\pi}{4} \right] & \text{for even } r \geq 6 \\ \cos \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] & \text{for odd } r \geq 9. \end{cases}$$

Proof. See Salié [10], p.98.

We recall from p. 101 of Salié that

$$\begin{aligned}
K'(a, b; \chi_1; 4) &= (-1)^{\frac{a+b}{2}} 2, \\
K'(a, b; \chi_1; 8) &= 0, \\
K'(a, b; \chi_1; 16) &= 4(-1)^{\frac{(ac)^2-1}{8}} \sqrt{2}, \\
K'(a, b; \chi_1; 32) &= 0, \\
&\text{and} \\
K'(a, b; \chi_1; 128) &= -32 \cos \left[\frac{4\pi ac}{128} + \frac{\pi ac}{4} \right].
\end{aligned}$$

4.2.1 Comparison with Salié

Referring to p.100 of Salié [10] we see that if $2 \nmid v$ we have

$$\begin{aligned}
S(1, k^2; 2^r) &= \sum_{0 < v < 2^r} \varepsilon^{v+k^2\bar{v}} = \sum_{0 < v < 2^r} \varepsilon^{kv+k\bar{v}} \\
&= \begin{cases} 2^{\frac{r+3}{2}} \cos \left[(-1)^{\frac{k-1}{2}} \frac{4\pi k}{2^r} + \frac{\pi}{4} \right] & \text{for even } r \geq 6; \\ 2^{\frac{r+3}{2}} (-1)^{\frac{k^2-1}{8}} \cos \left[(-1)^{\frac{k-1}{2}} \frac{4\pi k}{2^r} + \frac{\pi}{4} \right] & \text{for odd } r \geq 9. \end{cases}
\end{aligned}$$

Using the properties of cosine this is exactly the same formula we derived with $k = ac$.

5 Evaluation of $K'(a, b; \chi_j)$ for $j = 2, 3, 4$

Recall that for $r \geq 3$ and for $j = 2, 3, 4$ where $\chi_2(v) = i^v$, $\chi_3(v) = \left(\frac{2}{v}\right)$, and $\chi_4(v) = i^v \left(\frac{2}{v}\right)$, we have

$$K'(a, b; \chi_j) = \sum_{\substack{2^s | v \\ 0 < v < 2^r}} \chi_j(v) \varepsilon^{av+b\bar{v}}.$$

Lemma 4 For $v = k + 2^{r-s}w$ where w is taken mod 2^s , $r \geq 2s \geq 0$ and k odd, we have

$$\bar{v} \equiv \bar{k} - 2^{r-s} \bar{k}^2 w \pmod{2^r}.$$

For $s = 1, 2, \text{ or } 3$, we have $\bar{k}^2 \equiv 1 \pmod{8}$. Thus

$$\bar{v} \equiv \bar{k} - 2^{r-s} w \pmod{2^r}.$$

Theorem 5 *Given a and b odd such that $\bar{a}b \equiv c^2 \pmod{2^r}$, we have for $j = 2$ when $r \geq 6$ and for $j = 4$ when $r \geq 8$,*

$$K'(a, b; \chi_j) = 2^{\frac{r+3}{2}} i \chi_j(c) \begin{cases} \sin \left[\frac{4\pi ac}{2^r} + (-1)^{\frac{ac-1}{2}} \frac{\pi}{4} \right] & \text{for } r \text{ even,} \\ \sin \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] & \text{for } r \text{ odd;} \end{cases}$$

and for $j = 3$ when $r \geq 8$,

$$K'(a, b; \chi_j) = 2^{\frac{r+3}{2}} \begin{pmatrix} 2 \\ c \end{pmatrix} \begin{cases} \cos \left[\frac{4\pi ac}{2^r} + (-1)^{\frac{ac-1}{2}} \frac{\pi}{4} \right] & \text{for } r \text{ even,} \\ \cos \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] & \text{for } r \text{ odd.} \end{cases}$$

Proof. Case 1: r even.

Notice that

$$i^{v_0 + 2^{\frac{r}{2}} k} = i^{v_0} \quad \text{for } r \geq 4$$

and

$$\left(\frac{2}{v_0 + 2^{\frac{r}{2}} k} \right) = \left(\frac{2}{v_0} \right) \quad \text{for } r \geq 6.$$

Using Lemma 4 with $s = \frac{r}{2}$ and $v = v_0 + 2^{\frac{r}{2}} k$, $\bar{v} = \bar{v}_0 - 2^{\frac{r}{2}} k \bar{v}_0^2 \pmod{2^r}$ and substituting into $K_{v_0}^*(a, b; \chi_j)$ we have

$$\begin{aligned} K_{v_0}^*(a, b; \chi_j) &= \sum_{\substack{v \equiv v_0 \pmod{2^{\frac{r}{2}}} \\ 2|v}} \chi_j(v_0) \varepsilon^{av+b\bar{v}} \\ &= \sum_{k \pmod{2^{\frac{r}{2}}}} \chi_j(v_0) \varepsilon^{a(v_0 + 2^{\frac{r}{2}} k) + b(\bar{v}_0 - k \bar{v}_0^2 2^{\frac{r}{2}})} \\ &= \chi_j(v_0) \varepsilon^{av_0 + b\bar{v}_0} \sum_{k \pmod{2^{\frac{r}{2}}}} e^{\frac{2\pi i k(a - b\bar{v}_0^2)}{2^{\frac{r}{2}}}} \\ &= \begin{cases} 2^{\frac{r}{2}} \chi_j(v_0) \varepsilon^{av_0 + b\bar{v}_0} & \text{if } a \equiv b\bar{v}_0^2 \pmod{2^{\frac{r}{2}}} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This means that we need only to calculate the solutions to $v_0^2 \equiv c^2 \pmod{2^{\frac{r}{2}}}$. For $r \geq 6$ these are given by $v_0 = \pm 1c$ and $\pm(2^{\frac{r}{2}-1} - 1)c$ with inverses $\bar{v}_0 = \pm 1c$ and $\mp(2^{r-2} + 2^{\frac{r}{2}-1} + 1)c$ respectively.

The sum over all v where $r \geq 6$ is

$$\begin{aligned}
K'(a, b; \chi_j) &= \sum_{\substack{2^{\frac{r}{2}} v \\ 0 < v < 2^r}} \chi_j(v) \varepsilon^{av+b\bar{v}} \\
&= 2^{\frac{r}{2}} \sum_{\substack{\bar{v}_0 \\ v_0^2 \equiv c^2 \pmod{2^{\frac{r}{2}}}}} \chi_j(v_0) \varepsilon^{av_0+b\bar{v}_0} \\
&= 2^{\frac{r}{2}} \left\{ \chi_j(c) \varepsilon^{ac+b\bar{c}} \left(1 + \varepsilon^{-2^{\frac{r}{2}-1}ac+(2^{r-2}+2^{\frac{r}{2}-1})b\bar{c}} \right) \right. \\
&\quad \left. + \chi_j(-c) \varepsilon^{-ac-b\bar{c}} \left(1 + \varepsilon^{2^{\frac{r}{2}-1}ac-(2^{r-2}+2^{\frac{r}{2}-1})b\bar{c}} \right) \right\}
\end{aligned}$$

Now $\bar{a}b \equiv c^2 \pmod{2^r}$ implies that $b\bar{c} = ac \pmod{2^r}$. Thus we substitute $b\bar{c} = ac + 2^r q$ into the above equation. When $j = 2$ for $r \geq 6$ or $j = 4$ for $r \geq 8$, we have

$$\begin{aligned}
K'(a, b; \chi_j) &= 2^{\frac{r+1}{2}} \left\{ \chi_j(c) \varepsilon^{2ac} e^{\frac{\pi i}{4}(-1)^{\frac{ac-1}{2}}} + \chi_j(-c) \varepsilon^{-2ac} e^{-\frac{\pi i}{4}(-1)^{\frac{ac-1}{2}}} \right\} \\
&= 2^{\frac{r+3}{2}} i \chi_j(c) \sin \left[\frac{4\pi ac}{2^r} + (-1)^{\frac{ac-1}{2}} \frac{\pi}{4} \right] \text{ for } r \text{ even and } i = 2, 4.
\end{aligned}$$

When $j = 3$ for $r \geq 8$, we have

$$\begin{aligned}
K'(a, b; \chi_j) &= 2^{\frac{r+1}{2}} \left(\frac{2}{c} \right) \left\{ \varepsilon^{2ac} e^{\frac{\pi i}{4}(-1)^{\frac{ac-1}{2}}} + \varepsilon^{-2ac-2^{\frac{r}{2}}k\bar{c}} e^{-\frac{\pi i}{4}(-1)^{\frac{ac-1}{2}}} \right\} \\
&= 2^{\frac{r+3}{2}} \left(\frac{2}{c} \right) \cos \left[\frac{4\pi ac}{2^r} + (-1)^{\frac{ac-1}{2}} \frac{\pi}{4} \right] \text{ for } r \text{ even and } i = 3.
\end{aligned}$$

Case 2: r odd.

Notice that

$$\begin{aligned}
i^{v_0+2^{\frac{r-1}{2}}k} &= i^{v_0} \quad \text{for } r \geq 5 \\
&\text{and} \\
\left(\frac{2}{v_0+2^{\frac{r-1}{2}}k} \right) &= \left(\frac{2}{v_0} \right) \quad \text{for } r \geq 7.
\end{aligned}$$

Again we use an argument similar to Case 1. Let $v = v_0 + 2^{\frac{r+1}{2}}k$ and $\bar{v} =$

$\bar{v}_0 - k\bar{v}_0^2 2^{\frac{r+1}{2}} \pmod{2^r}$. Substituting into $K_{v_0}^{**}(a, b; \chi_j)$ gives us

$$\begin{aligned}
K_{v_0}^{**}(a, b; \chi_j) &= \sum_{\substack{v \equiv v_0 \pmod{2^{\frac{r+1}{2}}} \\ 2 \nmid v}} \chi_j(v) \varepsilon^{av+b\bar{v}} \\
&= \sum_{k \pmod{2^{\frac{r-1}{2}}}} \chi_j(v_0) \varepsilon^{a(v_0+2^{\frac{r+1}{2}}k)+b(\bar{v}_0-k\bar{v}_0^2 2^{\frac{r+1}{2}})} \\
&= \chi_j(v_0) \varepsilon^{av_0+b\bar{v}_0} \sum_{k \pmod{2^{\frac{r-1}{2}}}} e^{\frac{2\pi i k(a-b\bar{v}_0^2)}{2^{\frac{r-1}{2}}}} \\
&= \begin{cases} 2^{\frac{r-1}{2}} \chi_j(v_0) \varepsilon^{av_0+b\bar{v}_0} & \text{if } a \equiv b\bar{v}_0^2 \pmod{2^{\frac{r-1}{2}}} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus the sum is zero if $b\bar{a}$ is not a square modulo $2^{\frac{r-1}{2}}$. We need only to calculate the solutions to $v_0^2 \equiv c^2 \pmod{2^{\frac{r-1}{2}}}$. These are given by $v_0 = \pm 1c, \pm(2^{\frac{r-3}{2}} - 1)c, \pm(2^{\frac{r-3}{2}} + 1)c$, and $\pm(2^{\frac{r-1}{2}} - 1)c$ with inverses $\bar{v}_0 = \pm 1c, \pm(-2^{r-3} - 2^{\frac{r-3}{2}} - 1)c, \pm(2^{r-3} - 2^{\frac{r-3}{2}} + 1)c$, and $\pm(-2^{r-1} - 2^{\frac{r-1}{2}} - 1)c$ respectively.

For $r \geq 9$ the sum over all v after reduction is

$$\begin{aligned}
K'(a, b; \chi_j) &= 2^{\frac{r-1}{2}} \{ \chi_j(c) \varepsilon^{ac+b\bar{c}} (1 + \varepsilon^{-2^{\frac{r-3}{2}} ac + (2^{r-3} + 2^{\frac{r-3}{2}}) b\bar{c}} \\
&\quad + \varepsilon^{2^{\frac{r-3}{2}} ac + (2^{r-3} - 2^{\frac{r-3}{2}}) b\bar{c}} + \varepsilon^{-2^{\frac{r-1}{2}} ac + (2^{r-1} + 2^{\frac{r-1}{2}}) b\bar{c}}) \\
&\quad + \chi_j(-c) \varepsilon^{-ac-b\bar{c}} (1 + \varepsilon^{2^{\frac{r-3}{2}} ac + (-2^{r-3} - 2^{\frac{r-3}{2}}) b\bar{c}} \\
&\quad + \varepsilon^{-2^{\frac{r-3}{2}} ac - (2^{r-3} - 2^{\frac{r-3}{2}}) b\bar{c}} + \varepsilon^{2^{\frac{r-1}{2}} ac + (-2^{r-1} - 2^{\frac{r-1}{2}}) b\bar{c}} \}
\end{aligned}$$

Substituting $b\bar{c} = ac + 2^r q$ into the above equation gives us

$$\begin{aligned}
K'(a, b; \chi_j) &= 2^{\frac{r-1}{2}} \{ \chi_j(-c) \varepsilon^{-2ac-2^r q} (1 + \varepsilon^{-2^{r-3} ac} + \varepsilon^{-2^{r-3} ac} + \varepsilon^{-2^{r-1} ac}) \\
&\quad + \chi_j(c) \varepsilon^{2ac+2^r q} (1 + \varepsilon^{2^{r-3} ac} + \varepsilon^{2^{r-3} ac} + \varepsilon^{2^{r-1} ac}) \} \\
&= 2^{\frac{r-1}{2}} \{ \chi_j(-c) \varepsilon^{-2ac} (1 + e^{\frac{-\pi i ac}{4}} + e^{\frac{-\pi i ac}{4}} + e^{-\pi i ac}) \\
&\quad + \chi_j(c) \varepsilon^{2ac} (1 + e^{\frac{\pi i ac}{4}} + e^{\frac{\pi i ac}{4}} + e^{\pi i ac}) \}.
\end{aligned}$$

Since ac is odd, we have for $j = 2, 4$

$$\begin{aligned}
K'(a, b; \chi_j) &= 2^{\frac{r-1}{2}} i \{ \chi_j(-c) \varepsilon^{-2ac} (2e^{\frac{-\pi i ac}{4}}) + \chi_j(c) \varepsilon^{2ac} (2e^{\frac{\pi i ac}{4}}) \} \\
&= 2^{\frac{r+3}{2}} i \chi_j(c) \sin \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] \text{ for } r \geq 9.
\end{aligned}$$

Substituting $b\bar{c} = ac + 2^r q$ into the above equation for $j = 3$ gives us

$$K'(a, b; \chi_j) = 2^{\frac{r+3}{2}} \left(\frac{2}{c}\right) \cos \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right].$$

6 Summary of Evaluations and Observations

Notice that for $r \geq 8$ we have $K'(a, b; \chi_3) = \left(\frac{2}{c}\right) K'(a, b; \chi_1)$ and $K'(a, b; \chi_4) = \left(\frac{2}{c}\right) K'(a, b; \chi_2)$. Below we summarize the results for the generalized Kloosterman sums $K'(a, b)$.

For all a and b where $\bar{a}b \equiv c^2 \pmod{2^r}$ and $r \geq 8$, we have:

for $n \equiv 0 \pmod{4}$,

$$K'(a, b) = 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n}{4}} K'(a, b; \chi_1);$$

for $n \equiv 1 \pmod{4}$,

$$K'(a, b) = 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-1}{4}} \left(\frac{2}{c}\right)^r \{K'(a, b; \chi_1) + K'(a, b; \chi_2)\};$$

for $n \equiv 2 \pmod{4}$,

$$K'(a, b) = 2^{\frac{n(r+1)}{2}} (-1)^{\frac{n-2}{4}} K'(a, b; \chi_2);$$

for $n \equiv 3 \pmod{4}$,

$$K'(a, b) = 2^{\frac{n(r+1)-1}{2}} (-1)^{\frac{n-3}{4}} \left(\frac{2}{c}\right)^r \{-K'(a, b; \chi_1) + K'(a, b; \chi_2)\}$$

where

$$K'(a, b; \chi_1) = 2^{\frac{r+3}{2}} \begin{cases} \cos \left[\frac{4\pi ac}{2^r} + \frac{\pi}{4} (-1)^{\frac{ac-1}{2}} \right] & \text{for even } r \geq 8 \\ \cos \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] & \text{for odd } r \geq 9; \end{cases}$$

and

$$K'(a, b; \chi_2) = 2^{\frac{r+3}{2}} \begin{cases} i^{-(1+c)} \sin \left[\frac{4\pi ac}{2^r} + \frac{\pi}{4} (-1)^{\frac{ac-1}{2}} \right] & \text{for even } r \geq 8 \\ i^{-(1+c)} \sin \left[\frac{4\pi ac}{2^r} + \frac{\pi ac}{4} \right] & \text{for odd } r \geq 9. \end{cases}$$

Notice that the simplification of these sums resulted in real values. Since these sums are related to the eigenvalues of the adjacency matrices of Euclidean graphs

which are real, symmetric matrices, we should expect the Kloosterman sums to be real.

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