Milnor's Fibration Theorem

What this webpage is showing

First the set-up:

- Let $U \subset \mathbb{C}^{n+1}$ be a neighborhood of the origin.
- Let $f : U \to \mathbb{C}$ be an holomorphic map such that $f(\overline{0}) = 0$ and f has an isolated singularity at the origin (a singularity is a point where all of the partials of f are zero).
- Let $V = f^{-1}(0)$.
- We can identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. Now let S_ϵ be the $(2n+1)$-sphere in $\mathbb{R}^{2n+2}$ centered at the origin with radius ϵ.
- Let $K = V \cap S_\epsilon$. If ϵ is small enough, then locally V is topologically a cone over K. We call K the *link* of f.
- Let $\phi : S_\epsilon - K \to S^1$ be the map defined by $f/|f|$.

With all of this set up, Milnor's Fibration Theorem tells us the following:

Theorem: *The map ϕ is a C^∞ fiber bundle.*

This webpage looks at the special case of when f is a Pham-Brieskorn polynomial with $n = 1$. Specifically, f takes on the form

$$f(z, w) = z^a - w^b$$

where a and b are positive integers. In this case, S_ϵ is a three-dimensional sphere, and the link K is a curve in this sphere. Each fiber $\phi^{-1}(\theta)$ will be a surface in the three-sphere that has K as its boundary. As we rotate the base point θ on S^1, we rotate the fiber about the link K. If we project the three-sphere via stereographic projection, then we can see this in action.

The link for $z^a - w^b$ is an (a, b)-torus link. In the pictures and in the movies, we are displaying two fibers at the same time: the fiber for θ and the fiber for $\theta + \pi$. The two fibers are distinguished by the red and green colors. The union of the two fibers and the link gives a surface of genus $(a-1)(b-1)$. If you view this as a single surface, you can view this as a rotation of a surface of genus $(a-1)(b-1)$ in the three-sphere with an (a, b)-torus link as an axis.

When we use stereographic projection, there is one point on the three-sphere that is mapped to infinity. There is one fiber that passes through this point, so there is one fiber in our movies that explodes out to infinity. By design, this fiber associates with the value $\theta = \pi/2$.