Divergence operators, transformations of measure, and the interpolation method

Denis Bell

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Transformations of measure

Given a Borel measure $\gamma$ on a Banach space $E$ and a measurable transformation $T : E \mapsto E$, define the induced measure $\nu_T$ by

$$\gamma_T(B) = \gamma(T^{-1}(B)), \quad B \in \mathcal{B}(E)$$

We are interested in finding conditions on $T$ such that $\gamma_T \ll \gamma$ and computing the Radon-Nikodym derivative (RND) $\frac{d\gamma_T}{d\gamma}$ when these conditions hold.
The Girsanov Theorem

Let $w$ be a standard Wiener process in $\mathbb{R}^n$, defined on the time interval $[0, T]$, and $\gamma$ its law (the Wiener measure). Let $h : [0, T] \times \Omega \mapsto \mathbb{R}^n$ be a real-valued continuous bounded adapted process. Then the process

$$\tilde{w}_t = w_t + \int_0^t h_s ds, \quad t \in [0, T]$$

is a standard Wiener process with respect to the measure $\tilde{\gamma}$, where

$$\frac{d\tilde{\gamma}}{d\gamma}(w) = e^{-\int_0^T h_s \cdot dw_s - \frac{1}{2} \int_0^T |h_s|^2 ds}$$
In the special case when \( h \) is a deterministic path, the Girsanov theorem reduces to the \textit{Cameron-Martin theorem}, which states that translating Wiener measure by a deterministic path \( h \) of finite energy, i.e. an absolutely continuous path such that

\[
\int_0^T |\dot{h}_s|^2 \, ds
\]

produces an equivalent measure.

(The set \( H \) of such paths \( h \) is known as the \textit{Cameron-Martin} space.)

It is interesting to note that translation of Wiener measure by a deterministic path in \( H^c \) results in a singular measure.
Abstract Gaussian measures

Let $H$ and $E$ be, respectively, Hilbert and Banach spaces and $i : H \mapsto E$ be an abstract Wiener space in the sense of Gross with Gaussian measure $\gamma$ on $E$.

We identify $E^*$ as a subspace of $E$ under the embeddings

$$i^* \quad \quad \quad i$$

$$E^* \mapsto H^* \cong H \mapsto E$$

There is a series of results concerning transformation of $\gamma$ under map $T : E \mapsto E$, due to Kuo, Ramer, Kusuoka, Ustunel & Zakai... with the seminal result in this area due to Ramer.
Kuo’s theorem

Let $T$ be a diffeomorphism of $E$ of the form

$$T = I + K$$

where $I$ is the identity map on $E$ and $K$ is a $C^1$ map from $E$ to $E^*$. Then $\gamma_T \sim \gamma$ and

$$\frac{d\gamma_T}{d\gamma}(x) = |\det DT(x)|^{-1}e^{-(K(x),x)-\frac{1}{2}|K(x)|_H^2}$$

where $(\cdot, \cdot)$ denotes the pairing between $E^*$ and $E$ and det is defined with respect to $H$.

Note the difference in hypotheses between this result and GT. The latter assumes the adapted condition, whereas in Kuo’s theorem (and like results), adaptedness is replaced by a $C^1$ assumption on $K$. 
Divergence operators

Let $\gamma$ denote a Borel measure on a Banach space $E$ and $U$ a subset of the set of vector fields on $E$.

**Definition**

A linear operator $L : U \mapsto L^2(\gamma)$ is said to be a *divergence operator* (DO) for $\gamma$ if the following relation holds for all $Z \in U$ and a dense class of test functions $\phi$ on $E$

$$\int_E DZ\phi(x)d\gamma = \int_E \phi(x)LZ(x)d\gamma.$$
Examples

**Theorem**  
(Gaveau-Trauber). The Itô integral

\[ Lh = \int_0^T \dot{h}_s \, dw_s \]

is a DO for the Wiener space. The domain of \( L \) is the set of adapted process \( h \) of finite energy such that

\[ E\left[ \int_0^T |\dot{h}_s|^2 \, ds \right] < \infty. \]
Theorem
(Gross). Let $(H, E, \gamma)$ be an abstract Wiener space. Then the operator
\[ LZ(x) = (Z(x), x) - \text{trace}_H DZ(x) \]
is a DO for $\gamma$, with domain the set of $C^1$ maps from $E$ to $E^\ast$.

In view of the Gaveau-Trauber theorem, this operator corresponds to the Itô integral in the classical adapted case. It can be identified with the Skorohod integral. Nualart & Pardoux used this operator to develop a theory of anticipating stochastic integration.
**Relationship with Malliavin calculus**

Suppose $L$ is a DO for the space $(E, \gamma)$ (e.g. a Gaussian space) and $g : E \mapsto K$ that is "differentiable". Assume that $Z$ is a vector field on $K$ for which there exists a lift $R$ of $Z$ to $E$, i.e.

$$Dg(x)R = Z.$$

Then

$$\int_K D_Z \phi(x) d\gamma_g = \int_E D \phi(g(x)) Z d\gamma = \int_E D(\phi \circ g)(x) R d\gamma$$

$$= \int_E (\phi \circ g) LR d\gamma = \int_K \phi E[LR/g] d\gamma_g.$$ 

i.e. $\text{Div}_{\gamma_g} Z = E[LR/g]$. 
The interpolation method

Let \((L, U)\) denote a DO for \(\gamma\) on \(E\) and let \(T : E \mapsto E\) be a map of the form \(T = I_E + K\), where \(K \in U\). Define

\[
T_t = I + tK, \quad t \in [0, 1].
\]

Suppose that \(T_t\) is invertible for all \(t\). Note that \(\gamma_{T_t} \ll \gamma\) if and only if there exists a family \(X_t\) of RND's \(\frac{d\gamma_{T_t}}{d\gamma}\).

Let \(\phi\) be a test function on \(E\). Then

\[
\int_E \phi \circ T_t(x) d\gamma = \int_E \phi X_t d\gamma
\]

Replacing \(\phi\) by \(\phi \circ T_t^{-1}\), we have

\[
f(t) \equiv \int_E \phi \circ T_t^{-1}(x) X_t(x) d\gamma = \int_E \phi(x) d\gamma.
\]

Thus \(f\) is constant.
Computing $f'$ by formally differentiating wrt in the first integral, we have

$$0 = \int_E \left\{ D\phi(T_t^{-1}(x))X_t(x) + \phi \circ T_t^{-1}(x)d/dtX_t(x) \right\} d\gamma.$$ 

We rewrite the first term in the integrand using the easily verified relation

$$D\phi(T_t^{-1}(x))d/dtT_t^{-1}(x) = -D(\phi \circ T_t^{-1})(x)K \circ T_t^{-1}(x)$$

to get

$$\int_E \left\{ \phi \circ T_t^{-1}(x)d/dtX_t(x) - D(\phi \circ T_t^{-1})(x)K \circ T_t^{-1}(x)X_t(x) \right\} d\gamma = 0.$$ 

Assume the term $(K \circ T_t^{-1})X_t \in U$. We use the defining property of $L$ to transform the second term in the integrand. This yields
\[
\int_E \phi \circ T_t^{-1}(x) \left\{ \frac{d}{dt}X_t(x) - L[(K \circ T_t^{-1})X_t](x) \right\} d\gamma = 0.
\]

This will hold for all test functions \( \phi \) if and only if \( X_t \) satisfies the differential equation

\[
\frac{d}{dt}X_t(x) = L[(K \circ T_t^{-1})X_t](x).
\]

We now make use of the following easily verified relation.

**Lemma**
Assume \( h \in U \cap L^2(\gamma) \), \( \psi : E \mapsto \mathbb{R} \in C^1 \), \( \psi h \in U \). Then

\[
L(\psi h)(x) = \psi(x)Lh(x) - D\psi(x)h(x), \text{ a.s.}(\gamma).
\]

Using this above, we have

\[
\frac{d}{dt}X_t(x) = X_t(x)L[K \circ T_t^{-1}](x) - DX_t(x))K \circ T_t(x).
\]
Denote \( X_t(x) \) by \( X(t, x) \), \( \partial X/\partial t \) by \( X_1 \) and \( \partial X/\partial x \) by \( X_2 \), and substitute \( x = T_t(y) \) to obtain

\[
X_1(t, T_t(y)) + X_2(t, T_t(y))K(y) = \frac{d}{dt}X(t, T_t(y))L[K \circ T_t^{-1}](T_t(y)).
\]

Since \( K = \frac{d}{dt}T_t \), this is equivalent to

\[
d/dtX(t, T_t(y)) = X(t, T_t(y))L[K \circ T_t^{-1}]T_t(y).
\]

In view of the fact that \( X(0, \cdot) = 1 \), we have

\[
X(t, T_t(y)) = \exp \int_0^t L[K \circ T_s^{-1}](T_s(y))ds.
\]

Finally,

\[
\frac{d\gamma_T}{d\gamma} = X(1, x) = \exp \int_0^1 L[K \circ T_s^{-1}](T_s \circ T^{-1}(x))ds.
\]
Using this formula we can recover the RND’s in both the Girsanov theorem and Kuo’s theorem, from the appropriate DO’s.
Note that, in principle, all the steps in the above argument are reversible. The idea is to start with a family of RND’s defined by

\[ X(t, T_t(y)) = \exp \int_0^t L[K \circ T_{s}^{-1}](T_{s}(y))ds, \quad t \in [0, 1] \]

and by reversing the steps, to obtain

\[ \int_E \phi \circ T_t(x)d\gamma = \int_E \phi X_t d\gamma \]

thereby proving that \( \gamma_{T_t} \ll \gamma \) and \( \frac{d\gamma_{T_t}}{d\gamma} = X_t \).
We used this idea to obtain a generalization (to non-Gaussian measures) of the Cameron-Martin Theorem.

Further results in this direction were proved by Daletskii-Belopolskaya, Weizsacker, Smolyanov and others.
One problem is the invertibility condition on the family $T_t, t \in [0, 1]$.

This holds if $K \in L(E)$ and is a strict contraction, i.e. there exists $c \in (0, 1)$ with

$$\|K(x)\|_E < c\|x\|_E.$$

In this case $T_t \in GL(E)$ and

$$(l + tK)^{-1} = \sum_{n=0}^{\infty} t^n K^n.$$
Change of measure under the flow of a vector field

Let $E$ be a Banach space or a smooth manifold, equipped with a finite Borel measure $\gamma$ and let $Z$ be a $C^1$ vector field on $E$.

Consider the corresponding flow on $E$, i.e. the maps $\rho_s : x \mapsto x_s$ defined by

$$\frac{dx_s}{ds} = Z(x_s),$$

$$x_0 = x.$$

These maps are naturally diffeomorphisms of $E$ ($\rho_s^{-1} = \rho_{-s}$).

Say that $Z$ is admissible if $Z$ admits a divergence $L$. We say that $\gamma$ is quasi-invariant under (the flow of) $Z$ if $\gamma_s \equiv \gamma_{\rho_s} \sim \gamma$ for all $s \in \mathbb{R}$. 
Formally, quasi-invariance implies admissibility: suppose q-i holds and denote \( \frac{d\gamma_s}{d\gamma} = X_s \). Since \( \rho(0) = I \) and \( \rho'(0) = Z \), we have

\[
\frac{d}{ds} \bigg|_{s=0} \int_E \phi d\gamma_s = \frac{d}{ds} \bigg|_{s=0} \int_E \phi \circ \rho_s d\gamma = \int_E D_Z \phi d\gamma.
\]

Also

\[
\frac{d}{ds} \bigg|_{s=0} \int_E \phi d\gamma_s = \frac{d}{ds} \bigg|_{s=0} \int_E \phi X_s d\gamma = \int_E \phi \frac{dX_s}{ds} \bigg|_{s=0} d\gamma.
\]

Hence \( Z \) is admissible and

\[
\text{Div} Z = \frac{dX_s}{ds} \bigg|_{s=0}
\]
The converse is not true in general.

*Example.* Consider the case where $E$ is $\mathbb{R}^n$, $d\gamma = Fdx$, were $F$ is $C^1$ with compact support, and $Z = h$, a constant vector in $\mathbb{R}^n$.

Then $Z$ is admissible with divergence $-D_hF/F$. But not q-i under the flow $\rho_s(x) = x + sh$.

Note, in this case $LZ(x)$ blows up as $x$ approaches the boundary of the support of $F$. 
Using the interpolation method, we proved (CRAS, 2006):

**Theorem**

Suppose $Z$ is admissible with divergence $LZ$. Assume that there exists $B \subset E$ with $\gamma_s(B) = 1, \forall s$, such that the function $s \mapsto (LZ)(x - s)$ is continuous for $x \in B$. Then $\gamma$ is quasi-invariant under $Z$ and

$$
\frac{d\gamma_s(x)}{d\gamma} = \exp \int_0^s (LZ)(x-u)du.
$$