The Gauss-Bonnet Theorem

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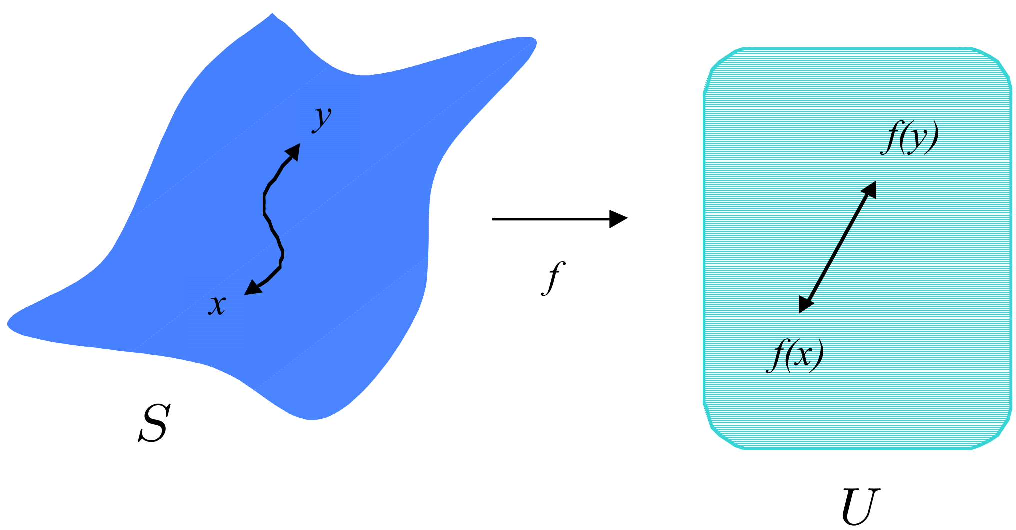
1. Gaussian curvature

Consider a smooth surface in $\mathbf{R}^3$. What does it mean to say the surface is flat?

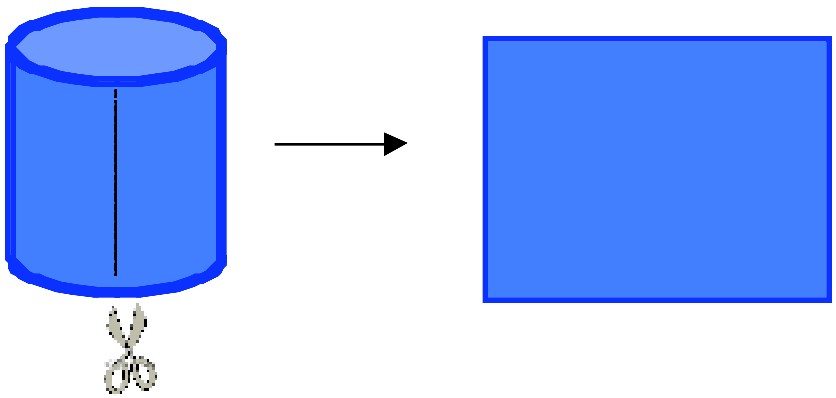
Definition A surface S is *flat* if there exists a distance preserving bijection between S and a subset U of the plane (say S and U are *isometric*).



$$d_S(x, y) = |f(x) - f(y)|$$

It would appear to 2-dimensional *residents* of S as if they were living in a plane.

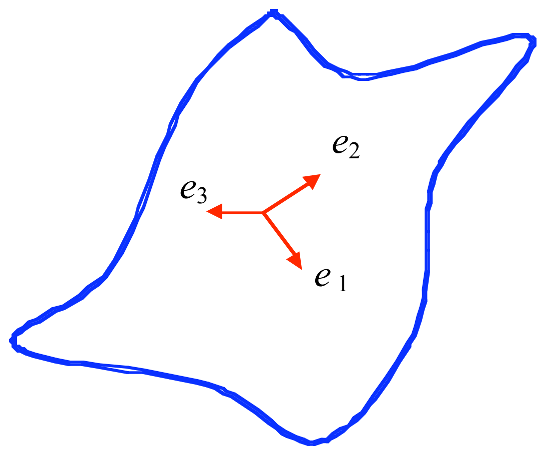
Example: a (portion of) a cylinder.



This is not possible with (any piece of) a sphere

Thus the sphere is *non-flat*. Gaussian curvature is a device for *quantifying* this behavior.

Let e_1 and e_2 be orthonormal tangent vectors to the surface S and e_3 a unit normal vector (defined at each point $x \in S$).



Then e_3 (the normal vector): $S \mapsto \mathbf{R}^3$, so $de_3(x) \in L(T_x, \mathbf{R}^3)$. In some sense, the magnitude of $de_3(x)$ is a measure of the curvature of S at x. *How is the magnitude to be determined*?

Note that since $< e_3, e_3 > \equiv 1$, for $v \in T_x S$, we have

$$0 = d_v < e_3, e_3 > = 2 < d_v e_3, e_3 > .$$

Thus $d_v e_3 \in T_x M$, i.e. $de_3(x) \in L(T_x S, T_x S)$.

Definition.The *Gaussian curvature* k of S at x is defined by $k = \text{Det } de_3(x)$

Since k is defined in terms of the normal vector, it looks like it is heavily dependent on the way in which S is embedded into $\mathbf{R}^3$. However, Gauss proved the following remarkable fact:

Gauss Theorem Egregium. *Two surfaces are isometric if and only if they have identical Gaussian curvatures at corresponding points (i.e. GC depends only on the metric structure of S and is independent of the embedding of S into $\mathbf{R}^3$).*

In particular

Corollary *A piece of a surface is flat if and only if its GC is identically zero*.

This implies the well-known fact that it is impossible to make a flat map of (part of) the earth that preserves distances.

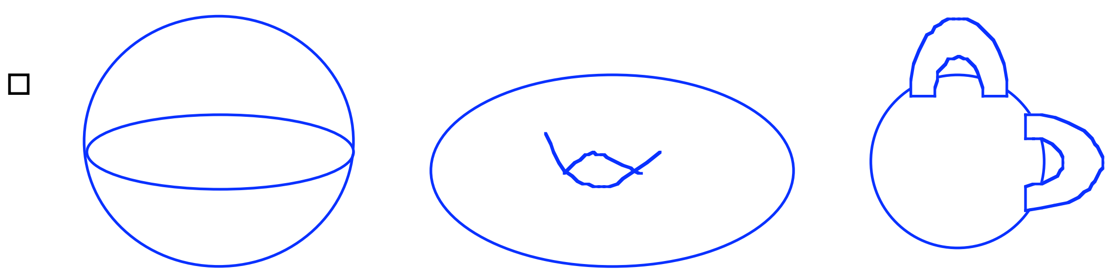
2. The Euler characteristic

Let S be a compact surface in (e.g. a balloon or a donut). Triangulate S into a grid of triangles. Suppose the triangulation contains f triangles (faces), e edges, and v vertices.

Definition. The *Euler characteristic* χ of S is defined by

$$\chi \equiv f - e + v.$$

χ is independent of the triangulation used and is a topological invariant. In fact, $\chi = 2(1 - g)$ where g is genus of S.



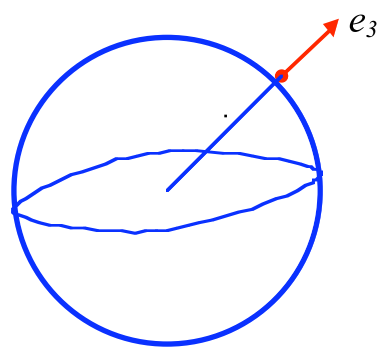
$\chi = 2$ $\chi = 0$ $\chi = -2$

3. The Gauss-Bonnet theorem

G-B Theorem (1850). *Let S be a closed orientable surface in $\mathbf{R}^3$ with Gaussian curvature k and Euler characteristic χ. Then*

$$\int_S k dA = 2\pi\chi.$$

Example. The sphere of radius a:



Since $e_3(x) = x/a$, we have $de_3(x) = \frac{1}{a}I$ where I is the 2×2 identity matrix. So $k = 1/a^2$. Note also that since the sphere has $g = 0$, $\chi = 2$. Hence

$$\int_S k dA = \int_S 1/a^2 dA = 1/a^2 A(S)$$

$$= 4\pi = 2\pi\chi.$$

4. An intrinsic description of GC

Again, let e_1, e_2, e_3 be an orthonormal frame on S with e_3 normal to S. We saw earlier that there exist 1-forms ω_{31} and ω_{32} on S such that

$$de_3 = \omega_{31} e_1 + \omega_{32} e_2.$$

Similarly

$$de_2 = \omega_{21} e_1 + \omega_{23} e_3$$

$$de_1 = \omega_{12} e_2 + \omega_{13} e_3.$$

It is easy to see that $\omega_{31} \wedge \omega_{32} = kdA$, where A is the area form on S. This is an extrinsic formula for k. However, there is the remarkable relation

$$\omega_{31} \wedge \omega_{32} = -d\omega_{12}.$$

Thus k can be computed (or defined) by

$$kdA = -d\omega_{12}.$$

The G-B formula can be expressed in the form

$$\chi = -\frac{1}{2\pi} \int_S d\omega_{12}.$$

5. A higher dimensional version of the Gauss-Bonnet theorem

Let M denote a compact oriented manifold of even dimension n. Let E be a real oriented Riemannian vector bundle of rank n over M.

Definition. A connection on E is a map: $Y \in T_xM, X \in \Gamma(E) \mapsto \nabla_Y X \in E_x$ satisfying the *product rule*: for $f \in C^\infty(M)$ and $X \in \Gamma(E)$,

$$\nabla_Y(fX) = Y(f)X + f\nabla_Y X.$$

We assume that ∇ is *metric*: i.e. for all X and $Y \in \Gamma(E)$,

$$d < X, Y > = < \nabla X, Y > + < X, \nabla Y > .$$

Let $e = \{e_1, \ldots, e_n\}^t$ denote an orthonormal frame of E, defined over a neighborhood U in M. Define a $n \times n$ matrix of *connection 1-forms* $\omega = [\omega_{ij}]$ on M by the relations

$$\nabla e_i = \sum_j \omega_{ij} e_j \quad (\nabla e = \omega e)$$

and a corresponding $n \times n$ matrix of *curvature 2-forms* Ω by

$$\Omega = d\omega - \omega^2$$

The metric property implies that both ω and Ω are *skew-symmetric*.

Suppose now that e and f are two orthonormal frames of E defined over neighborhoods U and V of M ($U \cap V \neq \phi$). Then there exists an orthogonal matrix-valued function A on $U \cap V$ such that $f = Ae$. with respective connection 1-forms and curvature 2-forms ω_e, Ω_e and ω_e, Ω_f. The following relation holds:

$$\Omega_f = A\Omega_e A^{-1}.$$

Thus unlike the 2-dimensional case Ω is frame-dependent, however it is a *tensor*.

The Pfaffian: There exists a map $Pf : so(n) \mapsto \mathbf{R}$ such that $Pf(A)$ is a homogeneous polynomial of degree $n/2$ in the entries of A and $Pf(A) = \sqrt{Det(A)}$. Furthermore,

$$Pf(ATA^{-1}) = Pf(T)$$

for orthogonal A and skew-symmetric T.

Now the set of even-degree differential forms is a commutative ring with $\wedge$ as multiplication. We define $Pf(\Omega_e)$ where, as before, Ω_e is the curvature matrix corresponding to an orthonormal frame e of E defined over a neighborhood U of M.

Note that if $Pf(\Omega_f)$ is another such expression defined over V then on $U \cap V$ we have $Pf(\Omega_e) = Pf(\Omega_f)$. Thus $Pf(\Omega_e)$ extends to a *globally* defined n-form on M. We denote this by $Pf(\Omega)$.

The Euler characteristic χ of E can be defined in the following way. Let $X : M \mapsto E$ denote a section of E and let E_0 denote the 0-section

$$\{0 \in E_x, x \in M\}.$$

Then generically, $X(M) \cap E_0$ will consist of a finite number of points and this number does not depend on X.

The number of points in the intersection is a topological invariant and is defined to be the Euler characteristic χ of E. It can be shown that when $E = TM$ then this definition coincides with the definition of EC defined by triangulation of M

$$\chi = \sum_{k=0}^{n} (-1)^k \Delta_k$$

where Δ_k is the number of k-simplices in the triangulation.

We have the following generalization of the G-B formula to this higher-dimensional vector bundle setting

Theorem

$$\chi = \left(\frac{-1}{2\pi}\right)^{n/2} \int_M Pf(\Omega).$$

In the case where M is orientable, $E = TM$ and ∇ is the Levi-Civita connection, this result reduces to the Gauss-Bonnet-Chern theorem (Chern, 1944).