FOOLPROOF ETERNAL DOMINATION IN THE ALL-GUARDS MOVE MODEL

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ABSTRACT. The eternal domination problem requires a graph be protected against an infinitely long sequence of attacks at vertices, by guards located at vertices, with the requirement that the configuration of guards induces a dominating set at all times. An attack is defended by sending a guard from a neighboring vertex to the attacked vertex. We allow all guards to move to neighboring vertices in response to an attack, but allow the attacked vertex to choose which neighboring guard moves to the attacked vertex. This is the all-guards move version of the “foolproof” eternal domination problem that has been previously studied. We present some results and conjectures on this problem.

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1. Introduction

Let $G = (V, E)$ be a simple, finite graph with $n$ vertices. Several recent papers have considered problems associated with using mobile guards to defend $G$ against an infinite sequence of attacks; see for instance [1, 3, 6, 9, 10]. Most of these papers consider attacks at vertices, while [5, 11, 12] consider the variation in which attacks are at edges. In this paper, we consider a variation on the vertex protection problem which was initially motivated by a desire to compare the vertex and edge protection parameters. This variation is analogous to the “foolproof” eternal domination problem considered, and characterized completely¹ in [3], but in this paper we allow all guards to move in response to an attack rather than just one.

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¹Exactly $n - Δ$ guards are needed to protect every connected graph in the one-guard moves foolproof model.
Graphs with equal eternal vertex cover and eternal domination numbers\textsuperscript{*}

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\textbf{Abstract}

Mobile guards on the vertices of a graph are used to defend it against an infinite sequence of attacks on either its vertices or its edges. If attacks occur at vertices, this is known as the eternal domination problem. If attacks occur at edges, this is known as the eternal vertex cover problem. We focus on the model in which all guards can move to neighboring vertices in response to an attack. Motivated by the question of which graphs have equal eternal vertex cover and eternal domination numbers, a number of results are presented; one of the main results of the paper is that the eternal vertex cover number is greater than the eternal domination number (in the all-guards move model) in all graphs of minimum degree at least two.

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1. Introduction

Let $G = (V, E)$ be a graph with $n$ vertices and minimum degree $\delta(G)$. A number of recent papers have considered problems associated with using mobile guards to defend $G$ against an infinite sequence of attacks; see for instance [1,3,6,7,10–12].

Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v \in V \mid \exists uv \in E\}$ and $N[x] = N(x) \cup \{x\}$. Further, for $S \subseteq V$, let $N(S) = \bigcup_{x \in S} N(x)$. For any $X \subseteq V$ and $x \in X$, we say that $v \in V - X$ is an external private neighbor of $x$ with respect to $X$ if $v$ is adjacent to $x$ but to no other vertex in $X$. The set of all such vertices $v$ is the external private neighborhood of $x$ with respect to $X$.

A dominating set of $G$ is a set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $x \in D$ adjacent to $u$. A dominating set $D$ is a connected dominating set if the subgraph $G[D]$ of $G$ induced by $D$ is connected. The minimum cardinality amongst all dominating sets of $G$ is the domination number $\gamma(G)$, while the minimum cardinality amongst all connected dominating sets is the connected domination number $\gamma_c(G)$ (see e.g. [9]).

A vertex cover of $G$ is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of $u$ and $v$ is in $C$. Let $\alpha(G)$ denote the vertex cover number of $G$, the minimum number of vertices required to cover all edges of $G$.

An independent set of $G$ is a set $I \subseteq V$ with the property that no two vertices in $I$ are adjacent. The maximum cardinality amongst all independent sets is the independence number $\beta(G)$. For all connected graphs $G$, it is well known that $n - \beta(G) = \alpha(G)$ (see e.g. [4, Theorem 9.12]). An independent set of edges of $G$ is a set of edges no two of which have a common end-vertex. The edge independence number $\beta_e(G)$ is the maximum cardinality among the independent sets of edges of $G$. It is also well known that $\alpha(G) \geq \beta_e(G)$ for all graphs, and that equality holds for bipartite graphs. The latter result is known as König's theorem (see e.g. [4, Theorem 9.13]).

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Vertex covers and eternal dominating sets

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**A B S T R A C T**

The eternal domination problem requires a graph to be protected against an infinitely long sequence of attacks on vertices by guards located at vertices, the configuration of guards inducing a dominating set at all times. An attack at a vertex with no guard is defended by sending a guard from a neighboring vertex to the attacked vertex. We allow any number of guards to move to neighboring vertices at the same time in response to an attack. We compare the eternal domination number with the vertex cover number of a graph. One of our main results is that the eternal domination number is less than the vertex cover number of any graph of minimum degree at least two having girth at least nine.

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1. Introduction

Let \( G = (V, E) \) be a graph with \( n \) vertices and minimum degree \( \delta(G) \). Several recent papers have considered problems associated with using mobile guards to defend \( G \) against an infinite sequence of attacks; see for instance [1,2,4-6,9-11,13].

Denote the open and closed neighborhoods of a vertex \( x \in V \) by \( N(x) \) and \( N[x] \), respectively. That is, \( N(x) = \{v \in V : uv \in E\} \) and \( N[x] = N(x) \cup \{x\} \).

A dominating set of \( G \) is a set \( D \subseteq V \) with the property that for each \( u \in V - D \), there exists \( x \in D \) adjacent to \( u \). A dominating set \( D \) is called a connected dominating set if the subgraph \( G \) induced by \( D \) is connected. The minimum cardinality amongst all dominating sets of \( G \) is the domination number \( \gamma(G) \), while the minimum cardinality amongst all connected dominating sets is the connected domination number \( \gamma_c(G) \). An excellent treatment of domination theory can be found in [8].

A vertex cover of \( G \) is a set \( C \subseteq V \) such that for each edge \( u \in E \), at least one of \( u \) and \( v \) is in \( C \). Let \( \alpha(G) \) denote the vertex cover number of \( G \), the size of a minimum vertex cover of \( G \). An independent set of \( G \) is a set \( I \subseteq V \) with the property that no two vertices in \( I \) are adjacent. The maximum cardinality amongst all independent sets is the independence number \( \beta(G) \). It is well known that \( n - \beta(G) = \alpha(G) \) for all graphs \( G \) (see e.g. [3, Theorem 9.12]).

An independent set of edges of \( G \) is a set of edges, no two of which have a common end-vertex. The edge independence number \( \beta_1(G) \) is the maximum cardinality among the independent sets of edges of \( G \). It is well known that \( \alpha(G) \geq \beta_1(G) \) for all graphs \( G \), and that equality holds for bipartite graphs. The latter result is known as König's theorem (see e.g. [3, Theorem 9.13]).

Let \( D_i \subseteq V, i \geq 1 \), be a set of vertices with a guard located on each vertex of \( D_i \). In this paper we allow at most one guard to be located on a vertex. The set \( D_i \) is also called a configuration of guards. The problems we study can be modeled as two-player games between a defender and an attacker: the defender chooses each \( D_i, i \geq 1 \), while the attacker chooses the locations of the attacks \( r_1, r_2, \ldots \), depending on the configuration \( D_1, D_2, \ldots \) of guards. Each attack \( r_i \) is handled by the defender by choosing the next \( D_i \) subject to some constraints that depend on the particular game (see below). The defender wins the game if they can successfully defend against any series of attacks, the attacker wins otherwise.

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