Interest and Equivalence

A. **Interest** is paid to the supplier of capital for the use of money. The interest rate, \( i \), is established based on the risk the supplier takes in making an investment.

B. **Simple Interest** is used for bank loans, mortgage loans, coupon and registered bonds and other investments where the interest is paid out and not reinvested at the end of each payment period.

C. **Compound Interest** is used for most other investments than those listed in B. Any interest not paid out at the end of a payment period is added to the capital investment (principal) to earn interest during the succeeding period.

D. **Cash Flow (vs. time) Diagrams**: Sign convention - Revenues are generally positive and costs are generally negative in sign. Sometimes both revenues and expenditures are shown and sometimes the yearly or monthly cash flows are netted in order to calculate internal rates of return as discussed in Chapter 7. If only equivalence is being demonstrated, the cash flows would all be of the same sign. This is implied in the equations in the following lesson.

E. **Cash Flow Equivalence** is illustrated in Tables 3.1 and 3.2 using various payment schedules for repaying a $5,000 loan in five years with interest at 8%. The cash flow diagrams are from the borrower's point of view.

**Plan 1:**
Pay an amount equal to the principal/n plus the interest due at the end of each payment period (simple interest).

Plan 1 is typical of bank loans.

**Plan 2:**
Pay interest due at the end of each payment period and the principal at the end of the loan term (simple interest).

Plan 2 is typical for coupon bonds, registered bonds, international loans.
Plan 3:
Pay equal amounts at the end of each payment period for the term of the loan (simple interest).

Plan 3 is typical of house and auto loans when the payment periods are stated in months.

Plan 4:
Borrower pays principal and accumulated interest in one payment at the end of the loan term (compound interest).

Plan 4 is typical of bank Certificates of Deposit (CD's) and IRA's.

Equivalence
When we are indifferent as to whether we have a quantity of money now or the assurance of some other sum of money in the future, or series of future sums of money, we say that the present sum of money is equivalent to the future sum or series of future sums.

If a firm believed 8% was an appropriate interest rate, it would have no particular preference whether it received $5000 now or was repaid by Plan 1 of Table 3-1. Thus $5000 today is equivalent to the series of five end of year payments. In the same fashion, the industrial firm would accept repayment Plan 2 as equivalent to $5000 now. Logic tells us that if Plan 1 is equivalent to $5000 now and Plan 2 is also equivalent to $5000 now, it must follow that Plan 1 is equivalent to Plan 2. In fact, all four repayment plans must be equivalent to each other and to $5000 now.
G. Single Payment Compound Interest Formulas  
(see inside front cover and yellow section tables)  
Note: All the formulas calculate absolute values based on equivalence. The sign convention 
used in the cash flow diagram depends on the viewpoint of the borrower or lender of money.

1. Single Payment Compound Amount Factor  
(Compund interest)  
\((F/P, i, n) = (1+i)^n = F/P\)  
thus \(F = P(F/P, i, n)\)

2. Single Payment Present Worth Factor  
(Compound interest)  
\((P/F, i, n) = 1/(1+i)^n = P/F\)

Notation:  
- \(i\) = interest rate per payment period  
- \(n\) = number of payment periods  
- \(P\) = present value of a sum of money  
- \(F_n\) = Future value of a sum of money in year \(n\)  
- \(A\) = Uniform end-of-period payment in a uniform series of \(n\) payments

Note: Three of the five variables must be known to solve time value of money problems!
Engineering Economic Analysis - Newnan, Lavelle, and Eschenbach Chapter 4
More Interest Formulas

A. Uniform Series Formulas

Conventions for uniform series payments:
   a. A occurs at the end of each period.
   b. P occurs one payment period before the first A.
   c. F occurs at the same time as the last A, and N periods after P.

1. Uniform Series Compound Amount Factor
   (Compound interest)
   \[
   (F/A, i, n) = [(1+i)^n - 1]/i
   \]
   \[= \frac{F}{A}\]

2. Uniform Series Sinking Fund Factor
   (Compound interest)
   \[
   (A/F, i, n) = \frac{i}{[(1+i)^n - 1]} = \frac{A}{F}
   \]

3. Uniform Series Capital Recovery Factor
   (Simple interest)
   \[
   (A/P, i, n) = \frac{i(1+i)^n}{[(1+i)^n - 1]} = \frac{A}{P} = \frac{A/F \times F/P}{A/P}
   \]

4. Uniform Series Present Worth Factor
   (Simple interest)
   \[
   (P/A, i, n) = \left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = \frac{P}{A}
   \]
   Note that: \((A/P, i, n) = (A/F, i, n)\) + i

B. Deferred Annuities (Uniform Series) - If the number of annuity payments is less than the number of analysis periods, the equivalent worth at the beginning or end of the annuity can be calculated and that value moved to a different location on the cash flow diagram.
For example, to find $P_0$ given $A$ at the end of years 5-9,

$$P_4 = A(P/A, i, 5) = F_4$$

$$P_0 = F_4(P/F, i, 4) = A(P/A, i, 5)(P/F, i, 4)$$

C. **Situations where $N$ is unknown** in uniform series calculations

Rearrange the equation to the form $(1+i)^N = ( )$. Take the logarithm of both sides and solve for $n$.

D. **Situations where $i$ is unknown** in uniform series calculations

To determine $i$, given $P$, $A$, and $n$, (or $F$, $A$, and $n$), rearrange the appropriate equation to the following forms and iterate.

$$i = \left[ A/P \right] \left[ \frac{(1+i)^n - 1}{(1+i)^n} \right] \quad i = \left( \frac{F}{A} + 1 \right)^{1/n} - 1$$

assume an $i$ and iterate \hspace{1cm} assume an $i$ and iterate

- The spreadsheet financial function @IRATE(Term, Pmt, PV) solves this situation efficiently.

E. **Arithmetic Gradient Series**

1. **Arithmetic Gradient Present Worth Factor**

$$(P/G, i, n) = \left[ (1+i)^n - 1 \right] / [i(1+i)^n] = P/G$$

2. **Arithmetic Gradient Uniform Series**

$$(A/G, i, n) = \left[ (1+i)^n - 1 \right] / [i(1+i)^n - i] = A/G$$

note that $(P/G, i, n) = [(P/A, i, n) - n(P/F, i, n)] / i$
F. **Geometric Gradients** (very complex unless programmed on a computer) will not be used as part of the normal course material, but may be appropriate for some study projects, especially those involving inflation. Geometric gradients are easily handled in spreadsheet analysis.

G. **Nominal and Effective Interest Rates**

- **Nominal Interest Rate per year,** \( r \), is the annual interest rate without considering the effect of more frequent compounding. The nominal rate is sometimes called the "stated rate."

- **Effective Interest Rate per year,** \( i_{eff} \), is the annual interest rate taking into account more frequent compounding (also called "yield"). Note: The interest rate per payment subperiod, \( i \) or \( i_m \), is an effective interest rate per subperiod.

\[
\text{effective} = (1 + \frac{r}{m})^m - 1
\]

\( m = \text{number of compounding subperiods/year} \)

\( i = \frac{i_m}{m} = \text{interest rate per compounding subperiod} \)

- "**Annual Percentage Rate**" (APR) is a term which the truth-in-lending laws require banks and other installment lenders to report to the consumer. In advertising, APR sometimes refers to nominal interest rates and sometimes refers to effective interest rates, depending on whose point of view is favored.

- **Table 4-1**, p125 compares nominal and effective interest rates for several compounding subperiods.

<table>
<thead>
<tr>
<th>Nominal interest rate per year</th>
<th>Effective interest rate per year, ( i_e ) when nominal rate is compounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>Yearly</td>
</tr>
<tr>
<td>1%</td>
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<td>15.000</td>
</tr>
</tbody>
</table>
H. Interest problems with uniform cash flows less often than compounding periods. Either
a) convert the uniform cash flow to an equivalent more frequent uniform cash flow, or
b) use the effective interest rate between cash flows as the basis for compounding.

I. Interest problems with uniform cash flows occurring more often than the compounding
periods: Use simple interest between compounding periods.

J. Continuous Compounding
Continuous compounding formulas are given on pp129-136. They will not be emphasized in
most Engineering Economics courses as daily compounding is generally the most frequent
compounding used by the banking system, i.e., checks and deposits are cleared once per day.
ATM machines could be programmed to compound continuously, but the cash flows are
almost never uniform, so the equations are of little use.