1. Division Algorithm

In what follows, the numbers \{0, \pm 1, \pm 2, \pm 3, \ldots\} will be referred to as **integers**. The numbers \{1, 2, 3, \ldots\} will be referred to as **positive integers**. The numbers \{0, 1, 2, 3, \ldots\} will be referred to as **nonnegative integers**.

**Division Algorithm:** Let \(a\) be an integer and let \(b\) be a positive integer. When we divide \(a\) by \(b\) (using long division), we get exactly one integer \(q\), called the **quotient**, and exactly one non-negative integer \(r\), called the **remainder**.

**Note:** The remainder \(r\) must be strictly less than \(b\).

**Example:**
(i) Let us divide \(a = 21\) by \(b = 3\). We get quotient \(q = 7\) and remainder \(r = 0\).
(ii) Let us divide \(a = 6\) by \(b = 3\). We get quotient \(q = 2\) and remainder \(r = 0\).
(iii) Let us divide \(a = 17\) by \(b = 3\). We get quotient \(q = 5\) and remainder \(r = 2\).
(iv) Let us divide \(a = 31\) by \(b = 3\). We get quotient \(q = 10\) and remainder \(r = 1\).

**Exercise:** Compute the quotient \(q\) and remainder \(r\) when \(a\) is divided by \(b\) in the following examples:
(i) \(a = 28\) and \(b = 5\).
(ii) \(a = 237\) and \(b = 11\).
(iii) \(a = 1134\) and \(b = 19\).

**Example:**
(i) What are all possible values of the remainder \(r\) when an integer \(a\) is divided by \(b = 3\)? Answer: All possible values of the remainder are \(r = 0, 1, 2\). (Look at the note after the description of division algorithm.)
(ii) What are all possible values of the remainder \(r\) when an integer \(a\) is divided by \(b = 5\)? Answer: All possible values of the remainder are \(r = 0, 1, 2, 3, 4\). (Look at the note after the description of division algorithm.)
(iii) What are all possible values of the remainder \(r\) when an integer \(a\) is divided by a positive integer \(b\)? Answer: All possible values of
the remainder are \( r = 0, 1, 2, \ldots, b - 1 \). (Look at the note after the description of division algorithm.)

**Exercise:** What are all possible values of the remainder \( r \) when an integer \( a \) is divided by \( b = 10 \)?

### 2. Operation “mod”

**Definition of “mod”**: Let \( a \) be an integer and let \( b \) be a positive integer. We define \( a \mod b \) as follows: \( a \mod b \) is the remainder that we get when we divide \( a \) by \( b \).

**Example:**

(i) \( 21 \mod 3 = 0 \) (when we divide \( a = 21 \) by \( b = 3 \) we get the remainder \( r = 0 \)).
(ii) \( 6 \mod 3 = 0 \) (when we divide \( a = 6 \) by \( b = 3 \) we get the remainder \( r = 0 \)).
(iii) \( 29 \mod 5 = 3 \) (when we divide \( a = 29 \) by \( b = 5 \) we get the remainder \( r = 4 \)).
(iv) \( 237 \mod 11 = 6 \) (when we divide \( a = 237 \) by \( b = 11 \) we get the remainder \( r = 6 \)).
(v) \( 2 \mod 15 = 2 \) (Be careful here: when we divide \( a = 2 \) by \( b = 15 \) we get the quotient \( q = 0 \) and the remainder \( r = 2 \)).

**Exercise:** Compute (i) \( 134 \mod 11 \), (ii) \( 535 \mod 7 \), (iii) \( 34 \mod 2 \), (iv) \( 5 \mod 12 \).

### 3. Clock problem

Let’s assume we have a 12-hour clock and it’s 5’oclock now. What will be the time on the clock in 321 hours?

The problem looks somewhat difficult. In fact, it is not so. To find the time in 321 hours, we move the hour hand of the clock through 321 spaces. Note that every time we move the hour hand through 12 spaces, we come back to where we started, namely at 5 o’clock. After the hour hand makes 26 complete revolutions, that is, after \( 12 \cdot 26 = 312 \) hours, it still comes to 5 o’clock. Note that \( 321 - 312 = 9 \). So, starting at 5 o’clock, we need to move the hour hand through 9 more spaces, and this will give us 2 o’clock. Thus, after 321 hours, the clock will show 2 o’clock.

There is an easier way to solve the problem. Note that the remainder of division 321 by 12 is 9, i.e. \( 321 \mod 12 = 9 \). Hence, we will need to move the hour hand through nine spaces, and this gives us the answer 2 o’clock.
An even faster calculation goes as follows: $321 + 5 = 326$ and $326 \mod 12 = 2$. This gives the answer 2 o’clock.

**Exercise:** Suppose you have a 12-hour clock. If it is 2 o’clock now, what will be the time in (i) 520 hours? (ii) 720 hours?

**Exercise:** Suppose you have a 24-hour clock. If it is 2 p.m. now, what will be the time in (i) 420 hours? (ii) 720 hours? (Hint: mod 24 here is relevant.)

### 4. Calendar problem

Assume today is Tuesday. What day of the week will it be in 129 days?

For convenience, let’s label Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday by numbers 0, 1, 2, 3, 4, 5, 6 respectively. To find the day 129 days from Tuesday (recall that Tuesday is labeled 2), we first compute $129 + 2 = 131$ and then compute $131 \mod 7 = 5$. We know that 5 means Friday. Thus, the answer is as follows: 129 days from Tuesday will be Friday.

**Exercise:** Mr. Smith left on a Friday for a world tour. He returned home 46 days from the day he departed. On what day did he come back?

**Exercise:** If today is Wednesday, what day will it be in (i) 120 days? (ii) in 210 days?