MAC 2313  HANDOUT 1

Note: These exercises are usually more involved than those from regular homework. They are intended for those who are interested in getting a more in-depth practice with vectors.

1.) Let $AA_1$, $BB_1$, and $CC_1$ be the medians of a triangle $ABC$. Prove that
$$\overrightarrow{AB} \cdot \overrightarrow{CC_1} + \overrightarrow{BC} \cdot \overrightarrow{AA_1} + \overrightarrow{CA} \cdot \overrightarrow{BB_1} = 0.$$ 

2.) Let $AA_1$, $BB_1$, and $CC_1$ be the altitudes of a triangle $ABC$. Denote by $a$, $b$, $c$ the lengths of sides $BC$, $CA$ and $AB$ respectively. Also, denote by $\vec{h}_a$, $\vec{h}_b$, $\vec{h}_c$ the altitude vectors $\overrightarrow{AA_1}$, $\overrightarrow{BB_1}$, and $\overrightarrow{CC_1}$ respectively. Prove that
$$a^2\vec{h}_a + b^2\vec{h}_b + c^2\vec{h}_c = \vec{0}.$$ (Hint: Write $\vec{h}_a$ in terms of $\overrightarrow{AB}$ and the projection of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$. Here you can use projection formula from the text. Do the same for other altitude vectors.)

3.) Suppose that $ABCD$ is a rectangle in $\mathbb{R}^3$. Let $M$ be an arbitrary point in $\mathbb{R}^3$. Prove that
$$\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}.$$ Using this equality, prove that
$$\|\overrightarrow{MA}\|^2 + \|\overrightarrow{MC}\|^2 = \|\overrightarrow{MD}\|^2 + \|\overrightarrow{MB}\|^2.$$

4.) Suppose that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar (i.e. lie in the same plane in $\mathbb{R}^3$). Prove that $\vec{a} + \vec{b} + \vec{c}$, and $\vec{c} + \vec{a}$ are also coplanar.

5.) Suppose that $A$, $B$, and $C$ are three points in $\mathbb{R}^3$ with position vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ respectively. Prove that the area of the triangle $ABC$ is given by
$$\frac{1}{2}\|\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1\|.$$ 

6.) Suppose that we are given two perpendicular vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$. We are also given a scalar $k$. Find a vector $\vec{r}$ satisfying the following system of equations:
$$\vec{a} \times \vec{r} = \vec{b}; \quad \vec{a} \cdot \vec{r} = k\|\vec{a}\|^2.$$ (Hint: Apply $\vec{a} \times$ from the left to both sides of the first equation, and use formulas from exercise 64 (page 751).)
7.) Prove that

\[(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0.\]

(Hint: You may want to use the formula from exercise 60 on page 751 together with exercise 64 on page 751.)