Let $R$ be the ring of continuous functions on $\mathbb{R}$, and let $A = \{f \in R : f(0) = 0\}$. Prove that $A$ is a maximal ideal.

**Proof:** First, we know that $A$ is an ideal, because if $f$ and $g$ are in $A$, then $(f-g)(0) = f(0) - g(0) = 0$, so $f-g \in A$. Also, if $f$ is in $A$ and $h$ in in $R$, then $fh(0) = f(0)h(0) = 0 \cdot h(0) = 0$. Now to show that $A$ is maximal, assume that $B$ is an ideal with $A \subseteq B$. Let $g \in B - A$. This means that $g(0) \neq 0$. We may assume $g(0) > 0$ (if it is not, use $-g(x)$), and since $g(x)$ is continuous, there is a $\delta > 0$ such that $g(x) > 0$ for all $x$ with $|x| \leq \delta$. Now define:

$$f(x) = \begin{cases} 2g(\delta)x/\delta & 0 \leq x \leq \delta \\ 2g(-\delta)x/\delta & -\delta \leq x \leq 0 \\ (g(x) + |g(x)|)/|x| > \delta & |x| > \delta \end{cases}$$

Note that $f(x)$ is in $A$, and that was constructed to make $f(x) + g(x) > 0$ for all $x$. Thus $f + g$ is a unit (its inverse is $1/(f + g)$, which is well-defined), and so there is a unit in $B$. We have seen that this implies that $B = R$. Hence the only ideal containing $A$ is $R$ itself, so $A$ is maximal.

**OR**

Let $f(x) \notin A$. Then $f(0) \neq 0$, and so the constant function $g(x) = f(0)$ is not in $A$. But $f(0) - g(0) = 0$, so the function $f - g$ is in $A$. This means the cosets $f + A$ and $g + A$ are the same. But the coset $g + A$ has an inverse $1/g + A$ (since $g(x)$ is never zero, $1/g(x)$ is well defined). Thus all nonzero elements of $R/A$ are units, which makes $R/A$ a field. By Theorem 14.4, this means that $A$ is maximal.

Find all maximal ideals of $\mathbb{Z}_8 \oplus \mathbb{Z}_{30}$.

**Answer:** The obvious choices: take a maximal ideal of one component and direct product it with the other component. The only maximal ideal of $\mathbb{Z}_8$ is $2\mathbb{Z}_8$, while $\mathbb{Z}_{30}$ has three maximal ideals: $2\mathbb{Z}_{30}$, $3\mathbb{Z}_{30}$, and $5\mathbb{Z}_{30}$. Combined, that gives us four maximal ideals: $2\mathbb{Z}_8 \oplus 2\mathbb{Z}_{30}$, $\mathbb{Z}_8 \oplus 3\mathbb{Z}_{30}$, $\mathbb{Z}_8 \oplus 5\mathbb{Z}_{30}$, and $\mathbb{Z}_8 \oplus 3\mathbb{Z}_{30}$, $\mathbb{Z}_8 \oplus 5\mathbb{Z}_{30}$. Is that it? Yes. If you take any two elements $(a, b)$ and $(c, d)$, then either $a$ and $c$ have a common factor or they don’t. Also $b$ and $d$ have a common factor or they don’t. If there is a common factor, then the two elements are in one of the options listed. If there are no common factors, then we can take a combination that gives a unit.

What about $\mathbb{Z}[i]/\langle 3 + i \rangle$?

**Answer:** Go look at the lattice! I drew a good one for you. The elements of the factor ring are represented by the distinct elements in each parallelogram (like the one colored yellow). But, of course, that’s not how you are going to do the problem...
the ring structure is the same as the ring structure of $\mathbb{Z}_{10}$. Note that this is not a field, so $\langle 3_i \rangle$ is not maximal.

38) Let $I = \langle 2 + 2i \rangle$ in the Gaussian integers. Show that $I$ is not prime. Determine the elements of $\mathbb{Z}[i]/I$ and its characteristic.

**Answer:** Go look at the lattice! Again! It looks good, and you can see the eight elements of the factor ring.

To see its not prime, note that $2 \cdot (1 + i) \in I$. If $2 \in I$, then $2 = (2 + i)(a + bi) = (2a - 2b) + (2a + 2b)i$. But this means $2a = -2b$ and $2a - 2b = 2$, which cannot be solved over the integers. Similarly, $1 + i \notin I$, since the components are odd while all components of the elements of $I$ are even.

Now note that $(2 + 2i)(2 - 2i) = 4$, so $0, 1, 2,$ and $3$ represent distinct cosets. Also, $1 + i$ is in none of these cosets (since $(1 + i) - a$ has an odd imaginary part, and thus not in $I$). Adding our coset representatives, we now have:

$$\{0, 1, 2, 3, 1+i, 2+i, 3+i, 4+i\}$$

Or if you would prefer, since $4 + i - i = 4 \in I$, we could do:

$$\{0, 1, 2, 3, i, 1+i, 2+i, 3+i\}$$

Is that it? Yes. Given any $a + bi$, if $b$ is even, then $(a + bi) - b/2(2 + 2i) = a - b$, and $a - b$ is in the same coset as $0, 1, 2,$ or $3$. If $b$ is odd, then $a + bi - (b - 1)/2(2 + i) = (a - b + 1) + i$, which is in the same coset as $i, 1 + i, 2 + i,$ or $3 + i$.

The characteristic is now easy to work out: the order of $1$ is $4$, so the characteristic is $4$.

Also, note that $(2 + I)(2 + I) = 0 + I$, so the factor ring is not an integral domain (and it shouldn’t be).

46) Let $R$ be a commutative ring, and $A$ an ideal. Define the nilradical of $A$ as $N(A) = \{r \in R : r^n \in A \text{ for some } n > 0\}$. Show that $N(A)$ is an ideal.

**Answer:** Let $a, b \in N(A)$. Then $a^m \in A$ and $b^n \in A$. We can raise $(a-b)$ to the $m-n$ power, use the binomial theorem, and that fact that $A$ is a ring, and we’ll see that $a - b \in N(A)$.

For the product, $(ab)^m = a^mb^m \in A$, since $a^m \in A$.

47) Let $R = \mathbb{Z}_{27}$. Find some nilradicals.

**Answer:** For $\langle 0 \rangle$, we need numbers raised to powers that give zero. Since $3^3 = 0$, we know $\langle 3 \rangle \subseteq N(\langle 0 \rangle)$. If $a$ is not a multiple of 3, then $a^n$ won’t be a multiple of 3, and especially not a multiple of 27. So $N(\langle 0 \rangle) = \langle 3 \rangle$. The same argument shows that the other two nilradicals are also $\langle 3 \rangle$.

56) Show $\mathbb{Z}[i]/(1-i)$ is a field.

**Answer:** Another lattice! Look at it. LOOK AT IT! Man, I really like these Gaussian integer problems, don’t I?

Since $1 - i$ is not a unit, we know $0$ and $1$ represent different cosets. Since $2 = (1-i)(1+i)$, all real integers are in the $0$ or $1$ coset. Finally, if we have $a+bi$, then $a+bi - (-b)(1-i) = a+b$, and so everything is contained in $\langle 1 - i \rangle$ or $1 + \langle 1 - i \rangle$. Hence our factor ring is just $\{0, 1\}$, which is a field.