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Modeling the change in electric potential due to lightning in a sphere

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ABSTRACT
The change in electric potential as a result of lightning in a sphere of radius 1 is evaluated. Eigenfunctions obtained by utilizing spherical Bessel functions are used to evaluate the new potential. The change in the electric potential is a constant along the lightning channel, and it is the same as the pre-flash potential outside the channel. The governing equation for the electric potential is obtained from Maxwell’s equations.

1. Introduction
Lightning is a process in which the electric charge build-up in two charge centers increase and eventually reach a breakdown threshold, leading to a lightning discharge. This forms a lightning channel, it is what we see as the lightning. During this process, the conductivity along the lightning channel becomes very large, therefore causing a change in the electric potential as well. In this study, we obtain a formula for the change in the electric potential immediately after lightning in a sphere of radius 1 with spherical symmetry using spherical bessel functions. To this end, we solve the following system in spherical coordinates:

\[
\begin{align*}
\frac{\partial \nabla^2 \phi}{\partial t} &= -\frac{1}{r^2} (\sigma r^2 \phi)' \\
\phi(0, t) &< \infty \quad t \in [0, \infty), \\
\phi(1, t) &= 0 \quad t \in [0, \infty), \\
\phi(r, 0) &= \phi_0 \quad r \in [0, 1],
\end{align*}
\]

The initial potential \( \phi_0 \) lies in the space

\[
L^2_0([0, 1]) = \left\{ f \in L^2([0, 1]) | \lim_{t \to 0} \int_0^t \frac{1}{s} |f(s)| \, ds = 0 \right\},
\]

where \( L^2([0, 1]) \) is the usual space of square-integrable functions on \([0, 1]\) and \( \sigma > 0 \) lies in the space \( L^\infty([0, 1]) \) of essentially bounded functions on \([0, 1] \).
In the moments after a lightning discharge, the conductivity along the lightning channel becomes infinitely large. In our domain, we assume lightning channel \( L \) is around the origin, \( L = [0, \Delta r] \), with \( L^c = [0, 1] \setminus [0, \Delta r] \). Therefore, we write \( \sigma = \sigma + \tau \Psi \), where \( \Psi \) is the characteristic function of \( L \), that is \( \Psi = 0 \) everywhere except on \( L \), and \( \tau \) is a large scalar. If lightning occurs at time \( t = 0 \), then in the moments after lightning, the electric potential is governed by

\[
\frac{\partial \nabla^2 \phi}{\partial t} = -\frac{1}{r^2}((\sigma + \tau \Psi)r^2 \phi)' \quad (r, t) \in [0, 1] \times [0, \infty),
\]

subject to the boundary conditions (1b)–(1d). If \( \phi^\tau(r, t) \) is the solution to (2), then the potential after the lightning is given by

\[
\phi^+(r) = \lim_{t \to 0^+} \lim_{\tau \to \infty} \phi^\tau(r, t)
\]

assuming lightning is very fast. We show that \( \phi^+ \) can be given by the following theorem.

**Theorem 1.1:**

\[
\phi^+(r) = \int_0^r \phi^{+\prime}(s) \, ds = \begin{cases} 
\phi_0(\Delta r) & \text{if } r \in L, \\
\phi_0(r) & \text{if } r \in L^c,
\end{cases}
\]

where \( \phi^{+\prime}(r) = (1 - \Psi(r))\phi'(r, 0) \).

In calculating the limit, we use functions \( \{\kappa_i\} \) and \( i \geq 1 \), where \( \kappa_i(r) = \sqrt{2}(i\pi r)j_1(i\pi r) \) and \( j_1 \) is the spherical Bessel function of order 1.

**Theorem 1.2:** \( \{\kappa_i\}, i \geq 1, \) is a basis for \( L_0^2([0, 1]) \).

Utilizing this result to write the solution \( \phi \) as an eigenexpansion leads to the following ordinary differential equation

\[
\dot{\beta} = -(S + \tau R)\beta,
\]

which has a unique solution. Using the functions \( \{\kappa_i\} \), we classify the eigenvectors of matrix \( R \) and we obtain another basis for \( L_0^2([0, 1]) \) via the following theorem.

**Theorem 1.3:** \( \{\Gamma_i\} \bigcup \{\Delta_i\}, i \geq 1 \) forms a basis for \( L_0^2([0, 1]) \).

We use the orthonormality properties of these new basis functions as we compute the limit and prove Theorem 1.1.

In 2008, Hager and Aslan [1] developed a continuous model to compute the change in the electric potential. The electric potential in a domain \( \Omega \) is modeled by the equations

\[
\Delta \frac{\partial \phi}{\partial t} = -\nabla \cdot (\sigma \nabla \phi) + \nabla \cdot J, \quad (x, t) \in \Omega \times [0, \infty),
\]

\[
\phi(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, \infty),
\]

\[
\phi(x, 0) = \phi_0(x), \quad x \in \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial \Omega \), the components of forcing term \( J \) lie in \( L^2(\Omega) \), and \( \sigma > 0 \) is as described earlier. They have established the following.

**Theorem 1.4:** If \( \partial \Omega \) is \( C^2 \) and \( \partial L \) is \( C^{2, \alpha} \), for some \( \alpha \in (0, 1) \) (the exponent of Hölder continuity for the second derivative), then the electric potential immediately after the lighting discharge is given by

\[
\phi^+(x) = \begin{cases} 
\phi_L & \text{if } x \in L, \\
\phi_0(x) + \xi(x) & \text{if } x \in L^c,
\end{cases}
\]

where

\[
\phi_L = \frac{\langle \nabla \phi_0, \nabla \Pi \rangle_\Omega}{\langle \nabla \Pi, \nabla \Pi \rangle_\Omega},
\]
and where \( \Pi \) and \( \xi \) are solutions to the boundary value problems

\[
\begin{align*}
\Delta \Pi &= 0 \text{ in } \mathcal{L}, & \Pi &= 0 \text{ on } \partial \Omega, & \Pi &= 1 \text{ in } \mathcal{L}, \\
\Delta \xi &= 0 \text{ in } \mathcal{L}, & \xi &= 0 \text{ on } \partial \Omega, & \xi &= \phi_L - \phi_0 \text{ on } \partial \mathcal{L}.
\end{align*}
\]

Here \( \langle \cdot, \cdot \rangle_{\Omega} \) is the \( L^2(\Omega) \) inner product

\[
\langle \nabla u, \nabla v \rangle_{\Omega} = \int_{\Omega} \nabla u \cdot \nabla v \, dx.
\]

The key to obtaining this result was utilizing the eigenfunctions of the following generalized eigenproblem: Find \( u \in H^1_0(\Omega), u \neq 0, \) and \( \lambda \in \mathbb{R} \) such that

\[
\langle \nabla u, \nabla v \rangle_{\mathcal{L}} = \lambda \langle \nabla u, \nabla v \rangle_{\Omega}
\]

for all \( v \in H^1_0(\Omega). \) Aslan et al. [2] showed that the eigenfunctions of (7) form a complete set and they classified the eigenfunctions in four groups. The function \( \Pi \) in Theorem 1.4 is an eigenfunction of (7) corresponding to eigenvalue 0. It is equal to 1 on the lightning domain and it is a harmonic function on the outside.

The change in the electric potential due to lightning can be used to compute lightning flash energy \( E \) given by the formula

\[
E = \frac{\varepsilon}{2} \int_{\Omega} \left( |\nabla \phi_0(x)|^2 - |\nabla \phi^+|^2 \right) \, dx,
\]

where \( \varepsilon \) is the permittivity of the free space, see [3]. The energy associated with a lightning flash is needed in various studies including climate change. Currently, constant assumptions are usually used or numerical estimates are obtained. Utilization of formula for \( E \) would lead to a more precise estimate for the flash energy.

The paper is organized as follows: In Section 2, we explain how the governing equation for the electric potential is obtained from Maxwell’s equations. In Section 3, we develop the problem in the sphere of radius 1. In Section 4, we obtain a basis for the space \( L^2_0([0,1]) \). In Section 5, we show that the matrices \( S \) and \( R \) are bounded and the ordinary differential equation involving them has a unique solution. In Section 6, we classify eigenvectors of matrix \( R \) and use them to obtain another basis for the space \( L^2_0([0,1]) \). In Section 7, we evaluate the limit and prove Theorem 1.1. Conclusions are given in Section 8.

2. Maxwell equations

The governing equations for the potential are derived from the Maxwell’s equations for linear materials. By Ampere’s law, the curl of the magnetic field strength is given by

\[
\nabla \times H = J_T + \varepsilon \frac{\partial E}{\partial t},
\]

where \( H \) is the magnetic field, \( J_T \) is the total current density, \( E \) is the electric field, and \( \varepsilon \) is the permittivity of the air. The total current density \( J_T \) is partly due to the movement of ice and water in the cloud and partly due to the conductivity of the cloud. Therefore, we can write \( J_T = J_p + \sigma_a E \), where \( \sigma_a \) is the conductivity of the atmosphere. Replacing \( J_T \) and taking the divergence gives

\[
\varepsilon \nabla \cdot \frac{\partial E}{\partial t} = -\nabla \cdot (\sigma_a E) - \nabla \cdot J_p.
\]

(8)
By Faraday’s law of induction,
\[ \nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
where \( \mathbf{B} \) is the magnetic flux density. In our model, we assume the time derivative of \( \mathbf{B} \) can be neglected, therefore obtaining \( \nabla \cdot \mathbf{E} = 0 \). This implies that \( \mathbf{E} = \nabla \phi \) or \( \mathbf{E} = -\nabla \phi \), where \( \phi \) is the electric potential. It is common to assume \( \mathbf{E} = -\nabla \phi \). Substituting this into (8), we obtain
\[ \frac{\partial \nabla^2 \phi}{\partial t} = -\nabla \cdot (\sigma \nabla \phi) + \nabla \cdot \mathbf{J}, \]
where \( \sigma = \sigma_a/\varepsilon \) and \( \mathbf{J} = J_p/\varepsilon \).

3. The problem in a sphere

We consider the problem in a sphere of radius 1 centered at the origin. In addition, we assume \( \mathbf{J} \) can be neglected. Therefore the following equations model our problem:

\[ \frac{\partial \nabla^2 \phi}{\partial t} = -\nabla \cdot (\sigma \nabla \phi), \quad (r, t) \in [0, 1] \times [0, \infty), \]
\[ \phi(0, \theta, \varphi, t) < \infty \quad t \in [0, \infty), \]
\[ \phi(1, \theta, \varphi, t) = 0 \quad t \in [0, \infty), \]
\[ \phi(r, \theta, \varphi, 0) = \phi_0 \quad r \in [0, 1], \]

where \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \varphi \leq \pi \), and \( \sigma > 0 \) lies in the space \( L^\infty([0,1]) \) of essentially bounded functions defined on \([0,1]\). In spherical coordinates, \( \phi = \phi(r, \theta, \varphi) \), we have
\[ \nabla^2 \phi = \phi_{rr} + \frac{2}{r} \phi_r + \frac{1}{r^2 \sin^2 \varphi} \phi_{\theta\theta} + \frac{\cot \varphi}{r^2} \phi_{\varphi\varphi} + \frac{1}{r^2} \phi_{\varphi\varphi}, \]
\[ \nabla \phi = \left( \phi_r, \frac{1}{r} \phi_\varphi, \frac{1}{r \sin \varphi} \phi_\theta \right). \]

Due to spherical symmetry, we have
\[ \nabla^2 \phi = \phi_{rr} + \frac{2}{r} \phi_r, \]
\[ \nabla \phi = (\phi_r, 0, 0). \]

We can also assume \( \phi(r, \theta, \varphi) = \phi(r) \). Therefore we can write
\[ \nabla \cdot (\sigma \nabla \phi) = \frac{1}{r^2} (\sigma r^2 \phi')', \]
and the problem can be rewritten as (1a)–(1d).

Solving the eigenvalue problem for the Laplacian with the boundary conditions (1b)–(1d), we see that \( j_0(i\pi r) \), spherical Bessel functions of order 0, are the eigenfunctions corresponding to the eigenvalues \( \lambda_i = -(i\pi)^2 \), see [4]. The normalized eigenfunctions, \( \{\sqrt{2}(i\pi)j_0(i\pi r)\} \), \( i \leq 1 \), form a complete set in \( L^2([0,1]) \) with weight function \( r^2 \), see [5]. Therefore, we can express the solution to (1a)–(1d) as
\[ \phi(r, t) = \sum_{i=1}^{\infty} \alpha_i(t) \phi_i(r), \quad \text{where} \quad \phi_i(r) = \sqrt{2}(i\pi)j_0(i\pi r). \]
Substituting this eigenexpansion into (1a), we obtain

\[
\sum_{i=1}^{\infty} \alpha_i(t) \nabla^2 \phi_i(r) = \sum_{i=1}^{\infty} \alpha_i(t) \frac{1}{r^2} (\sigma(r) r^2 \phi'_i(r)).
\]

Using the eigenproblem equation, \( \nabla^2 \phi = \lambda \phi \), multiplying both sides by \( \phi_j(r) \), and integrating from 0 to 1, we obtain

\[
\sum_{i=1}^{\infty} \alpha_i(t)(i\pi)^2 \langle r \phi_i(r), r \phi_j(r) \rangle = \sum_{i=1}^{\infty} \alpha_i(t) \langle \sigma(r) r \phi'_i(r), r \phi'_j(r) \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) represents the usual \( L^2 \) inner product on \([0, 1]\). Utilizing the orthonormality of eigenfunctions, this can be written as the linear system

\[
A \dot{\alpha} = -B \alpha, \quad \alpha(0) = \alpha_0,
\]

where

\[
A = \begin{bmatrix} \cdot & (i\pi)^2 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \quad (B)_{ij} = -\langle \sigma(r) r \phi'_i(r), r \phi'_j(r) \rangle.
\]

Note that \( f_0(i\pi r) = -(i\pi) j_1(i\pi r) \), where \( j_1 \) is the spherical Bessel function of order 1. Therefore \( \phi'_i(r) = -\sqrt{2}(i\pi)^2 j_1(i\pi r) \).

In the moments after a lightning discharge, the conductivity along the lightning channel \( \mathcal{L} \) becomes very large. Therefore, we write \( \sigma = \sigma + \tau \Psi \), where \( \Psi \) is the characteristic function of \( \mathcal{L} \), and we let \( \tau \) tends to infinity right after lightning. If lightning occurs at time \( t = 0 \), then in the moments after lightning

\[
(B)_{ij} = -\langle (\sigma(r) + \tau \Psi(r)) r \phi'_i(r), r \phi'_j(r) \rangle
= -\langle \sigma(r) r \phi'_i(r), r \phi'_j(r) \rangle - \langle \Psi(r) r \phi'_i(r), r \phi'_j(r) \rangle
= -\langle \sigma(r) r \phi'_i(r), r \phi'_j(r) \rangle - \langle \tau \phi'_i(r), r \phi'_j(r) \rangle \mathcal{L}
= -(B + \tau W)_{ij},
\]

where \( \langle \cdot, \cdot \rangle \mathcal{L} \) represents the \( L^2 \) inner product over \( \mathcal{L} \), and

\[
(W)_{ij} = \langle r \phi'_i(r), r \phi'_j(r) \rangle \mathcal{L}.
\]

Therefore, (11) becomes

\[
A \dot{\alpha} = -(B + \tau W) \alpha.
\]

By letting \( \beta = A^{\frac{1}{2}} \alpha, S = A^{-\frac{1}{2}} BA^{-\frac{1}{2}}, \) and \( R = A^{-\frac{1}{2}} WA^{-\frac{1}{2}} \), we can rewrite (12) as

\[
\dot{\beta} = -(S + \tau R) \beta.
\]

where

\[
(S)_{ij} = \langle \sigma(r) \kappa_i(r), \kappa_j(r) \rangle, \\
(R)_{ij} = \langle \kappa_i(r), \kappa_j(r) \rangle \mathcal{L}, \\
\kappa_i(r) = \frac{1}{i\pi} r \phi'_i, \quad i \geq 1.
\]
4. The space $L^2_0([0, 1])$

Let $L^2_0([a, b]) = \{ f \in L^2([a, b]) | \lim_{t \to a} \int_a^t \frac{1}{s} |f(s)| \, ds = 0 \}$ and consider the set of functions $\{ \kappa_i \}$, $i \geq 1$. Since $\kappa_i$ is continuous and $\kappa_i(0) = 0$ for $i \geq 1$, $\kappa_i \in L^2_0([0, 1])$. We now prove Theorem 1.2.

By the definition of $\kappa_i(r)$, we have

$$\kappa_i(r) = -\frac{1}{i\pi} r \phi_i'(r) = \sqrt{2}(i\pi r) j_1(i\pi r).$$

By using Mathematica, we obtain

$$\int \kappa_i^2(r) \, dr = \frac{-2 + 2i^2\pi^2 r^2 + 2 \cos(2i\pi r) + i\pi r \sin(2i\pi r)}{2i^2\pi^2 r},$$

and by evaluating the integral from 0 to 1 we obtain

$$\| \kappa_i(r) \|_2 = 1$$

Similarly, we have

$$\int \kappa_i(r) \kappa_j(r) \, dr = \frac{2i^2j\pi r \cos(j\pi r) \sin(i\pi r) - 2(ij^2\pi r \cos(i\pi r) + (i^2 - j^2) \sin(i\pi r)) \sin(j\pi r)}{ij(i^2 - j^2)\pi^2 r},$$

and therefore

$$\langle \kappa_i(r), \kappa_j(r) \rangle = 0.$$

Hence $\{ \kappa_i \}, i \geq 1$, forms an orthonormal set in $L^2([0, 1])$.

Now, let $f \in L^2_0([0, 1])$, then by Evans [6], Section 5.3, there exist functions $u_n \in C^\infty([0, 1])$ such that $u_n \to f$ in $L^2_0([0, 1])$. Choose smooth function $\zeta_N$ satisfying

$$0 \leq \zeta_N \leq 1, \quad \text{and} \quad \zeta_N = \begin{cases} 0 & \text{at } 0, \\ 1 & \text{on } [\frac{1}{N}, 1]. \end{cases}$$

Then $\zeta_N u_n$ is a smooth function such that

$$0 \leq \zeta_N u_n < u_n \text{ on } [0, 1] \setminus \left[ \frac{1}{N}, 1 \right], \quad \text{and} \quad \zeta_N u_n = \begin{cases} 0 & \text{at } 0, \\ u_n & \text{on } [\frac{1}{N}, 1]. \end{cases}$$

We can write

$$\| f - \zeta_N u_n \|^2 \leq \| f - u_n \|^2 + \| u_n - \zeta_N u_n \|^2$$

$$\leq \| f - u_n \|^2 + \int_0^{\frac{1}{N}} u_n^2(r) \, dr$$

$$\leq \| f - u_n \|^2 + \int_0^{\frac{1}{N}} (u_n(r) - f(r))^2(r) \, dr + \int_0^{\frac{1}{N}} f^2(r) \, dr$$

$$+ 2 \left( \int_0^{\frac{1}{N}} (u_n(r) - f(r))^2(r) \, dr \right)^{1/2} \left( \int_0^{\frac{1}{N}} f^2(r) \, dr \right)^{1/2}.$$

Since $u_n$ converges to $f$, for given $\epsilon > 0$, there exists $M > 0$ such that

$$\| f - u_n \|^2 \leq \frac{\epsilon}{2}, \quad \text{whenever} \quad n \geq M. \quad (14)$$
Also, for fixed $N$, 

$$
\int_0^1 \left( u_n(r) - f(r) \right)^2 \, dr \leq \int_0^1 \left( u_n(r) - f(r) \right)^2 \, dr \leq \frac{\epsilon}{2}. \tag{15}
$$

On the other hand, by using Lebesgue measure properties, since $f^2 \in L^2([0, 1])$ we have

$$
\lim_{N \to \infty} \int_0^1 f^2(r) \, dr = 0. \tag{16}
$$

Combining (14), (15), and (16), we obtain

$$
\| f - \zeta N u_n \|_2^2 \leq \epsilon, \quad \text{for } n \geq M \text{ and } N \to \infty.
$$

For convenience, let $f_n := \lim_{N \to \infty} \zeta N u_n$. Then, $f_n$ is a smooth function converging to $f$ in $L^2([0, 1])$. Let

$$
g_n(r) = \int_1^r \frac{1}{s} f_n(s) \, ds \quad \text{with} \quad g_n'(r) = \frac{1}{r} f_n(r).
$$

Since $\frac{1}{r} f_n(r)$ is continuous, $g_n \in C^1([0, 1])$, and since $g_n(1) = 0$ and since $\{\phi_i\}$ forms a complete set, we obtain the following:

$$
g_n(r) = \sum_{i=1}^{\infty} a_i^* \phi_i(r).
$$

Differentiating, we obtain

$$
g_n'(r) = \frac{1}{r} f_n(r) = \sum_{i=1}^{\infty} a_i^* \phi_i'(r),
$$

which gives

$$
rg_n'(r) = f_n(r) = \sum_{i=1}^{\infty} a_i^* r \phi_i'(r) = \sum_{i=1}^{\infty} a_i^* \kappa_i(r).
$$

Since $f_n$ converges to $f$ in $L^2([0, 1])$, for given $\epsilon > 0$, there exists $n \geq N$ such that

$$
\int_0^1 |f(r) - f_n(r)|^2 \, dr < \frac{\epsilon}{4}.
$$

Let’s choose $N_0$ such that

$$
\int_0^1 |f(r) - f_{N_0}(r)|^2 \, dr < \frac{\epsilon}{4}, \tag{17}
$$

where

$$
f_{N_0} = \sum_{i=1}^{\infty} a_i^{N_0} \kappa_i(r).
$$

Now consider

$$
s_m(r) = \sum_{i=1}^{m} a_i^{N_0} \kappa_i(r).
$$

Since this series is convergent, there exists $M > 0$ so that for $m \geq M$, we have

$$
\int_0^1 \left| f_{N_0}(r) - s_m(r) \right|^2 \, dr = \int_0^1 \left| \sum_{i=m+1}^{\infty} a_i^{N_0} \kappa_i(r) \right| \, dr < \frac{\epsilon}{4}. \tag{18}
$$
Combining (17) and (18), for \( m \geq M \) we have

\[
\int_0^1 |f(r) - sm(r)|^2 \, dr \leq \int_0^1 (|f(r) - f_{N_0}(r)| + |f_{N_0}(r) - s_m(r)|)^2 \, dr \\
\leq 2 \int_0^1 |f(r) - f_{N_0}(r)|^2 \, dr + 2 \int_0^1 |f_{N_0}(r) - s_m(r)|^2 \, dr \\
\leq \epsilon.
\]

Therefore, any function \( f \in L^2_0([0, 1]) \) can be written as

\[
f(r) = \sum_{i=1}^{\infty} \alpha_i \kappa_i(r),
\]

and hence \( \{\kappa_i\}, i \geq 1 \) forms a basis for \( L^2_0([0, 1]) \).

5. Existence and uniqueness

The main goal of this section includes establishing the existence and uniqueness of the solution to

\[
\dot{\beta} = -(S + \tau R) \beta.
\]

We first prove the following Lemma.

Lemma 5.1: The matrices \( S \) and \( R \) are both positive semi-definite. Moreover,

\[
\|S\|_2 \leq \|\sigma\|_{\infty}, \quad \text{and} \quad \|S\|_2 \leq 1.
\]

Proof: Let \( \ell^2 \) be the usual space of square summable vectors with the norm

\[
\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}.
\]

By the definition of \( S \) and using \( \sigma \geq 0 \), for all \( x \in \ell^2 \) we have

\[
x^T S x = \left( \sigma \sum_{i=1}^{\infty} x_i \kappa_i, \sum_{i=1}^{\infty} x_i \kappa_i \right) \geq 0.
\]

Therefore, \( S \) is positive semi-definite. Then using Cauchy–Schwarz inequality, we obtain

\[
|x^T S x| \leq \sqrt{x^T S y} \sqrt{x^T S x}
\]

for all \( x \) and \( y \) in \( \ell^2 \). On the other hand,

\[
x^T S x = \langle \sigma(r) x_i \kappa_i(r), x_i \kappa_i(r) \rangle \leq \|\sigma\|_{\infty} \langle x_i \kappa_i(r), x_i \kappa_i(r) \rangle = \|\sigma\|_{\infty}
\]

if \( \|x\| = 1 \). Therefore for \( \|x\| = 1 \) and \( \|y\| = 1 \), we have

\[
\|S\| = \max \max (y^T S x) \leq \|\sigma\|_{\infty}
\]

by combining (19) and (20). Similarly \( R \) is positive semi-definite and
\[ \| \mathbf{R} \| = \max \max (y^T \mathbf{R} x) \leq 1. \]

**Theorem 5.2:** The Equation (13) has a unique solution \( \beta(\cdot): [0, \infty) \rightarrow \ell^2 \) with \( \beta(\cdot) \) and \( \dot{\beta}(\cdot) \) continuous on \([0, \infty)\).

**Proof:** Since \( \| S \| \) and \( \| R \| \) are both bounded, by Curtain and Zwart [7], Corollary 2.2.3, \(-(S + \tau R)\) is the infinitesimal generator for a strongly continuous semigroup \( e^{-(S + \tau R)} \) on \( \ell^2 \). Therefore by Adams [8], Theorem 7.10, the Equation (13) has a unique continuous solution with a continuous derivative on \([0, \infty)\).

\[ \square \]

### 6. Eigenfunctions

In this section, we will classify the eigenvalues and eigenvectors of matrix \( \mathbf{R} \) and prove Theorem 1.3. Let \( \{ \Gamma_i \}, i \geq 1 \), be an orthonormal basis for \( L^2_0(\mathcal{L}) \) with \( \Gamma_i(r) = 0 \) on \( \mathcal{L}^c \), where \( \mathcal{L} = [0, \Delta r] \) and \( \mathcal{L}^c = [0, 1] \setminus \mathcal{L} \). We can expand each of these functions using the basis \( \{ \kappa_i \} \) to obtain

\[ \Gamma_i(r) = \sum_{j=1}^{\infty} \langle \Gamma_i(r), \kappa_j(r) \rangle \kappa_j(r) = \sum_{j=1}^{\infty} \langle \Gamma_i(r), \kappa_j(r) \rangle \mathcal{L} \kappa_j(r). \]

Let \( \mathbf{G} \) be the matrix with entries \( G_{ij} = \langle \Gamma_i(r), \kappa_j(r) \rangle _\mathcal{L} \). Then, we can write \( \Gamma_i = \mathbf{G}_i^T \kappa \) or \( 0 = \mathbf{G}^T \kappa \).

**Proposition 6.1:** \( G_i \) is an eigenvector of the matrix \( \mathbf{R} \) corresponding to the eigenvalue 1.

**Proof:** Using the eigenvector relation we obtain

\[ (\mathbf{R} \mathbf{G}_i)_j = \left( \sum_{k=1}^{\infty} \langle \Gamma_i(r), \kappa_k(r) \rangle _\mathcal{L} \kappa_k(r), \kappa_j(r) \right)_{\mathcal{L}} = \langle \Gamma_i(r), \kappa_j(r) \rangle _\mathcal{L} = \langle G_i)_j, \]

and therefore \( G_i \) is an eigenvector or matrix \( \mathbf{R} \) corresponding to eigenvalue 1. Hence for each \( \Gamma_i(r) \), we obtain an eigenvector of \( \mathbf{R} \).

Now let \( \{ \Delta_i \}, i \geq 1 \), be an orthonormal basis for \( L^2_0(\mathcal{L}^c) \) with \( \Delta_i(r) = 0 \) on \( \mathcal{L} \). Expand each of these functions using the basis \( \{ \kappa_i \} \), we obtain

\[ \Delta_i(r) = \sum_{j=1}^{\infty} \langle \Delta_i(r), \kappa_j(r) \rangle \kappa_j(r) = \sum_{j=1}^{\infty} \langle \Delta_i(r), \kappa_j(r) \rangle _\mathcal{L} \kappa_j(r). \]

Let \( \mathbf{D} \) be the matrix with entries \( D_{ij} = \langle \Delta_i(r), \kappa_j(r) \rangle _\mathcal{L} \). Then we can write \( \Delta_i = \mathbf{D}_i^T \kappa \) or \( 1 = \mathbf{D}^T \kappa \).

**Proposition 6.2:** \( D_i \) is an eigenvector of the matrix \( \mathbf{R} \) corresponding to the eigenvalue 0.

**Proof:** The eigenvector relation grants us the following:

\[ (\mathbf{R} \mathbf{D}_i)_j = \left( \sum_{k=1}^{\infty} \langle \Delta_i(r), \kappa_k(r) \rangle _\mathcal{L} \kappa_k(r), \kappa_j(r) \right)_{\mathcal{L}} = \langle \Delta_i(r), \kappa_j(r) \rangle _\mathcal{L} = 0 \]
since $\Delta_i = 0$ on $\mathcal{L}$. Therefore, $D_i$ is an eigenvector or matrix $R$ corresponding to eigenvalue 0. Hence, for each $\Delta_i(r)$, we obtain an eigenvector of $R$.

Now we prove Theorem 1.3. Let $f \in L^2_0([0, 1])$, and therefore $f(0) = 0$. We can write $f = f_1 + f_2$ where $f_1(r) = 0$ on $\mathcal{L}^c$ and $f_2(r) = 0$ on $\mathcal{L}$. Therefore $f_2(0) = 0$ and $f_2 \in L^1_0([0, 1])$. This gives

$$f_1(0) = f_1(0) + f_2(0) = f(0) = 0,$$

and hence $f_1 \in L^2_0([0, 1])$. Therefore, $f_1(r)$ and $f_2(r)$ can be written as a linear combination of the basis functions $\Gamma_i(r)$ and $\Delta_i(r)$, respectively. We obtain

$$f(r) = \sum_{i=1}^{\infty} \gamma_i \Gamma_i(r) + \sum_{i=1}^{\infty} \delta_i \Delta_i(r),$$

where $\gamma_i$ and $\delta_i$ are constants for $i \geq 1$, showing that any function in $L^2_0([0, 1])$ can be written as a linear combination of $\{\Gamma_i\} \bigcup \{\Delta_i\}$, $i \geq 1$. On the other hand, to show linear independence, consider

$$\sum_{i=1}^{\infty} \gamma_i \Gamma_i(r) + \sum_{i=1}^{\infty} \delta_i \Delta_i(r) = 0.$$

Then we have

$$\sum_{i=1}^{\infty} \gamma_i \Gamma_i(r) = 0 \text{ on } \mathcal{L} \quad \text{and} \quad \sum_{i=1}^{\infty} \delta_i \Delta_i(r) = 0 \text{ on } \mathcal{L}^c.$$ 

Since $\Gamma_i(r)$ is a linearly independent set on $\mathcal{L}$ and $\Delta_i$ is a linearly independent set on $\mathcal{L}^c$, we obtain $\gamma_i = 0$ and $\delta_i = 0$ for $i \geq 1$. Therefore, $\{\Gamma_i\} \bigcup \{\Delta_i\}$, $i \geq 1$ is linearly independent set. Hence $\{\Gamma_i\} \bigcup \{\Delta_i\}$, $i \geq 1$ forms a basis for $L^2_0([0, 1])$.

### 7. Potential change via Bessel functions

In this section we prove Theorem 1.1. Using (10), we have

$$\phi(r, t) = \sum_{i=1}^{\infty} \alpha_i(t) \phi_i(r).$$

Differentiating with respect to $r$ and multiplying by $r$, we obtain

$$-r \phi'(r, t) = -\sum_{i=1}^{\infty} \alpha_i(t) r \phi_i'(r) = \sum_{i=1}^{\infty} \beta_i(t) \kappa_i(r) = \kappa^T(r) \beta(t),$$

where $\beta(t)$ is the solution to (13). Since $\kappa_i(0) = 0$, $-r \phi'(r, t) = 0$, and therefore $-r \phi'(r, t) \in L^2_0([0, 1])$. Then, using Theorem 1.3, we can write

$$- r \phi'(r, t) = \sum_{i=1}^{\infty} \gamma_i(t) \Gamma_i(r) + \sum_{i=1}^{\infty} \delta_i(t) \Delta_i(r)$$

Combining (22c) with (21), we obtain

$$\beta(t) = G \gamma(t) + D \delta(t).$$
Taking the time derivative of (23) and combining with (13) yields

$$\dot{\beta}(t) = -S\beta(t) - \tau R\beta(t)$$

$$= -S\beta(t) - \tau (RG\gamma(t) + RD\delta(t))$$

$$= -S\beta(t) - \tau G\gamma(t),$$

since the columns of $G$ are eigenvectors of $R$ corresponding to eigenvalue 1 and the columns of $D$ are eigenvectors of $R$ corresponding to eigenvalue 0. By substituting the derivative of (23), we obtain the following system of differential equations:

$$\begin{bmatrix} G & D \end{bmatrix} \begin{bmatrix} \dot{\gamma}(t) \\ \dot{\delta}(t) \end{bmatrix} = -S \begin{bmatrix} G & D \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \delta(t) \end{bmatrix} - \tau \begin{bmatrix} G \\ 0 \end{bmatrix} \begin{bmatrix} \gamma(t) \\ 0 \end{bmatrix}$$

(24)

**Lemma 7.1:** $[G \ D]$ is an orthonormal matrix.

**Proof:** Combining (21) and (22a), we obtain

$$\sum_{j=1}^{\infty} \gamma_j(t)\Gamma_j(r) + \sum_{j=1}^{\infty} \delta_j(t)\Delta_j(r) = \sum_{j=1}^{\infty} \beta_j(t)\kappa_j(r).$$

Taking the inner product of both sides with $\Gamma_i(r)$ over $[0,1]$, and utilizing the orthonormality of $\{\Gamma_i\} \cup \{\Delta_i\}, i \geq 1$ gives

$$\gamma_i(t) = \sum_{j=1}^{\infty} \beta_j(t)\langle \Gamma_i, \kappa_j(r) \rangle = \sum_{j=1}^{\infty} \beta_j(t)(G_i)j = G_i^T \beta(t),$$

and therefore

$$\gamma(t) = G^T \beta(t).$$

Similarly,

$$\delta(t) = D^T \beta(t).$$

Therefore we can write

$$\begin{bmatrix} \gamma(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} G^T \\ D^T \end{bmatrix} \beta(t).$$

Combining the last equality with the matrix form of (23),

$$\beta(t) = \begin{bmatrix} G & D \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \delta(t) \end{bmatrix},$$

shows $[G \ D]$ is an orthonormal matrix.

Now, we multiply (24) on the left by $[G \ D]^T$ to obtain

$$\begin{bmatrix} \dot{\gamma}(t) \\ \dot{\delta}(t) \end{bmatrix} = -\begin{bmatrix} G^TSG & G^TSD \\ D^TSG & D^TSD \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \delta(t) \end{bmatrix} - \tau \begin{bmatrix} \gamma(t) \\ 0 \end{bmatrix}.$$

Since matrix $G$ is supported on $\mathcal{L}$ and matrix $D$ is supported on $\mathcal{L}^c$, this becomes

$$\begin{bmatrix} \dot{\gamma}(t) \\ \dot{\delta}(t) \end{bmatrix} = -\begin{bmatrix} G^TSG & 0 \\ 0 & D^TSD \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \delta(t) \end{bmatrix} - \tau \begin{bmatrix} \gamma(t) \\ 0 \end{bmatrix}.$$

(25)
Let \( \mathbf{F} = \begin{bmatrix} \mathbf{G} & \mathbf{D} \end{bmatrix} \) and \( \mathbf{z} = \begin{bmatrix} y(t) & \mathbf{d}(t) \end{bmatrix} \), and multiply both sides by \( \mathbf{z}^T \) on the left to get

\[
\mathbf{y}^T \dot{\mathbf{y}} + \mathbf{d}^T \dot{\mathbf{d}} = - (\mathbf{F} \mathbf{z})^T \mathbf{S} (\mathbf{F} \mathbf{z}) - \tau \begin{bmatrix} \mathbf{y} & \mathbf{d} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}.
\]

Therefore

\[
\frac{1}{2} \frac{d}{dt} \left( \| \mathbf{y} \|^2 + \| \mathbf{d} \|^2 \right) \leq \| \mathbf{S} \|_2 \| \mathbf{F} \mathbf{z} \|^2 - \tau \| \mathbf{y} \|^2 \\
\leq \| \sigma \|_\infty \| \mathbf{z} \|^2
\]

by using Lemma 5.1 and orthonormality of the matrix \( \mathbf{F} \). Hence we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \mathbf{y} \|^2 + \| \mathbf{d} \|^2 \right) \leq c (\| \mathbf{y} \|^2 + \| \mathbf{d} \|^2),
\]

where \( c = 2 \| \sigma \|_\infty > 0 \). Multiplying both sides by the integrating factor \( e^{-ct} \) gives

\[
\frac{d}{dt} \left( e^{-ct} (\| \mathbf{y} \|^2 + \| \mathbf{d} \|^2) \right) \leq 0
\]

Integrating both sides from 0 to \( \Delta t \), we obtain

\[
e^{-c \Delta t} \left( \| \mathbf{y} \|^2 + \| \mathbf{d} \|^2 \right) - \left( \| \mathbf{y}(0) \|^2 + \| \mathbf{d}(0) \|^2 \right) \leq D,
\]

and thus

\[
\| \mathbf{y} \|^2 + \| \mathbf{d} \|^2 \leq De^{c \Delta t},
\]

where \( D \) is a constant. Therefore, as \( \Delta t \) approaches zero, \( \| \mathbf{y} \|^2 + \| \mathbf{d} \|^2 \) will remain bounded.

Now, we multiply (25) on the left by \( \begin{bmatrix} \mathbf{y} & \mathbf{0} \end{bmatrix} \) to get

\[
\mathbf{y}^T \dot{\mathbf{y}} = - \begin{bmatrix} \mathbf{y} & \mathbf{0} \end{bmatrix} (\mathbf{F}^T \mathbf{S} \mathbf{F} \mathbf{z} - \tau \begin{bmatrix} \mathbf{y} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}).
\]

Therefore

\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{y} \|^2 \leq 2 \| \mathbf{S} \|_2 \| \mathbf{F} \mathbf{z} \|^2 - 2 \tau \| \mathbf{y} \|^2 \\
\leq 2 \| \sigma \|_\infty \| \mathbf{z} \|^2 - 2 \tau \| \mathbf{y} \|^2 \\
\leq 2 \| \sigma \|_\infty De^{c \Delta t} - 2 \tau \| \mathbf{y} \|^2,
\]

by Lemma 5.1. Multiplying both sides by the integrating factor \( e^{\tau t} \) gives

\[
\frac{d}{dt} \left( e^{\tau t} \| \mathbf{y} \|^2 \right) \leq E e^{c \Delta t + \tau t}.
\]

Integrating both sides from 0 to \( \Delta t \), we obtain

\[
e^{\tau \Delta t} \| \mathbf{y} \|^2 - \| \mathbf{y}(0) \|^2 \leq \frac{E}{\tau} \left( e^{(c+\tau) \Delta t} - e^{c \Delta t} \right),
\]
and thus
\[\|\gamma\|_2^2 \leq e^{-\tau \Delta t} \left( \frac{E}{\tau} e^{(\gamma + \tau) \Delta t} - e^{\gamma \Delta t} + \|\gamma(0)\|_2^2 \right) \]
\[\leq E - e^{-\tau / \Delta t} \left( E e^{(\gamma - \tau) \Delta t} + e^{-\tau / \Delta t} \|\gamma(0)\|_2^2 \right).\]

Therefore, as \(\tau\) goes to \(\infty\), \(\|\gamma\|_2^2\) will go to 0, and thus \(\gamma\) goes to 0. Using this in (23) gives
\[
\lim_{\Delta t \to 0^+} \lim_{\tau \to \infty} \beta(\Delta t) = \beta(0^+) = D\delta(0).
\]

Using this result in (21), we obtain
\[
-r\phi'(r, 0^+) = \lim_{\Delta t \to 0^+} \lim_{\tau \to \infty} -r\phi'(r, \Delta t)
\]
\[= \lim_{\Delta t \to 0^+} \lim_{\tau \to \infty} k^T \beta(\Delta t)
\]
\[= k^T \beta(0^+)
\]
\[= k^T D\delta(0)
\]
\[= \Delta^T \delta(0).
\]

By the definition of \(\Delta\),
\[r \phi'(r, 0^+) = 0 \quad \text{on} \quad \mathcal{L}.
\]

(26)

On the other hand, using the basis functions for \(\mathcal{L}^c\), we can write
\[-r\phi'(r, 0) = \Delta^T \delta(0) \quad \text{on} \quad \mathcal{L}^c.
\]

Therefore,
\[-r\phi'(r, 0^+) = -r\phi'(r, 0) \quad \text{on} \quad \mathcal{L}^c,
\]

and dividing by \(-r\) gives
\[\phi'(r, 0^+) = \phi'(r, 0) \quad \text{on} \quad \mathcal{L}^c.
\]

(27)

Combining (26) and (27), and using the notation \(\phi^+(r) = \phi(r, 0^+)\) we have
\[\phi^+(r) = \left(1 - \Psi(r)\right)\phi'(r, 0),
\]

where \(\Psi\) is the characteristic function on \(\mathcal{L}\). Integrating gives
\[\phi^+(r) = \int_1^r \phi^+(s) \, ds.
\]

Furthermore, for \(r \in \mathcal{L}^c\)
\[\phi^+(r) = \int_1^r \phi^+(s) \, ds = \phi_0(r).
\]

For \(r \in [0, \Delta r]\), \(\phi^+(r)\) is constant since \(\phi^+_r(r)\) vanishes on this interval. Thus, \(\phi^+(r) = \phi^+(\Delta r)\) due to continuity of \(\phi^+\) and
\[\phi^+(r) = \phi_0(\Delta r).
\]

Hence, we have
\[\phi^+(r) = \int_1^r \phi^+(s) \, ds = \begin{cases} \phi_0(\Delta r) & \text{if } r \in \mathcal{L}, \\ \phi_0(r) & \text{if } r \in \mathcal{L}^c, \end{cases}
\]

In Figure 1, we observe this result for a sample pre-flash potential.
8. Conclusions

We evaluated the new electric potential immediately after the lightning discharge. Theorem 1.1 shows that the electric potential is constant along the lightning domain and it is equal to the pre-flash potential elsewhere. Potential is evaluated using an eigenexpansion for the space $L^2_0([0,1])$. Eigenfunctions, $\kappa_i(r)$, are given in terms of spherical Bessel functions of order 1. These functions are utilized to obtain another basis for the same space. The second basis functions are also used to classify the eigenvectors of the matrix $R$, which have a critical role in evaluating the new potential. A sample plot for the result is also given.

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