Divergence theorems in path space III: hypoelliptic diffusions and beyond

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Abstract. Let $x$ denote a diffusion process defined on a closed compact manifold. In an earlier article, the author introduced a new approach to constructing admissible vector fields on the space of paths associated to $x$, under the assumption that $x$ is elliptic. In this article, this method is extended to yield similar results for degenerate diffusion processes. In particular, these results apply to non-elliptic diffusions satisfying Hörmander’s condition.

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1 Introduction

Let \( X_1, \ldots, X_n \) and \( V \) denote smooth vector fields on a closed compact manifold \( M \) such that \( V \) lies within the span of the vectors \( X_1, \ldots, X_n \) at every point in \( M \). Fix a point \( o \in M \) and a positive time \( T \) and consider the Stratonovich stochastic differential equation (SDE)

\[
dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + V(x_t)dt, \quad t \in [0, T]
\]

\( x_0 = o \).

where \( w = (w_1, \ldots, w_n) \) is a standard Wiener process in \( \mathbb{R}^n \). Then the solution process \( x \) is a random variable taking values in the space of paths

\[
C_o(M) = \{ \sigma : [0, T] \mapsto M/\sigma(0) = o \},
\]

an infinite-dimensional manifold with tangent bundle consisting of fibers

\[
T_o C_o(M) = \{ r : [0, T] \mapsto TM/ r_0 = 0, \ r_t \in T_oM \forall t \in [0, T] \}.
\]

The law \( \gamma \) of \( x \), as a measure on \( C_0(M) \), can be considered as a generalized version of Wiener measure on \( C_0(\mathbb{R}^n) \). A major goal in stochastic analysis is to extend the rich body of results that have been developed for the Wiener measure to this more general setting.

The Cameron-Martin space, i.e. the space of paths \( \{ \sigma : [0, T] \mapsto \mathbb{R}^n, \ \sigma_0 = 0 \} \) with finite energy

\[
\int_0^T ||\dot{\sigma}_t||^2 dt
\]

provides a geometrical framework for the Wiener measure and plays a central role in its analysis. Therefore, in addressing the problem raised above, it is natural to seek an analogue of the Cameron-Martin space for the measure \( \gamma \). A reasonable candidate for such an analogue is the set of vector fields on the space \( C_o(M) \) that admit an “integration by parts” formula of the type described in the following definition.

**Definition 1.1** A vector field \( \eta \) on \( C_o(M) \) is admissible (with respect to \( \gamma \)) if there exists an \( L^1 \) function \( \text{Div}(\eta) \) such that the relation

\[
\int_{C_o(M)} \eta(\Phi)d\gamma = \int_{C_o(M)} \Phi \text{Div}(\eta)d\gamma
\]

holds for a dense class of smooth functions \( \Phi \) on \( C_o(M) \).

The construction of admissible vector fields is an important problem that has been studied by many authors in the last three decades. A breakthrough in the problem was achieved by Driver [6] in 1992, following important partial results by Bismut [5]. Driver proved that parallel translation along \( x \) of Cameron-Martin paths in \( T_oM \) produces admissible vector fields on \( C_o(M) \). A fundamental innovation in [6] is the use of the rotation-invariance of the Wiener process. This property also plays a crucial role in the present work.
The work of Bismut and Driver stimulated a great deal of activity in this area and the problem is still being widely studied (cf., e.g. Driver [7], Hsu [9] and [10], Enchev & Stroock [8], Elworthy, Le Jan & Li [7]). Much of this work has dealt with the elliptic case, where the vector fields \( X_1, \ldots, X_n \) in (1.1) are assumed to span \( TM \) at all points of \( M \). In [1], the author introduced a new approach to the problem of constructing admissible vector fields on the space of paths defined by the diffusion process (1.1), again in the elliptic setting. The purpose of the present article, the third in a series of papers on this theme (cf. [1] and [2]), is to extend this approach to degenerate (i.e. non-elliptic) diffusions.

The central object of study in the author’s approach is the Itô map \( g : w \mapsto x \) defined by equation (1.1). This is used to lift the problem from the manifold \( M \) to \( \mathbb{R}^n \), where classical integration by parts theorems can be applied. The lifting method had previously been used by Malliavin in his probabilistic approach to the hypoellipticity problem [12]. “Lifting” is defined as follows.

**Definition 1.2** A process \( r \) taking values in \( \mathbb{R}^n \) is said to be a lift of \( \eta \) to \( C_0(\mathbb{R}^n) \) (via the Itô map) if the following diagram commutes

\[
\begin{array}{ccc}
TC_0(\mathbb{R}^n) & \xrightarrow{dg} & TC_0(M) \\
\uparrow r & & \uparrow \eta \\
C_0(\mathbb{R}^n) & \xrightarrow{g} & C_0(M)
\end{array}
\]

Since \( g \) is non-differentiable in the classical sense the derivative \( dg \) must be interpreted in the extended sense of the Malliavin calculus\(^2\). The idea in [1] is to simultaneously construct a vector field \( \eta \) on \( C_0(M) \) and an admissible lift \( r \) of \( \eta \) to \( C_0(\mathbb{R}^n) \). In particular (cf. Theorems 2.1 and 2.2 below), this requires that \( r \) take the form

\[
r_t = \int_0^t A(s)dw_s + \int_0^t B(s)ds
\]

where \( A \) and \( B \) are continuous adapted processes taking values in \( so(n) \) (the space of skew-symmetric \( n \times n \) matrices) and \( \mathbb{R}^n \) respectively. Processes of this form thus comprise the tangent bundle \( TC_0(\mathbb{R}^n) \) in the above diagram.

For test (i.e. smooth cylindrical) functions \( \Phi \) on \( C_0(M) \), one then has

\[
E[(\eta \Phi)(x)] = E[r(\Phi \circ g)(w)] = E[\Phi \circ g(w)Div(r)] = E[\Phi(x)E[Div(r)/x]].
\]

where \( Div \) denotes the divergence operator in the classical Wiener space. Thus \( \eta \) is admissible with divergence

\[
Div(\eta)(x) = E[Div(r)/x].
\]

\(^2\)As this type of regularity is now generally well-understood by stochastic analysts, this point will not be emphasized in the paper. See e.g. the monographs [3], [13], [14], [15] for an introduction to the Malliavin calculus.
An important consequence of the ellipticity assumption is the fact that every non-anticipating vector field on $C_0(M)$ can be written in the form

$$\eta_t = \sum_{i=1}^{n} h_i(t)X_i(x_t) \quad (1.3)$$

where $h_i, i = 1, \ldots, n$ are real-valued processes, adapted to the filtration of $x$. In the highly non-generic situation where the vector fields $\{X_i\}$ commute, then for every $t > 0$, $x_t$ becomes a function of $w_t$ and the problem trivializes. The argument in [1] sets up a duality between the processes $h$ and $r$, the lift of $\eta$, in which (in the non-commuting case) the commutators $[X_i, X_j]$ play an explicit role.

It was shown in [2] that in the hypoelliptic case (where the diffusion process (1.1) is degenerate but Hörmander’s condition holds), then generic vector fields of the form (1.3) do not admit lifts to $C_0(\mathbb{R}^n)$ under the Itô map. In particular, admissible vector fields on $C_0(M)$ consisting of linear combinations of $X_1, \ldots, X_n$ cannot be constructed by the author’s method in this case. The point of departure for the present work is the a priori selection of an additional collection of vector fields $\{V_I : I \in \mathcal{I}\}$ on $M$ such that

$$\{V_I(x) : I \in \mathcal{I}\} \text{ span } T_xM, \ \forall x \in M. \quad (1.4)$$

Thus in the elliptic case $\{V_I\}$ can be taken to be the set $\{X_1, \ldots, X_n\}$, whereas in the hypoelliptic case, one can choose $\{V_I\} = \text{Lie}(X_1, \ldots, X_n)$, the Lie algebra generated by the vector fields $X_1, \ldots, X_n$. We construct admissible vector fields on $C_0(M)$ in the form

$$\eta_t = \sum_{I \in \mathcal{I}} h_I(t)V_I(x_t).$$

Somewhat surprisingly, it proves to be possible to trade ellipticity in $\{X_1, \ldots, X_n\}$ for condition (1.4). This enables us to establish our results under very general hypotheses.

The layout of the paper is as follows. Section 2 contains background material. The results here are well-known, for the most part. Theorem 2.1 asserts that Riemann integrals of continuous adapted paths have divergence given by an Itô integral, while Theorem 2.2 states that Itô integrals with continuous adapted skew-symmetric integrands are divergence-free. The former result follows easily from the Girsanov theorem, the latter from the infinitesimal rotation-invariance of the Wiener measure. Theorem 2.6 gives a relationship between a vector field $\eta$ along the path $x$ and the lift of $\eta$ to the Wiener space. This relationship, expressed in terms of the derivative of the stochastic flow of the SDE (1.1) and the inverse flow, plays a key role throughout. The required geometric machinery and notations are also introduced in this section of the paper.

Section 3 contains the main results. Theorem 3.1 produces a class of admissible vector fields on $C_0(M)$, under hypotheses that allow the SDE (1.1) to be degenerate. The proof of Theorem 3.1 follows the above outline and is an extension of the argument in [1]. An essential step in the proof is the decomposition of non-tensorial terms in the lift obtained from Theorem 2.6, into tensorial plus skew-symmetric parts.

Theorem 3.2 is a variation on Theorem 3.1 that exhibits a vector field on $C_0(M)$ with given divergence. In particular, we obtain a class of vector fields with divergence expressed in terms of Ricci curvature. The interest of this result lies in the fact that formulae of this type appear in the work of other authors, e.g Driver [6] and Elworthy, Le Jan & Li [8], where they are obtained using different methods. In Example 3.3,
Theorem 3.2 is applied to yield vector fields on $C^o(M)$ with divergence having no extraneous dependence on the Wiener path $w$. This property is important in applications of the theorem that require a degree of regularity of the divergence such as the study of quasi-invariance. Theorem 3.4 is an intrinsic formulation of Theorem 3.1 that does not depend on the choice of a basis $\{V_I\}$. We assume here that $M$ is a Riemannian manifold. The proof of Theorem 3.4 requires the introduction of a tensor that enables us to express the Levi-Civita connection on $M$ in terms of a connection intrinsic to the diffusion process (1.1). In Theorem 3.6, we apply our theory to gradient systems. As a consequence (Corollary 3.7 and Theorem 3.12), we obtain Driver’s result cited above.

In Section 4, we consider the special case where the vector fields $X_1, \ldots, X_n$ are linearly independent. In this case, the problem under consideration simplifies considerably and our argument simplifies accordingly. We conclude with an example where the SDE (1.1) takes values in the Heisenberg group $G$. In this case we obtain explicit formulae for a class of admissible vector fields $\eta$ on the path space $C^o(G)$.

2 Background material

2A Divergence theorems for Wiener space

We present two such results. These concern the transformation of Wiener measure under Euclidean motions; the first under translations, the second under rotations.

Let $\Omega$ denote the measure space for the Wiener process, equipped with the filtration

$$\mathcal{F}_t = \sigma\{w_s/ \ s \leq t\}.$$ 

**Theorem 2.1** Let $h : \Omega \times [0, T] \mapsto \mathbb{R}^n$ be a continuous adapted path. Then the process $\int_0^T h_s \, dw_s$ is admissible (with respect to the Wiener measure) and

$$\text{Div} \left[ \int_0^T h_s \, dw_s \right] = \int_0^T h_s \cdot dw_s$$

where $\cdot$ on the right of the equation denotes the Euclidean inner product.

**Proof.** The result follows easily from the Girsanov theorem, which implies that for $\Phi \in C^\infty_b(C_0(\mathbb{R}^n))$ and $\epsilon \in \mathbb{R}$,

$$E\left[ \Phi(w + \epsilon \int_0^T h_s \, ds) \right] = E[\Phi(w)G_\epsilon(w)] \quad (2.1)$$

where

$$G_\epsilon(w) \equiv \epsilon \int_0^T h_s \cdot dw_s - \frac{\epsilon^2}{2} \int_0^T ||h_s||^2 \, ds.$$ 

Differentiating each side of (2.1) wrt $\epsilon$ and setting $\epsilon = 0$ gives the theorem.

**Theorem 2.2** Let $A : \Omega \times [0, T] \mapsto so(n)$ be a continuous adapted process. Then the process $\int_0^T Adw$ is admissible and

$$\text{Div} \left[ \int_0^T Adw \right] = 0.$$
Proof. Define a process $\theta^\epsilon_t = \exp \epsilon A_t$ where $\exp$ denotes matrix exponentiation. Then $\theta^\epsilon_t$ is an adapted $O(n)$-valued matrix process with $\theta^\epsilon_0 = I$. It follows from the infinitesimal rotation-invariance of the Wiener measure that the law of the process

$$w^\epsilon = \int_0^t \theta^\epsilon_t dw_t$$

is invariant under $\epsilon$. Hence for $\Phi \in C^\infty_0(C_0(\mathbb{R}^n))$, we have

$$E[\Phi(w^\epsilon)] = E[\Phi(w)].$$

As before, differentiating in $\epsilon$ and setting $\epsilon = 0$ gives the result.

2.B Geometric preliminaries

In this section we introduce the geometric machinery that will be needed in Section 3. We adopt the summation convention throughout the paper: whenever an index in a product (or a bilinear form) is repeated, it will be assumed to be summed on.

First, let $[g^j_k]$ be the Riemannian metric defined on $M$ by

$$g^{j_k} = a^j_I a^k_I$$

where

$$V_I = a^j_I \frac{\partial}{\partial x_j}, \ I \in \mathcal{I}$$

is the expression of the vector fields in local coordinates (note that the matrix $[g^{j_k}]$ is non-degenerate by the spanning condition (1.4)).

Let $(.,.)$ denote the inner product structure on $TM$ defined by the metric $[g^{j_k}]$.

Then we have

$$V = (V, V_I) V_I, \ \forall V \in TM. \quad (2.2)$$

To see this, let $V = b_J V_J$ and write $V_J = a^j_J \frac{\partial}{\partial x_j}$ for each $J$, as above. Then

$$(V, V_I) V_I = (b_J a^j_J \frac{\partial}{\partial x_j}, a^k_J \frac{\partial}{\partial x_k}) a^l_I \frac{\partial}{\partial x_l}$$

$$= b_J a^j_J g^{j_k} g_{k_l} = b_J a^j_J \delta_{jl} \frac{\partial}{\partial x_l}$$

$$= b_J a^j_J \frac{\partial}{\partial x_l} = V$$

as claimed.

We denote the Levi-Civita covariant derivative associated with this metric by $\tilde{\nabla}$.

The following constructions were introduced by Elworthy, Le Jan and Li (cf. [8]). Assume the set of vectors $\{X_1(x), \ldots, X_n(x)\}$ span a subspace $E_x$ of $T_x M$ of constant dimension as $x$ varies in $M$ and define $E$ to be the subbundle of $TM$

$$E = \bigcup_{x \in M} E_x.$$ 

Then $E$ becomes a Riemannian bundle under the inner product induced on $E$ by the linear maps

$$X(x) : (h_1, \ldots, h_n) \in \mathbb{R}^n \mapsto h_i X_i(x) \quad (2.3)$$
from the Euclidean space \( \mathbb{R}^n \).

There is a metric connection \( \nabla \) on \( E \) compatible with the metric \(< \ldots, \cdot >\). This connection (termed the Le Jan-Watanabe connection in [8]), is defined by

\[
\nabla_V Z = X(x) d_V (X^* Z), \quad Z \in \Gamma(E), V \in T_x M,
\]

where \( d \) is the standard derivative, applied the function

\[
x \in M \mapsto X(x) Z(x) \in \mathbb{R}^n.
\]

**Lemma 2.3** For all \( x \in M \) and \( V \) and \( W \) in \( T_x M \), we have

\[
< \nabla_V X_j, W > X_j = 0.
\]

**Proof.** Let \( P = P(x) \) denote orthogonal projection in \( \mathbb{R}^n \) onto the subspace \( \text{Ker} \ X(x)^\perp \) and \( \{e_1, \ldots, e_n\} \) the standard orthonormal basis of \( \mathbb{R}^n \). Then

\[
X^* [ < \nabla_V X_j, W > X_j ] = X^* [ < X d_V P e_j, W > X_j ]
\]

\[
= < d_V P e_j, X^* W > P e_j
\]

(where \(< \ldots, \cdot >\) denotes the Euclidean inner product)

\[
= < e_j, d_V P X^* W > P e_j
\]

\[
= P( d_V P ) X^* W
\]

\[
= P( d_V P ) P X^* W.
\]

On the other hand, differentiating the relation \( P^2 = P \) gives

\[
d_V PP + P d_V P = d_V P.
\]

Thus

\[
d_V PP = d_V P - P d_V P = Q d_V P
\]

where \( Q = I - P \). Hence

\[
P d_V PP = PQ d_V P = 0
\]

and we have

\[
X^* [ < \nabla_V X_j, W > X_j ] = 0.
\]

The Lemma now follows from the fact that \( XX^* = I \).

The **Riemann curvature** tensor \( R \) corresponding to this connection is defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

The **Ricci tensor** is defined by

\[
\text{Ric}(X) = R(X, e_i) e_i
\]

where \( \{e_i\} \) is a (locally defined) orthonormal frame in \( E \).

The next result shows that the vector fields \( \{X_i\} \) play the role of a (generalized) orthonormal basis of \( E \) and, in particular, the Ricci tensor can be computed using these vector fields.
Lemma 2.4
(i) $< Y, X_i > X_i = Y, \forall Y \in E$.
(ii) $\text{Ric}(Y) = R(Y, X_i)X_i, \forall Y \in TM$.

We omit the proofs of these statements since they are elementary (the proof of (i) is similar to that of (2.2) above. A proof of (ii) can be found in [3. Sec. 2]).

2.C Flow-related theorems

Lemma 2.5 Let $g_t : M \mapsto M$ denote the stochastic flow $x_0 \mapsto x_t$ defined by the SDE (1.1). Define $Y_t : T_{x_0} M \mapsto T_{x_t} M$ and $Z_t : T_{x_t} M \mapsto T_{x_0} M$ by $Y_t \equiv dg_t$ and $Z_t \equiv Y_t^{-1}$.

Let $B$ denote a vector field on $M$ and $d$ the stochastic time differential. Then

$$d[Z_t B(x_t)] = Z_t \left( [X_i, B](x_t) \circ dw_i + [V, B](x_t) dt \right).$$

Proof. Let $D_t$ denote the stochastic covariant differential along the path $x_t$, with respect to the Levi-Civita $\nabla$ connection defined above. Then differentiating with respect to the initial point $o$ in (1.10) gives

$$D_t Y = \nabla_{Y_t} X_i \circ dw_i + \nabla_{Y_t} V dt.$$ 

We then have

$$D_t Z = D_t(Y_t^{-1})$$
$$= -Z_t D_t Y Z_t$$
$$= -Z_t \left( \nabla_{Id_t} X_i \circ dw_i + \nabla_{Id_t} V dt \right)$$

where $Id_t$ denotes the identity map on $T_{x_t} M$. Thus

$$d(Z_t B) = D_t Z B + Z_t \nabla_{dx_t} B$$
$$= -Z_t \left( \nabla_{B} X_i \circ dw_i + \nabla_{B} V dt \right) + Z_t \left( \nabla_{X_i} B \circ dw_i + \nabla_{V} B dt \right)$$

$$d[Z_t B(x_t)] = Z_t \left( [X_i, B](x_t) \circ dw_i + [V, B](x_t) dt \right).$$

as required.

Theorem 2.6 Let $r : \Omega \times [0, T] \mapsto \mathbb{R}^n$ be an Itô process. Then the path $\eta \equiv dg(w)r$ is given by

$$\eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_i$$ (2.4)

Proof. Note that $\eta$ is a vector field along the path $x$. Let $U_s : T_o \mapsto T_{x_s} M$ denote stochastic parallel translation along $x$.

Differentiating in (1.1) with respect to $w$ gives the following covariant equation for $\eta$

$$D_t \eta = \nabla_\eta X_i(x_t) \circ dw_i + X_i(x_t) \circ dr_i + \nabla_\eta V(x_t) dt$$ (2.5)

$$\eta_0 = 0.$$  

3Here and in the sequel, we assume that all vector fields appearing in the equations are evaluated at $x_t$. 
We write (2.5) as
\[ d(U_t^{-1}\eta) = U_t^{-1}\tilde{\nabla}_\eta X_i(x_t) \circ dw_i + U_t^{-1}X_i(x_t) \circ dr_i + U_t^{-1}\tilde{\nabla}_\eta V(x_t)dt. \]
Denoting the path \( t \mapsto U_t^{-1}\eta \) by \( y \), we note that the equation for \( y \) has the form
\[ dy = M_i(t)y_t \circ dw_i + M_0(t)y_t + U_t^{-1}X_i(x_t) \circ dr_i \]
(2.6)
where \( M_j(t), j = 1, \ldots, n \) are linear operators on \( T_oM \).

On the other hand, differentiation in (1.1) with respect to the initial point \( o \) gives
the following equation for \( \tilde{Y} \equiv U_t^{-1}Y_t \)
\[ d\tilde{Y} = M_i(t)\tilde{Y}_t \circ dw_i + M_0(t)\tilde{Y}_t dt \]
(2.7)
\[ \tilde{Y}_0 = I. \]
Equation (2.6) can be solved in terms of \( \tilde{Y} \) using an operator version of the familiar
"integrating factor" method used to solve first order linear ODE's. Noting, then, that
\( \tilde{Y}^{-1} \) is an integrating factor for (2.6) and using this to solve for \( y \) gives
\[ y_t = \tilde{Y}_t \int_0^t \tilde{Y}_s^{-1}U_s^{-1}X_i(x_s) \circ dr_i. \]
(2.8)
Writing (2.8) in terms of \( \eta \) and \( Y \), we obtain (2.4).

**Remarks**
1. Theorem 2.6 gives an alternative proof of the "lifting" equation (3.2) in [1].
2. Suppose \( \eta \) in (2.4) has the form \( \eta_t = X_i(x_t)h_i(t) \) for an \( \mathbb{R}^n \)-valued process \( h = (h_1, \ldots, h_n) \). Then, writing
\[ X = [X_1 \ldots X_n] \]
and solving for \( dr \) in (2.4), we have
\[ Z_tX(x_t) \circ dr = d[Z_tX(x_t)h_t]. \]
This equation suggests that \( r \) can be considered as a type of "covariant derivative" of
\( h \) along \( x \), where the operator \( Z_tX(x_t) \) plays the role of backward parallel translation.

## 3 Divergence theorems

**3.A First result**
Let \( X \) be as defined in (2.3). Then the SDE (1.1) may be written
\[ dx = X(x_t) \circ d\tilde{w} \]
where
\[ d\tilde{w} = dw + X(x_t)^*V(x_t)dt \]
and the adjoint map is defined using the metric \( < \cdot, \cdot > \) on \( E \) (so \( X(x)^* \) is a left
inverse for \( X(x) \)). By the Girsanov theorem, the law \( \tilde{\nu} \) of of \( \tilde{w} \) is equivalent to the
law \( \nu \) of \( w \), with Radon-Nikodym derivative \( \frac{d\tilde{\nu}}{d\nu} \) given by
\[ G(w) = \exp \left( \int_0^T X(x_t)^*V(x_t) \cdot dw - \frac{1}{2} \int_0^T ||X(x_t)^*V(x_t)||^2 dt \right). \]
Suppose that \( r \) is an admissible lift for the vector field \( \eta \) under the map \( \tilde{g} : \tilde{w} \mapsto x \). Then
\[
E[\eta \phi(x)] = E \left[ G(w) \cdot r(\Phi \circ \tilde{g})(w) \right]
= E[\Phi \circ \tilde{g}(w) \text{Div}(G \cdot r)],
E[\Phi \circ \tilde{g}(w) \{ G \cdot \text{Div}(r) - r(G) \}].
\]
Thus \( \eta \) is admissible.

In view of this discussion, there is no loss in generality in assuming \( V = 0 \) and we shall assume in the sequel that this is the case.

We introduce the following tensors \( \{ T_I \} \) associated to the vector fields \( \{ V_I \} \)
\[
T_I(X) = \nabla V_I X + [X, V_I], \quad X \in E.
\]

**Theorem 3.1** Let \( r = (r_1, \ldots, r_n) \) be a path in the Cameron-Martin space of \( \mathbb{R}^n \) and define \( \{ h_I : I \in \mathcal{I} \} \) by the linear stochastic system
\[
dh_I = (X_I, V_I) \dot{r}_I dt - (T_J(\circ dx), V_I) h_J dt
\]
\[
h_I(0) = 0.
\]
Then the vector field \( \eta_t \equiv h_I(t)V_I(x_t), \ t \in [0, T] \) is admissible on \( C_o(M) \).

**Proof.** We first note that Theorem 2.6 implies that \( r \) is lift of \( \eta \) if \( r \) satisfies
\[
X_I dr_I = V_I \circ dh_I + [X_I, V_I] h_I \circ dw_j
\]
Substituting \( \eta_t = h_I(t)V_I(x_t) \) into (3.2) and using Lemma 2.5, we have
\[
X_I dr_I = V_I \circ dh_I + G_{ij}^I h_I \circ dw_j + T_I(\circ dx) h_I,
\]
Writing the Lie bracket term involving \( X_j \) in terms of the connection \( \nabla \) and using Lemma 2.4 (i) gives
\[
[X_J, V_I] = T_I(X_J) - \nabla V_I X_J
= T_I(X_J) - < \nabla V_I X_J, X_i >
\]
Denote
\[
G_{ij}^I = < \nabla V_I X_i, X_j > - < \nabla V_I X_j, X_i >
\]
Combining the previous two lines with Lemma 2.3, we have
\[
[X_J, V_I] = G_{ij}^I X_i + T_I(\circ dx) h_I.
\]
Substituting this into (3.3) gives
\[
X_I dr_I = V_I \circ dh_I + G_{ij}^I h_I X_i \circ dw_j + T_I(\circ dx) h_I.
\]
We note that, more generally, a semimartingale path \( \tilde{r} \) is a lift of \( h_I V_I \) if equation (3.5) holds with the left hand side replaced by the Stratonovich differential \( X_i \circ d\tilde{r}_i \).

Suppose now the coefficient functions \( \{ h_I \} \) satisfy the system
\[
X_I dr_I = V_I \circ dh_I + T_I(\circ dx) h_I,
\]
\[
h_I(0) = 0.
\]
Then
\[ X_i [dr_i + G_i^j h_I \circ dw_j] = V_I \circ dh_I + G_i^j X_i h_I \circ dw_j + T_I(X_j)h_I \circ dw_j \]
So if we define
\[ \tilde{r_i} = r_i + \int_0^t G_i^j h_I \circ dw_j. \] (3.7)
then (3.3) holds with \( r \) replaced by \( \tilde{r} \). It follows that \( \tilde{r} \) is a lift of \( \eta \), where
\[ \eta_t = h_I(t)V_I(x_t). \] (3.8)
Furthermore, the the skew-symmetry of the functions \( G_i^j \) in the upper indices and Theorem 2.2 imply that the Stratonovich integral in (3.7) can be written as a Riemann integral plus a divergence-free Itô integral. It follows from Theorems 2.1 and 2.2 that \( \tilde{r} \) is admissible. Note also that by (2.2), the processes \( h_I \) defined by (3.1) satisfy equation (3.3).

We have thus shown that \( \tilde{r} \) is an admissible lift to the Wiener space of the vector field \( \eta \) in (3.8). In view of Definition 1.2, we have for any test function \( \Phi \) on \( C_0(M) \)
\[ E[(\eta \Phi)(x)] = E[r(\Phi \circ g)(w)] \]
\[ = E[\Phi \circ g(w) Div(r)] \]
\[ = E[\Phi(x)E[Div(r)/x]]. \]
Thus \( \eta \) is admissible and
\[ Div(\eta)(x) = E[Div(r)/x]. \]

3.B Computation of the divergence
In order to compute the divergence of the vector field \( \eta \) in Theorem 3.1, it is necessary to convert the Stratonovich integral in (3.7) into Itô form. The relation between the Stratonovich and Itô differentials is formally
\[ dG_i^j h_I \circ dw_j = G_i^j h_I dw_j + \frac{1}{2} d(G_i^j h_I) dw_j. \] (3.9)
Write
\[ \alpha_{kij} = \nabla_{X_k} \nabla_{V_i} X_j > + \nabla_{V_i} \nabla_{X_k} X_j > \]
\[ - \nabla_{X_k} \nabla_{V_j} X_i > - \nabla_{V_j} \nabla_{X_k} X_i > \] (3.10)
and
\[ \beta_i^k = -(T_j(X_k), V_I) h_J. \] (3.11)
Then by (3.1) and (3.4)
\[ dG_i^j = \alpha_{kij} dw_k + \{\ldots\} dt \]
and
\[ dh_I = \beta_i^k dw_k + \{\ldots\} dt. \]
Substituting these into (3.9) and using the Itô rules
\[ dw_i dw_j = \delta_{ij} dt, \quad dw_i dt = 0 \]
we see that the Ito-Stratonovich correction term in (3.9) is
\[
\frac{1}{2} \left( \alpha_I^{kik} h_I + G_I^{ik} \beta_I^k \right) dt.
\] (3.12)
Thus (3.7) becomes
\[
\tilde{r}_i = r_i + \int_0^T G_I^{ij} h_I dw_j + \frac{1}{2} \int_0^T \left( \alpha_I^{kik} h_I + G_I^{ik} \beta_I^k \right) dt.
\]
As remarked in the proof of Theorem 3.1, the Itô integral has divergence zero and using Theorem 2.1 we obtain
\[
\text{Div}(\tilde{r}) = \int_0^T \left( \dot{r}_i + \frac{1}{2} \left( \alpha_I^{kik} h_I + G_I^{ik} \beta_I^k \right) \right) dw_i
\]
Hence
\[
\text{Div}(\eta) = E\left[ \int_0^T \left( \dot{r}_i + \frac{1}{2} \left( \alpha_I^{kik} h_I + G_I^{ik} \beta_I^k \right) \right) dw_i / x \right]
\] (3.13)
where the $\alpha$’s and $\beta$’s are given in (3.10) and (3.11).

By adjusting the right hand side in equation (3.1) by the addition of a suitably chosen drift term, the above argument can easily be modified to give

**Theorem 3.2** Let $\gamma : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ be a $C^1$ adapted process and define \( \{ h_I \} \) by
\[
h_I(0) = 0
\]
\[
dh_I = \left( (d\gamma_i - \frac{1}{2} G_I^{ik} \beta_I^k dt) X_i + (T_J (\text{od}x) - \frac{1}{2} \alpha_J^{kik} X_i dt) h_J, V_I \right).
\]
Then the vector field $\eta = h_I V_I$ is admissible and for every test function $\Phi$ on $C_0(M)$, we have
\[
E\left[ (\eta \Phi)(x) \right] = E\left[ \Phi(x) \int_0^T \dot{r}_i dw_i \right].
\] (3.14)

The proof of Theorem 3.2 is an easy modification of the argument above, where we replace $r$ by the path
\[
\tilde{r}_i = \gamma_i - \frac{1}{2} \int_0^t \left( \alpha_I^{kik} h_I + G_I^{ik} \beta_I^k \right) dt.
\]
The essential point is that the correction term (3.12) in the computation of the divergence does not explicitly involve the path $r$.

**Corollary** Given any path $r$ in the Cameron-Martin space of $\mathbb{R}^n$, we can construct an admissible vector field $\eta$ on $C_0(M)$ such that
\[
E\left[ (\eta \Phi)(x) \right] = E\left[ \Phi(x) \int_0^T \left( \dot{r}_i + \frac{1}{2} < \text{Ric}(\eta), X_i > (x_i) \right) dw_i \right].
\] (3.15)

**Remarks**
1. Formula (3.15) is similar to those appearing in the work of Driver [6], [7] and Elworthy, Le Jan & Li [8].
2. Choosing $\gamma = 0$ in Theorem 3.2, we see that the path $\tilde{\eta} \equiv V_I h_I$, where
\[
dh_I = (h_J T_J(\omega dx) - \frac{1}{2} X_i (G^{ik}_j \beta^j_k + \alpha^{ik}_j h_J)) dt, V_I, h_I(0) = 0
\]
is divergence-free with respect to the law of $x$. In this sense $\tilde{\eta}$ is analogous to a vector field on Wiener space of the form $\int_0^T Adw$, where $A$ is a continuous adapted so$(n)$-valued process.

3. The appearance of the conditional expectation in (3.13) entails a loss of information concerning the regularity of the function $\text{Div}(\eta)$. This point is crucial in certain applications of the results presented here. For example, the regularity of $\text{Div}(\eta)$ plays a major role in recent work of the author [4] in which the admissibility of $\eta$ is used, in the elliptic setting, to establish quasi-invariance of the law of $x$ under the flow generated by $\eta$ on $C_o(M)$.

With this in mind, we note that by choosing the process $\gamma$ in (3.14) appropriately, we can eliminate the extraneous dependence of the stochastic integrals on $w$ and thus circumvent this problem. The next example illustrates this point.

**Example 3.3** Suppose $B$ is a smooth vector field on $M, \rho$ is a deterministic $C^1$ real-valued function, and define
\[
\gamma_i(t) = \int_0^T \rho_t(B, X_i)(x_i) dt
\]
so
\[
\int_0^T \gamma_i dw_i = \int_0^T \rho_t(B, X_i) dw_i.
\]
Using the Levi-Civita connection $\tilde{\nabla}$ to write this in Stratonovich form we have
\[
\int_0^T \rho_t(B, X_i) dw_i =
\]
\[
\int_0^T \rho_t(B, X_i) \circ dw_i - \frac{1}{2} \int_0^T \rho_t\left(\left(\tilde{\nabla}_B X_i + (B, \tilde{\nabla}_X_i)\right) dt =
\]
\[
\int_0^T \rho_t(B, \omega dx) - \frac{1}{2} \int_0^T \rho_t\left(\left(\tilde{\nabla}_B X_i + (B, \tilde{\nabla}_X_i)\right) dt
\]
(3.16)
Since (3.16) is measurable with respect to $x$, (3.14) becomes
\[
\text{Div}(\eta) = \int_0^T \rho_t(B, \omega dx) - \frac{1}{2} \int_0^T \rho_t\left(\left(\tilde{\nabla}_B X_i + (B, \tilde{\nabla}_X_i)\right) dt.
\]
In particular, $\text{Div}(\eta)$ is an explicit function of the path $x$.

3.C A basis-free formulation of the argument
Assume now that $M$ is a Riemannian manifold. In this case we can formulate the preceding argument intrinsically, i.e. in a way that does not depend on the choice of a basis $\{V_i\}$. 

Let $\tilde{\nabla}$ denote the Levi-Civita covariant derivative with respect to the Riemannian metric on $M$ and $\tilde{D}$ the corresponding covariant stochastic differential. As before, $\langle ., . \rangle$ and $\nabla$ will denote the inner product and the connection on the subbundle $E$ introduced in Section 2.B.

We define

\[ T(X, Y) = \tilde{\nabla}_Y X - \nabla_Y X, \quad Y \in TM, X \in E, \quad (3.17) \]

noting that $T$ is tensorial in both arguments.

Let $r : [0, T] \times \Omega \mapsto \mathbb{R}^n$ be an Itô semimartingale

\[ dr_k(t) = b^{kj}(t) dw_j + c^k(t) dt \]

where $b^{kj}$ and $c^k$ are adapted continuous processes. Then differentiation in equation (1.1) gives the following covariant equation for the path $\eta \equiv dg(w) r$

\[ \tilde{D}_t \eta = \tilde{\nabla}_\eta X_i \circ dw_i + \nabla \eta (\circ dr_i) \]

where

\[ G^{ij}_\eta \equiv \langle \nabla V X_i, X_j \rangle - \langle \nabla V X_j, X_i \rangle. \]

In view of Lemma 2.3, we have

\[ \tilde{D}_t \eta = G^{ij}_\eta X_j \circ dw_i + \nabla \eta (\circ dr_i) \]

Thus

\[ \tilde{D}_t \eta = (T(X_i, \eta) \circ dw_i + X_i \circ dr_i) \]

\[ \tilde{D}_t \eta = T(X_i, \eta) \circ dw_i + X_i (\circ dr_i + G^{ij}_\eta \circ dw_j). \quad (3.18) \]

**Theorem 3.4** Let $r$ be any Cameron-Martin path in $\mathbb{R}^n$ and define a vector field $\eta$ along $x$ by the covariant SDE

\[ \tilde{D}_t \eta = T(\circ dx, \eta) + X_i \dot{r}_i dt \]

\[ \eta(0) = 0. \]

Then $\eta$ is an admissible vector field on $C^0(M)$. For vector fields $X$ and $Y$ on $M$, define the differential operator

\[ L_{Y,X} \equiv \nabla_Y \nabla_X - \nabla_{\tilde{\nabla}_Y X}. \]

For test functions $\Phi$ on $C^0(M)$, we then have

\[ E[(\eta \Phi)(x)] = E\left[ \Phi(x) \int^T_0 (\dot{r}_i + \frac{1}{2} \alpha_i) dw_i \right], \quad (3.20) \]

where

\[ \alpha_i(t) = \langle L_{\eta, X_i} X_i, X_i \rangle - \langle L_{\eta, X_j} X_j, X_i \rangle + \langle \nabla_{\eta X_i} X_i, X_j \rangle >
- \langle \nabla_{\eta X_j} X_i, X_i \rangle + \langle \nabla T(X_i, \eta) X_i, X_j \rangle - \langle \nabla T(X_j, \eta) X_j, X_i \rangle. \]
Proof. Note that equation (3.18) implies \( \tilde{r} \) is a lift of \( \eta \), where
\[
\tilde{r}_i = r_i - \int_0^\cdot G^i_{\eta j} \circ dw_j.
\]
(3.21)

Since the functions \( G^i_{\eta j} \) are skew-symmetric in the indices \( j \) and \( i \), Theorems 2.1 and 2.2 imply that \( \tilde{r} \) is an admissible vector field on the Wiener space. As before, for any test function \( \Phi \) on \( C_c(M) \), we have
\[
E[D\Phi(x)\eta] = E[\Phi(x)\text{Div}(\tilde{r})].
\]
and it follows that \( \eta \) is admissible as claimed.

We now derive the formula for the divergence of the vector field \( \eta \). As before, this requires the computation of the Stratonovich-Itô correction term in (3.21). We now proceed to do this.

Note that the operator-valued map \( (X,Y) \mapsto L_{Y,X} \) is tensorial in both \( X \) and \( Y \).

We have
\[
\nabla X \nabla Y = R(X,Y) + \nabla Y \nabla X + \nabla [X,Y] \\
= R(X,Y) + \nabla Y \nabla X - \nabla \nabla X Y + \nabla \nabla X Y.
\]

In particular
\[
D_t \nabla \eta X_i = [R(\circ dx_t,\eta) + L_{\eta,\circ dx_t} + \nabla \tilde{D}_t \eta] X_i.
\]

Thus, neglecting differentials of terms of bounded variation (which will not affect the present calculation)
\[
D_t \nabla \eta X_i = [R(X_k,\eta) + L_{\eta,X_k}] X_i dw_k + \nabla \tilde{D}_t \eta X_i.
\]

This yields
\[
d_t G^i_{\eta j} = <D_t \nabla \eta X_i, X_j> - <D_t \nabla \eta X_j, X_i> + <\nabla \eta X_i, D_t X_j> - <\nabla \eta X_j, D_t X_i> = \\
= \left\{ <[R(X_k,\eta) + L_{\eta,X_k}] X_i, X_j> - <[R(X_k,\eta) + L_{\eta,X_k}] X_j, X_i> \right. \\
+ <\nabla \eta X_i, \nabla X_k X_j> - <\nabla \eta X_j, \nabla X_k X_i> \right\} dw_k + <\nabla \tilde{D}_t \eta X_i, X_j> - <\nabla \tilde{D}_t \eta X_j, X_i>.
\]

Substituting for \( \tilde{D}_t \eta \) from equation (3.19) and using Lemma 2.4(b) and the symmetry of the Ricci tensor, we obtain
\[
d_t G^i_{\eta j} dw_j = \left\{ <L_{\eta,X_j} X_i, X_j> - <L_{\eta,X_i} X_j, X_i> + <\nabla \eta X_i, \nabla X_j X_i> - <\nabla \eta X_j, \nabla X_i X_j> \right. \\
- <\nabla \eta X_j, \nabla X_i X_i> + <\nabla T(X_j,\eta) X_i, X_j> - <\nabla T(X_i,\eta) X_j, X_i> \right\} dt \\
= \alpha_i(t) dt.
\]
Thus (3.9) gives
\[ \tilde{r}_i = r_i + \int_0^t G_{ij}^i \eta_j \, dw_j + \frac{1}{2} \int_0^t \alpha_i \, dt \]
Formula (3.21) now follows from Theorems 2.1 and 2.2, as before.

**Remark** It is clear that the argument used to prove Theorem 3.4 is valid in more generality, with the deterministic Cameron-Martin path \( r \) replaced by an \( \mathcal{X} \)-measurable random path of the form
\[ r = \int_0^t A(s) \, dw_s + \int_0^t B(s) \, ds. \tag{3.22} \]
where \( A : \Omega \times [0, T] \to so(n) \) and \( B : \Omega \times [0, T] \to \mathbb{R}^n \) are continuous adapted processes. We note that it is easy to construct examples of \( \mathcal{X} \)-measurable processes of the form (3.22). A large class of such examples is obtained by choosing a 2-form \( \lambda \) on \( M \) and a deterministic continuous real-valued function \( f \) and defining \( r = (r_1, \ldots, r_n) \) where
\[ r_i = \int_0^t f(s)\lambda(X_i(x_s), X_j(x_s)) \, dw_j. \]
In view of Theorems 2.1 and 2.2, it is natural to consider the Wiener space \( C_0(\mathbb{R}^n) \) as a manifold with tangent bundle \( \cup_w T_w C_0(\mathbb{R}^n) \), where each fiber \( T_w C_0(\mathbb{R}^n) \) consists of paths of the form (3.22).

For each such path \( r = r(x) \), equation (3.19) produces a vector field \( \eta \) on \( C_0(M) \) that is then lifted to a vector field \( \tilde{r} \) on \( C_0(\mathbb{R}^n) \) by equation (3.21). We summarize this construction as follows. Define
\[ H(r) = (r, \eta), \quad r \in TC_0(\mathbb{R}^n) \]
and let
\[ \pi : TC_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n) \]
denote the bundle projection. Then the chain of maps in Theorem 3.4 and its proof is illustrated by the following commutative diagram

**Remark** We note that every adapted vector field on \( C_0(M) \) with an admissible lift to the Wiener space is obtained from Theorem 3.4. Denote the process \( \eta \) in Theorem 3.4 by \( \eta'. \) Then we have

**Proposition 3.5** Suppose \( \eta \) is an adapted vector field on \( C_0(M) \) such that
\[ \eta = dg(w)r \]
for some \( r \in TC_0(\mathbb{R}^n) \). Then there exists \( \tilde{r} \in TC_0(\mathbb{R}^n) \) such that \( \eta = \eta^{\tilde{r}} \).

**Proof.** This follows immediately from equations (3.18) and (3.19). We define \( \tilde{r} \) by

\[
\tilde{r}_i = r_i + \int_0^t G_{ij}^{\eta} \circ dw_j, \quad i = 1, \ldots, n.
\]

### 3.D Gradient systems

Suppose \( M \) is an isometrically embedded submanifold of a Euclidean space \( \mathbb{R}^N \) (by Nash’s embedding theorem, every finite-dimensional Riemannian manifold can be realized this way). Define \( X_i = P e_i, \ 1 \leq i \leq N \) where \( e_1, \ldots, e_N \) is the standard orthonormal basis of \( \mathbb{R}^N \) and \( P(x) \) is orthogonal projection onto the tangent space \( T_x M \). Then the infinitessimal generator of the process \( x \) defined by

\[
dx_t = \sum_{i=1}^n X_i(x_t) \circ dw_i
\]

is \( 1/2 \Delta_B \), where \( \Delta_B \) is the Laplace-Beltrami operator on \( M \). Thus \( x \) is a Brownian motion in \( M \).

The purpose of this subsection is to recover Driver’s integration by parts formula for Brownian motion on a Riemannian manifold (cf. [6]), in this setting.

We note first that in this case the connection \( \nabla \) coincides with the Levi-Civita connection on \( M \) (cf. [8]), hence the tensor \( T \) defined in (3.17) vanishes. In this case Theorem 3.4 yields the following result.

**Theorem 3.6** If \( r \) is any (random, \( x \)-adapted) path such that \( \dot{r} \in L^2[0, T] \) then the vector field \( \eta \) defined by the covariant equation

\[
\frac{D\eta}{dt} = X_i \dot{r}_i
\]

is admissible.

In particular, let \( h \) be any path in the Cameron-Martin space of \( T_o(M) \) and define

\[
r_i = \int_0^t < U_t h_t, X_i > dt, \quad i = 1, \ldots, N
\]

where \( U_t \) denotes stochastic parallel translation along the path \( x \). Then the process \( \eta \) in (3.24) becomes \( U_t h_t \) and we obtain Driver’s result:

**Corollary 3.7** For every path \( h \) in the Cameron-Martin space of \( T_o(M) \), the vector field \( \eta_h \equiv U_t h_t \) is admissible.

The remainder of this section is concerned with computing the divergence of \( \eta \) and showing that it coincides with the formula involving Ricci curvature given in [6]. Rather than using formula (3.21) in Theorem 3.4, we rework the computation, which admits of a simpler formulation in the present setting.

According to (3.21), the process \( \tilde{r} \) defined by

\[
\tilde{r}_j = r_j - \int_0^t G_{ij}^{\eta} \circ dw_i
\]

is 1/2\( \Delta_B \), where \( \Delta_B \) is the Laplace-Beltrami operator on \( M \). Thus \( x \) is a Brownian motion in \( M \).

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According to (3.21), the process \( \tilde{r} \) defined by

\[
\tilde{r}_j = r_j - \int_0^t G_{ij}^{\eta} \circ dw_i
\]
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is a lift of $\eta$, where

$$G_{ij}^\eta = \langle \nabla_{\eta_i} X_i, X_j \rangle - \langle \nabla_{\eta_j} X_j, X_i \rangle.$$  \hspace{1cm} (3.26)

The following result was proved in [1]. We give here a simpler, coordinate-free version of the proof in [1].

**Lemma 3.8** The vector fields $X_i$ satisfy the symmetry property

$$\langle \nabla_Y X_i, Z \rangle = \langle \nabla_Z X_i, Y \rangle$$

for any vector fields $Y$ and $Z$ on $M$.

**Proof.** Combining

$$Z \langle X_i, Y \rangle = \langle (ZP)e_i, Y \rangle + \langle Pe_i, Z(Y) \rangle$$

and

$$Z \langle X_i, Y \rangle = Z \langle e_i, Y \rangle = \langle e_i, Z(Y) \rangle$$

we have

$$\langle e_i, Z(Y) \rangle = \langle (ZP)e_i, Y \rangle + \langle Pe_i, Z(Y) \rangle .$$

Interchanging $Z$ and $Y$ and subtracting gives

$$\langle e_i, [Z, Y] \rangle = \langle \nabla_Z X_i, Y \rangle - \langle \nabla_Y X_i, Z \rangle + \langle Pe_i, [Z, Y] \rangle$$

and the result follows.

Using Lemma 3.8, we may write (3.26) in the form

$$G_{ij}^\eta = \langle \nabla_{X_j} X_i, \eta \rangle - \langle \nabla_{X_i} X_j, \eta \rangle .$$

We use this form to compute the Itô-Stratonovich correction term in the stochastic integral in (3.25), namely $\frac{1}{2} dG_{ij}^\eta dw_i$. In view of (3.24), we have

$$dt \langle \nabla_{X_j} X_i, \eta \rangle = \langle \nabla_{X_i} \nabla_{X_j} X_i, \eta \rangle dt + \langle \nabla_{X_j} X_i, X_k \rangle \dot{r}_k dt$$

Substituting this, and a similar expression into (7) and ignoring the terms in $dt$ (which will not affect the result), gives

$$\frac{1}{2} dG_{ij}^\eta dw_i = \frac{1}{2} \{ \langle \nabla_{X_i} \nabla_{X_j} X_i, \eta \rangle - \langle \nabla_{X_j} \nabla_{X_i} X_i, \eta \rangle \} dt$$

$$= \frac{1}{2} \langle \nabla_{X_i} [X_j, X_i], \eta \rangle dt .$$

Hence our lift $\tilde{r}$ of $\eta$ is the process

$$r_j - \int_0^T G_{ij}^\eta dw_i - \frac{1}{2} \int_0^T \langle \nabla_{X_i} [X_j, X_i], \eta \rangle dt$$

Applying Theorems 2.1 and 2.2 as in the proof of Theorem 3.4, then conditioning with respect to $x$ thus gives

$$\text{Div}(\eta) = E \left[ \int_0^T \left( \dot{r}_j - \frac{1}{2} \langle \nabla_{X_i} [X_j, X_i], \eta \rangle \right) dw_j / x \right].$$
Divergence theorems in path space III

\[
= \int_0^T \left( < \frac{D\eta}{dt}, X_j(x_t) > - \frac{1}{2} < \eta_t, \nabla_{X_i}[X_j, X_i](x_t) > \right) dE[w_j/x]. \tag{3.27}
\]

The remaining arguments in this section are due to Bruce Driver. I am greatly obliged to Bruce for his help in this matter.

**Lemma 3.9** (Driver)

\[
\nabla_{X_i}[X_i, X_j] = \text{Ric}(X_j) - P(LQ)Qe_j
\]

where

\[
L = \sum_{i=1}^n X_i^2.
\]

**Proof.** Define

\[
I \equiv \nabla_{X_i}[X_i, X_j] = \nabla_{X_i}[\nabla_{X_i}X_j - \nabla_{X_j}X_i]
\]

and

\[
II \equiv \text{Ric}(X_j) = R(X_j, X_i)X_i
= (\nabla_{X_i}\nabla_{X_i} - \nabla_{X_i}\nabla_{X_j} - \nabla_{[X_j, X_i]}X_i). \tag{3.28}
\]

Differentiating the obvious relation \(< X_i, X_i > = d\) and using Lemma 3.8, we have \(\forall Z \in T_x M\),

\[
< \nabla_Z X_i, X_i > = < \nabla_{X_i}X_i, Z > = 0,
\]

hence

\[
\nabla_{X_i}X_i \equiv 0. \tag{3.29}
\]

Substituting this into (3.28), we have

\[
II = -\nabla_{X_i}\nabla_{X_i}X_i - \nabla_{[X_j, X_i]}X_i
\]

thus

\[
I - II = \nabla_{X_i}\nabla_{X_j}X_j + \nabla_{[X_j, X_i]}X_i. \tag{3.30}
\]

Let \(Q\) denote the projection \(I - P\). Then we have

\[
\nabla_{X_i}\nabla_{X_i}X_j = (PX_iPX_i)X_j
= PX_i(P(X_iX_i)e_j)
= PX_i((X_iX_i)e_j).
\]

(here, and repeatedly in the sequel, we use the simple identity \(P(X_iP) = (X_iP)Q\))

\[
= P(LPQ)e_j + P(X_iP)(X_iQ)e_j
= -P(LQ)e_j + P(X_iP)(X_iQ)e_j. \tag{3.31}
\]

Furthermore,

\[
\nabla_{[X_j, X_i]}X_i = P([X_j, X_i]P)e_i
= ([X_j, X_i]P)Qe_i - ([X_jX_i] - X_i[X_j])PQe_i
= ([X_jX_i]P)e_i = P([X_jX_i]P)e_i, \tag{3.32}
\]

where \(L \equiv \sum_{i=1}^n X_i^2\).
where we used the fact that \([X_i(X_j)]Q|e_i = 0\), as can be seen by evaluating the expression on an orthonormal basis \(\{e_i\}\) such that for each \(i\), either \(e_i \in T_xM\) or \(e_i \in T_xM^\perp\). An easy calculation now shows that (3.32) is equal to

\[
P(X_iP)(X_jP)e_i = (X_iP)(X_jP)X_i
\]

(3.33)

\[
= -P(X_iP)(X_iQ)e_j
\]

(3.34)

where (3.33) follows from the relation

\[
(X_iP)(X_jP) = (X_jP)(X_iP)
\]

(3.35)

The proof of (3.35) is as follows

\[
0 = Q[X_i,X_j] = Q[(X_iP)e_j - (X_jP)e_i]
\]

\[
= [(X_iP)Pe_j - (X_jP)Pe_i].
\]

Combining (3.33) with (3.31), we have the Lemma.

**Proposition 3.10** Suppose \(F(x)\) is an \(L^2\) functional of \(x\). Then \(F\) has the representation

\[
F = EF + \int_0^T \alpha_s \cdot P(x_s)dw_s
\]

(3.36)

where is a non-anticipating \(x\)-measurable process.

**Proof.** We consider the anti-development \(b\) of \(x\) onto \(T_xM\), defined by the functional

\[
\text{Stratonovich equation}
\]

\[
\begin{align*}
\text{db} &= U^{-1}_t \circ dx_t = U^{-1}_t P(x_t) \circ dw \\
\alpha_s &= \text{Stratonovich drift}
\end{align*}
\]

(3.37)

where \(u\) denotes stochastic parallel translation along \(x\). As is well known, the map \(x \mapsto b\) is a measure theoretic isomorphism. Hence \(F(x)\) is a function of \(b\). Furthermore the process \(b\) is a standard Brownian motion in \(T_xM\) and thus has zero drift\(^4\). It follows that the Itô correction term in (3.37) vanishes and the equation can be written

\[
\text{db} = U^{-1}_t P(x_t)dw.
\]

By the standard martingale representation theorem, we have

\[
F(x) = f(b) = Ef(b) + \int_0^T \tilde{\alpha}_s \cdot \text{db}_s
\]

\[
= EF(x) + \int_0^T \tilde{\alpha}_s \cdot U^{-1}_s P(x_s)dw
\]

and the Proposition follows.

**Proposition 3.11**

\[
E[w_t/x] = \beta_t \equiv \int_0^t P(x_s)dw.
\]

\(^4\)It is a straightforward, though somewhat tedious, exercise to give a direct proof of this fact. The proof uses the fact that the Itô-Stratonovich term in equation (3.23) lies in the normal bundle to \(M\).
Proof. Note firstly, the process \( \beta \) is \( x \)-measurable, since equation (3.23) expressed in Ito form, yields
\[ \beta_t = x_t - \frac{1}{2} \int_0^t DX_i(x_s)X_i(x_s)ds. \]
Let \( F \) be an arbitrary \( L^2 \) functional of \( x \). It suffices to show that
\[ E[F(x)w_t] = E[F(x)\beta_t]. \tag{3.38} \]
We start with the RHS of (3.38). According to (3.36) and the isometry property of Itô integrals, we have
\[
E[F(x)\beta_t] = E\left[ \left( EF + \int_0^T \alpha_s \cdot \alpha_s P(x_s)dw \right) \cdot \int_0^t P(x_s)dw \right] \\
= E\left[ \int_0^t \alpha_s P(x_s)dw \cdot \int_0^T P(x_s)dw \right] \\
= E\left[ \int_0^t <\alpha_s, P(x_s)e_k > P(x_s)e_k ds \right]
\]
(where \( \{e_k\} \) is a standard orthonormal basis of \( \mathbb{R}^n \) )
\[
= E\left[ \int_0^t P(x_s)\alpha_s ds \right] \\
= E[F(x)w_t]
\]
and (3.38) is proved.

Combining Lemma 3.9, Proposition (3.11), and (3.27), we obtain Driver’s formula for the vector field obtained by parallel translation of a Cameron-Martin path in \( T_0M \) along the diffusion process \( x \).

**Theorem 3.12** The divergence of the vector field \( \eta_t = U_t h_t \) is given by
\[
\text{Div}(\eta) = \int_0^T \frac{D\eta}{dt} < \frac{1}{2} \text{Ric}(\eta), X_j(x_t) > dw_j \\
= \int_0^T \frac{D\eta}{dt} < \frac{1}{2} \text{Ric}(\eta), d\beta >.
\]

**Remark** With hindsight and a little reverse engineering, we can modify the proof of Theorem 3.12 so as to avoid the use of Proposition 3.11. Define the process \( \rho \) by
\[
\rho_j = \tilde{r}_j - \frac{1}{2} \int_0^t < P(LQ)e_j, \eta_s > ds
\]
where \( \tilde{r} \) is as in (3.25). Then, according to Theorem 2.6 (eq. (2.4)) which we may write in the form
\[
\eta_t = Y_t \int_0^t Z_s P(x_s) \circ dr,
\]
we will have
\[
Dg(w)\rho = Dg(w)\tilde{r} - \frac{1}{2} Y \int_0^t Z_s P e_j < P(LQ)e_j, \eta_s > ds \tag{3.39}
\]
\[ Dg(w)\tilde{r} \]
since the integrand in (3.39) vanishes, i.e. \( \rho \) is also a lift of \( \eta \). Furthermore the divergence of \( \rho \) in Wiener space is given by

\[ E\left[ \int_0^T (\dot{r}_j - \frac{1}{2} <P(LQ)Qe_j, \eta_t> - \frac{1}{2} <\nabla X_i[X_j, X_i], \eta_t>)dw_j \right] \]
(by Proposition 3.9)

\[ = E\left[ \int_0^T (\dot{r}_j - \frac{1}{2} <\text{Ric}(X_j), \eta_t>)dw_j \right] . \]

Conditioning on \( x \), we obtain the formula

\[ \text{Div}(\eta) = E\left[ \int_0^T (\dot{r}_j - \frac{1}{2} <\text{Ric}(X_j), \eta_t>)d\beta_j \right] \]
where, as before, \( \beta \) is the process \( E[w/x] \).

It seems worthwhile to state the construction employed above as a separate conclusion.

**Proposition 3.13.** The process \( \tilde{r} \) defined by

\[ \tilde{r}_j = \int_0^t \langle Uh_t + \frac{1}{2} \text{Ric}(\eta_t), X_j(x_t) \rangle dt - \int_0^t G^0_i dw_i \]
is a lift of the vector field \( \eta_t \equiv U_t h_t \) to \( C_0(\mathbb{R}^n) \).

### 4 Linearly independent diffusion coefficients

In this section we consider the special case where the vectors \( X_1(x), \ldots, X_n(x) \) are linearly independent at every point \( x \in M \) (in the elliptic case there is a topological obstruction to this condition, i.e. if \( M \) has non-zero Euler characteristic then it is impossible. However, the condition is reasonable in the non-elliptic case). As we shall see, this implies that the Wiener path \( w \) is a function of the solution \( x \) of the SDE (1.1) i.e.

\[ w = \Theta(x) \]
where \( \Theta \) is a measurable function on \( C_0(M) \). In this case the following simplified version of the method used in Section 3 produces admissible vector fields on \( C_0(M) \).

Choose \( r \) to be any process of the form

\[ r_t = \int_0^t A(s)dw_s + \int_0^t B(t)dt, \quad t \in [0, T] \tag{4.1} \]

where \( A \) and \( B \) are continuous adapted processes with values in \( so(n) \) and \( \mathbb{R}^n \) and define \( \eta \) by (2.4), i.e.

\[ \eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_s. \]
By Theorems 2.1, 2.2 and 2.5, \( r \) is an admissible lift of \( \eta \), hence \( \eta(w) = \eta(\Theta(x)) \) is an admissible vector field on \( C_o(M) \).

We now study how the formulae in Section 3 reduce in the linearly independent case. As before, define \( X(x) : \mathbb{R}^n \to T_xM \) by

\[
X(x)(h_1, \ldots, h_n) = X_i(x)h_i.
\]

We will need the following result.

**Lemma 4.1** The vectors \( X_1(x), \ldots, X_n(x) \) are linearly independent if and only if

\[
X(x)^*X(x) = I_{\mathbb{R}^n}.
\]

Since Lemma 4.1 is elementary, the proof will be omitted.

Assume now that \( \{X_1, \ldots, X_n\} \) are linearly independent. Then Lemma 4.1 enables us to solve the SDE (1.1) for \( w \) in terms of \( x \) and obtain

\[
dw = X(x)^* \circ dx,
\]

thus \( w = \theta(x) \), as claimed above. We also have

**Corollary to Lemma 4.1** For \( a_i \in C^\infty(M), i = 1, \ldots, n \) and \( V \in TM \)

\[
\nabla_V(a_iX_i) = V(a_i)X_i.
\]

In particular

\[
\nabla_V X_i = 0, \ i = 1, \ldots, n.
\]

The corollary implies that the functions \( G^{ij} \) in (3.4) are all zero. Furthermore, the tensors \( T_I \) in Section 3 take the form

\[
T_I(aX_i) = a[X_i, V_I], \ i = 1, \ldots, n
\]

for \( a \in C^\infty(M) \). Theorem 3.1 then becomes

**Theorem 4.2** Suppose the process \( r \) is defined as in (4.1) and the functions \( h_I \) are chosen to satisfy

\[
dh_I = (X_i, V_I) \circ dr_i - ([X_i, V_J], V_I)h_J \circ dw_i \tag{4.2}
\]

\[
h_I(0) = 0.
\]

Then the vector field \( \eta = h_I V_I \) is admissible and

\[
\text{Div}(\eta) = \int_0^T B_i(t)dw_i.
\]

**Example 4.3**

Let \( M \) be the Heisenberg group, i.e. the Lie group \( \mathbb{R}^3 \) with group multiplication

\[
(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}(a_1b_2 - b_1a_2)).
\]
Let

\[
X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}
\]

\[
X_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z}
\]

and define \( V_1 = X_1, V_2 = X_2, \) and

\[
V_3 = [V_1, V_2] = \frac{\partial}{\partial z}.
\]

Then

\[
[X_1, V_2] = V_3
\]

\[
[X_2, V_1] = -V_3
\]

\[
[X_i, V_j] = 0, \quad i + j \neq 3.
\]

Thus equation (4.2), which we write in the form

\[
V_I \circ dh_I = X_i \circ dr_i - [X_i, V_I] h_I \circ dw_i
\]

becomes

\[
V_1 \circ dh_1 + V_2 \circ dh_2 + V_3 \circ dh_3
\]

\[
= X_1 \circ dr_1 + X_2 \circ dr_2 + V_3 (h_1 \circ dw_2 - h_2 \circ dw_1).
\] (4.3)

Since the vectors \( \{V_1, V_2, V_3\} \) are linearly independent equation (4.3) has a unique solution, given by

\[
h_1 = r_1
\]

\[
h_2 = r_2
\]

\[
h_3 = \int_0^1 r_1 \circ dw_2 - r_2 \circ dw_1.
\] (4.4)

As point of interest, we note that if \((w_1, w_2)\) is substituted for \((r_1, r_2)\) then the preceding integral becomes the Levey area (it should be noted, however, that in the present context \((w_1, w_2)\) is not an allowable choice for \(r\)).

References


