Non-Ramanujancy of Euclidean graphs of order $2^r$

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Abstract

Graphs are attached to the $n$-dimensional space $\mathbb{Z}_2^n$ where $\mathbb{Z}_2$ is the ring with $2^r$ elements using an analogue of Euclidean distance. The graphs are shown to be non-Ramanujan for $r = 4$. Comparisons are made with Euclidean graphs attached to $\mathbb{Z}_p^n$ for $p$ an odd prime. The percentage of non-zero eigenvalues of the adjacency operator attached to these finite Euclidean graphs is shown to tend to zero as $n$ tends to infinity.

Keywords: Ramanujan graph; Euclidean graph; Spectra of graphs

1. Introduction

We study a finite analogue of real symmetric spaces, namely the finite Euclidean space $\mathbb{Z}_2^n$ over the finite ring $\mathbb{Z}_2$ using a finite analogue of the usual Euclidean distance. The paper of Medrano et al. [6] addresses the question of how graphs attached to $\mathbb{Z}_p^n$ for odd prime $p$ behave but avoided the case $p = 2$ as the case is unwieldy. At the end of this introduction, comparisons are made between the graphs where $p = 2$ and $p$ an odd prime.

Part of the motivation to study the graphs of order $2^r$ is due to the binary nature of computers and to the search for bipartite graphs. Another motivating factor is to find finite analogues of real Euclidean space $\mathbb{R}^n$ and analogies to the harmonic analysis on $\mathbb{R}^n$ as discussed in Terras [9]. This has been of interest to physicists for some...
time in areas such as the statistical theory of the energy levels of a complex physical
system. Here, we study analogous properties: the distribution of the eigenvalues and
eigenfunctions associated to Euclidean graphs.

As in Medrano et al. [6], we would like to know if replacing \( \mathbb{R} \) by the ring \( \mathbb{Z}_2 \) is
better or worse than replacing \( \mathbb{R} \) by \( \mathbb{F}_p \) or \( \mathbb{Z}_p \) where \( p \) is an odd prime.

The measure of better is tied to the search to find new examples of Ramanujan
graphs as defined by Lubotzky et al. [4]. A connected, \( k \)-regular graph is Ramanujan
iff for all eigenvalues \( \lambda \) of the adjacency matrix with \( |\lambda| \neq k \), we have

\[
|\lambda| \leq 2\sqrt{k-1}.
\]

See Li [3] and Lubotzky et al. for more information on Ramanujan graphs.

Ramanujan graphs are of interest as they provide the best family of explicitly known
expander graphs and as their properties are similar to random regular graphs. They are
also of interest to number theorists as their associated Ihara zeta functions satisfy the
Riemann hypothesis (see [8]) and as they lead to rapid confusion for the random walker
(see [7]).

1.1. Outline

Theorem 3 relates the degree of the graph modulo 8 to the degree of the correspond-
ingraphs modulo 4. Theorem 8 proves that Euclidean graphs modulo 2\(^r\) are bipartite
and Theorem 4 shows that graphs of the same order are isomorphic for \( a \equiv 1 \mod 4 \) in
even dimension.

Theorem 9 relates the eigenvalues of the adjacency matrices of the graphs \( X_{2^r}(n,a) \)
(defined after equation 1 below) to the eigenvalues of the graphs \( X_{2^{r-1}}(n,a) \) while
Corollary 10 extends this property to lower \( n \)-dimensional graphs. Theorem 11 shows
that when \( a \equiv 1 \mod 4 \), the graphs \( X_{2^r}(n,a) \) are not Ramanujan for \( r \geq 4 \). We conclude
with Theorem 12 which shows that the percentage of non-zero eigenvalues tends to
zero as \( n \) tends to infinity.

1.2. Comparison of Euclidean graphs over the finite ring \( \mathbb{Z}_{2^r} \) and \( \mathbb{Z}_{p^r} \) where \( p \) is an
odd prime

Similarities:

1. Euclidean graphs over \( \mathbb{Z}_{2^r} \) are non-Ramanujan for \( r \geq 4 \). For \( p \) an odd prime,
Euclidean graphs over \( \mathbb{Z}_{p^r} \) are non-Ramanujan for \( r \geq 2 \) unless \( p = 3 \) when \( r = 2 = n \).
See Medrano et al. [6], Theorem 2.5.

2. There is only one isomorphism class of these Euclidean graphs for all \( p \) and
chosen \( a \) in even dimensions. See Theorem 4 below and Theorem 5 in Medrano
et al. [6].

3. The percentage of non-zero eigenvalues of the adjacency matrix associated to
both types Euclidean graphs tends to 100% as \( n \) tends to infinity. Although observed
in Medrano et al. it is proved below for \( p = 2 \).
Differences:

1. Over \( \mathbb{Z}_{2^r} \), the graphs are bipartite unlike the graphs over \( \mathbb{Z}_{p^r} \) where \( p \) is an odd prime.
2. Although the degree of the graphs of \( X_{p^r}(n, a) \) can be related to the degree of \( X_p(n, a) \) for odd prime \( p \), the degree of \( X_{2^r}(n, a) \) can only be related to \( X_8(n, a) \) in a nice way. This bars us from proving that \( X_{2^r}(n, a) \) is non-Ramanujan for \( r = 2 \) and 3 as mentioned below.

2. Finite Euclidean graphs over rings

Let \( \mathbb{Z}_{2^r} \) be the ring \( \mathbb{Z}/2^r \mathbb{Z} \). The finite Euclidean space \( \mathbb{Z}_{2^r}^n \) consists of column vectors \( x \) with the \( j \)th entry \( x_j \) in \( \mathbb{Z}_{2^r} \). Define the distance between \( x \) and \( y \) in \( \mathbb{Z}_{2^r}^n \) by

\[
d(x, y) = (x - y)^t \cdot (x - y) = \sum_{j=1}^{n} (x_j - y_j)^2.
\]

This distance has values in \( \mathbb{Z}_{2^r} \) and is point-pair invariant.

Given \( a \) in \( \mathbb{Z}_{2^r} \) define the Euclidean graph \( X_{2^r}(n, a) \) with vertices the vectors in \( \mathbb{Z}_{2^r}^n \) and two vectors being adjacent if \( d(x, y) = a \).

A Cayley graph \( X(G, S) \) for an additive group \( G \) and symmetric edge set \( S \subseteq G \) has vertices the elements of \( G \) and edges between vertices \( x \) and \( y = x + s \) for \( x, y \in G \), \( s \in S \). The set \( S \) is symmetric if \( s \in S \) implies \( -s \in S \). If \( S \) is a set of generators of \( G \) then the Cayley graph is connected.

Let

\[
S_{2^r}(n, a) = \{ x \in \mathbb{Z}_{2^r}^n \mid d(x, 0) = a \}.
\]

Thus, the Euclidean graph \( X_{2^r}(n, a) \) is a Cayley graph for the additive group \( \mathbb{Z}_{2^r}^n \) with edge set \( S_{2^r}(n, a) \).

**Theorem 1.** For the Euclidean graph \( X_{2^r}(2, a) \), the order of \( S_{2^r}(2, a) \) is given by

\[
|S_{2^r}(2, a)| = \begin{cases} 
2^{r+1} & \text{for } a \equiv 1 \mod 4, \\
0 & \text{for } a \equiv 3 \mod 4.
\end{cases}
\]

We note that the cases where \( a \equiv 0 \mod 2 \) are not considered as the graphs are not connected as shown in Theorem 7 below.

**Proof.** Case 1: \( a \equiv 1 \mod 4 \). Since the formula holds for \( r = 2 \) and 3, assume that \( r \geq 4 \) and that it is true for \( r - 1 \). Let \( s = [s_1, s_2] \) in \( S_{2^{r-1}}(2, a) \) solve \( s_1^2 + s_2^2 \equiv a \mod 2^{r-1} \). Then exactly one of the \( s_i \) for \( i = 1, 2 \) is odd. Without loss of generality assume it is \( s_2 \). Define \( s' = [s_1, s_2 + 2^{r-2}] \) in \( S_{2^{r-1}}(2, a) \). For \( r \geq 4 \),

\[
 s' \cdot s - s' \cdot s' \equiv 2^{r-1} s_2 \equiv 0 \mod 2^r.
\]
Since $s_2$ is odd, $2^{r-1}s_2$ is not congruent to 0 modulo $2^r$. This implies that only one solution of $s, s'$ lifts to a solution modulo $2^r$. Thus of the four possible solutions modulo $2^{r-1}$, only two of them lift to a solution modulo $2^r$. This gives

$$|S_{2^r}(2, a)| = 2|S_{2^{r-1}}(2, a)| = 2^{r+1}.$$  

Case 2: $a \equiv 3 \mod 4$. It is easily verified that $|S_{4}(2, a)| = 0$ and $|S_{8}(2, a)| = 0$. Since any solution modulo $2^r$ can be reduced to a solution modulo 4, there are no solutions $s = [s_1, s_2]^t$ to $s_1^2 + s_2^2 \equiv a \mod 2^r$.  

**Theorem 2.** Given the Euclidean graph $X_4(n, a)$ where $n \geq 3$ and $a \equiv 1 \mod 2$, we have

$$|S_4(n, a)| = \begin{cases} 2^n \sum_{j=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4j+1} & \text{if } a \equiv 1 \mod 4, \\ 2^n \sum_{j=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4j+3} & \text{if } a \equiv 3 \mod 4. \end{cases}$$  

**Proof.** Let $s = [s_1, \ldots, s_n]^t \in S_4(n, a)$ solving $s^t \cdot s \equiv a \equiv m \mod 4$ where $m$ is either 1 or 3. Then for some $j \in \{0, \ldots, \lfloor (n-m)/4 \rfloor \}$ we have $4j + m$ of the entries of $s$ being odd. Thus, $n - (4j + m)$ of the remaining entries of $s$ are even. Since there are two possible choices for each of these entries, there are $2^n$ solutions $s \in S_4(n, a)$ for a given $j$.  

**Theorem 3.** For the Euclidean graph $X_8(n, a)$ where $a \equiv 1 \mod 4$ and $n \geq 2$, 

$$|S_8(n, a)| = 2^{n-1}|S_4(n, a)| + \begin{cases} 0 & \text{if } n \not\equiv a \mod 4, \\ 2^{n-1} & \text{if } n \equiv a \mod 8, \\ -2^{n-1} & \text{if } n \equiv a + 4 \mod 8. \end{cases}$$  

**Proof.** The proof is similar to that of Theorem 2.  

Let $s \in S_4(n, a)$ solving $s^t \cdot s \equiv a \equiv m \mod 4$ where $m$ is 1 or 3. Then for $j \in \{0, \ldots, \lfloor (n-m)/4 \rfloor \}$ we have $4j + m$ of the $s_j$ congruent to 1 or 3 mod 4 and $n - (4j + m)$ of the remaining $s_j$ congruent to 0 or 2 mod 4. Modulo 8, there are four solutions to $s_j^2 \equiv 1 \mod 8$ and four solutions to $s_j^2 \equiv 0$ or 4 mod 8.

Case 1: $n \not\equiv a \mod 4$. This means $j \not= (n-m)/4$. Since $4j + 1$ of the $s_j$ are odd all possible $s$ have an even entry $s_j$. Replacing the even $s_j$ with $s_j + 2$ gives another solution modulo 4. Only one of the solutions, $s$ or $s$ with its $i$th entry replaced with $s_i + 2$, gives $s^t \cdot s - m \equiv 0 \mod 8$. Since we have to restrict one even $s_i$, we have $4^n/2 = 2^{n-1}$ solutions modulo 8. Thus, for $n \not\equiv a \mod 4$, we have

$$|S_8(n, a)| = 2^{n-1} \sum_{j=0}^{\lfloor (n-m)/4 \rfloor} \binom{n}{4j+m} = 2^{n-1}|S_4(n, a)|.$$
Cases 2 and 3: $n \equiv a \mod 4$. The argument follows Case 1 except for the term where $j = (n - m)/4$. At this $j$ there exist solutions $s$ with all odd $s_i$. If $n \equiv a \mod 8$, then we obtain an additional $2^2n$ solutions modulo 8. Otherwise, if $n \equiv a + 4 \mod 8$ we obtain no additional solutions modulo 8. Since we have accounted for these solutions modulo 4, we need to add or subtract these $2^2n/2 = 2^{n-1}$ additional solutions which gives the stated result.

We reserve relating the order of $S_2(n, a)$ to $S_8(n, a)$ until Corollary 10 as the corollary relates not only to the order of the symmetric sets and to the value of the largest eigenvalue, but relates all higher $n$-dimensional eigenvalues to lower dimensional ones.

**Theorem 4.** Let $\mathbf{c} = [c_1c_2]^t$ solving $c \equiv c_1^2 + c_2^2 \mod 2^r$ for all $c_1, c_2$ in $\mathbb{Z}_{2^r}$. Then for $a \equiv 1 \mod 4$, every Euclidean graph $X_2^r(n, a)$ is isomorphic to $X_2^r(n, ca)$ for fixed $r$ and even $n$.

**Proof.** The proof follows the one for the field case in Medrano et al. [5].

Assume $n = 2t$.

For $\mathbf{c} = [c_1c_2]^t$ solving $c \equiv c_1^2 + c_2^2 \mod 2^r$, set

$$M_c = \begin{bmatrix}
    c_1 & c_2 & 0 \\
    -c_2 & c_1 & 0 \\
    & & \ddots \\
    0 & c_1 & c_2 \\
    & -c_2 & c_1
\end{bmatrix} \quad (t - 2 \times 2 \text{ blocks down the diagonal}).$$

(Note: The $\det(M_c) \neq 0$ since $c \equiv 1 \mod 4$. Thus, $M_c^{-1}$ exists.)

Given a solution $\mathbf{y} = [y_1 \ldots y_n]^t$ in $X_2^r(n, a)$, let $\mathbf{y}^* = M_c \mathbf{y}$. Then $M_{\mathbf{y}^*} = M_c M_{\mathbf{y}}$ and $\det(M_{\mathbf{y}^*}) = \det(M_c) \det(M_{\mathbf{y}})$ imply that

$$d(\mathbf{y}^*, \mathbf{0}) = (y_1^2 + y_2^2) + \cdots + (y_n^2 + y_1^2) = c(y_1^2 + y_2^2 + \cdots + y_{n-1}^2 + y_n^2) = c \cdot d(\mathbf{y}, \mathbf{0}).$$

Thus, the map $\mathbf{y} \rightarrow \mathbf{y}^* = M_c \mathbf{y}$ gives a graph isomorphism from $X_2^r(n, a)$ onto $X_2^r(n, ca)$ since if $\mathbf{y}$ and $\mathbf{z}$ are adjacent vertices of $X_2^r(n, a)$ we have

$$d(\mathbf{y}^*, \mathbf{z}^*) = d(M_c \mathbf{y}, M_c \mathbf{z}) = d(M_c \mathbf{y} - M_c \mathbf{z}, \mathbf{0}) = d(M_c (\mathbf{y} - \mathbf{z}), \mathbf{0}) = ca. \quad \square$$

2.1. Eigenvalues of Euclidean graphs

One of the main goals is to study the spectrum of the adjacency operator $A_a$ acting on functions $f : \mathbb{Z}_{2^r}^n \rightarrow \mathbb{C}$ by

$$A_a f(x) = \sum_{d(x, y) = a} f(y).$$
Lemma 5. The function $e^{2\pi ib's/2'}$ where $b \in \mathbb{Z}_2^n$ is an eigenfunction of the adjacency operator $A_a$ of $X_{2'}(n, a)$ corresponding to the eigenvalue

$$\lambda_b^{(r)} = \sum_{d(s, b) = a} e^{2\pi ib's/2'}.$$  

Moreover, as $b$ runs through $\mathbb{Z}_2^n$, the $e^{2\pi ib's/2'}$ form a complete orthogonal set of eigenfunctions of $A_a$. Using the inner product on $f, g: \mathbb{Z}_2^n \rightarrow \mathbb{C}$ defined by

$$\langle f, g \rangle = \sum_{x \in \mathbb{Z}_2^n} f(x)g(x)$$

it follows that every eigenvalue of $X_{2'}(n, a)$ has the form $\lambda_b$ for some $b \in \mathbb{Z}_2^n$.

Proof. See Medrano et al. [5, Proposition 2.2]. □

Theorem 6. All Euclidean graphs $X_{2'}(n, a)$ where $a \equiv 1 \mod 4$ for all $n \geq 2$ and where $a \equiv 3 \mod 4$ for $n \geq 4$ are connected.

Proof. Claim. The multiplicity of $k$, the degree of the graph, is one.

Let $s = [s_1, \ldots, s_m]$ in $S_{2'}(n, a)$. Then we need to find $b$ such that

$$\lambda_b^{(r)} = \sum_{d(s, b) = a} e^{2\pi ib's/2'} = k = |S_{2'}(n, a)|.$$  

This implies that

$$e^{2\pi i(b_1s_1 + \cdots + b_ns_n)/2'} = 1 \quad \text{for all } s \in S_{2'}(n, a).$$

Thus we need $b_1s_1 + \cdots + b_ns_n \equiv 0 \mod 2'$ for all $s$ such that $s_1^2 + \cdots + s_m^2 \equiv a \mod 2'$.

Choose $s' = [s'_1, \ldots, s'_n]$ in $S_{2'}(n, a)$ where $s'_1$ is odd. (We can choose $s'_1$ odd as there exists an odd entry in every element of $S_{2'}(n, a)$). Then $s'' = [-s'_1, \ldots, s'_n]$ is in $S_{2'}(n, a)$. Since both of these satisfy $b's \equiv 0 \mod 2'$, we have $b's' - b's'' \equiv 2b_1s_1 \equiv 0 \mod 2'$. Since $s_1$ is odd, $2'^{-1}b_1$. Similarly, since permutations of the coordinates of $s'$ preserve solutions, we have $2'^{-1}b_i$ for all $i = 1, \ldots, n$. Since $b_i$ is in $\mathbb{Z}_{2'}$, we have that $b_i \equiv 0$ or $2'^{-1}$ for all $i$. If $b_j = 2'^{-1}$ for $j = i_1, \ldots, i_m$ and $b_i = 0$ for all other $i$, then $b_1s_1 + \cdots + b_ns_n \equiv 0 \mod 2'$ becomes

$$2'^{-1}(s_{i_1} + \cdots + s_{i_m}) \equiv 0 \mod 2'$$  

for all $s' \in S_{2'}(n, a)$. Again since elements which are permutations of the coordinates of $s'$ are also in $S_{2'}(n, a)$, we can choose a solution such that $s_{i_1} + \cdots + s_{i_m}$ is odd. This is due to the fact that if $a \equiv 1 \mod 4$ or if $a \equiv 3 \mod 4$ where $n \geq 4$, there exists at least one solution to $s_1^2 + \cdots + s_m^2 \equiv a \mod 2'$ with $s_i$ even for some $i$. See DeDeo [2, p. 26]. This is a contradiction to Eq. (3). Thus, we must have that $b = 0$ which gives the multiplicity of $k$ as one. □
Theorem 7. If \( a \equiv 0 \mod 2 \) or if \( a \equiv 3 \mod 4 \) and \( n \leq 3 \), then the Euclidean graph \( X_2(n, a) \) is not connected.

Proof. Claim. The multiplicity of \( k \), the degree of the graph, is greater than one.

For at least one fixed \( b \) there exists an eigenvalue for which \( \lambda_b = k \). This occurs at \( b = 0 \). Thus, we must find a \( b = [b_1, \ldots, b_n]^t \neq 0 \).

Case 1: \( a \equiv 0 \mod 2 \). Take \( b' = [2^r-1, \ldots, 2^r-1]^t \). Since \( s_1^2 + \cdots + s_n^2 \equiv 0 \mod 2 \), we must have an even number of odd \( s_i \). Then

\[
\sum_{d(s,0)=a} c^{2 a b_s / 2^r} = c^{a (s_1 + \cdots + s_n)} = 1 \quad \text{for all } s \in S_{2r}(n, a).
\]

Case 2: \( a \equiv 3 \mod 4 \). We showed in Theorem 1 that the order of \( S_{2r}(n, a) \) was 0 when \( n = 2 \). Thus, the graph does not have any adjacent points and is not connected. If \( n = 3 \), then it is easy to show that \( b = [2^r-1, 2^r-1, 0]^t \) gives another eigenvalue \( k \).

From here we assume that \( X_2(n, a) \) is a connected graph (Fig. 1).

Theorem 8. All Euclidean graphs \( X_2(n, a) \) where \( a \equiv 1 \mod 4 \) for all \( n \) and where \( a \equiv 3 \mod 4 \) for \( n \geq 4 \) are bipartite.

Proof. We need only to show that \( -k \) is an eigenvalue of the adjacency matrix associated to \( X_2(n, a) \). Let \( b_0 = [2^{r-1}, \ldots, 2^{r-1}]^t \) and \( s = [s_1, \ldots, s_n]^t \) in \( S_{2r}(n, a) \). Then

\[
\lambda_{b_0}(r) = \sum_{d(s,0)=a} c^{2 a b_0 / 2^r} = \sum_{d(s,0)=a} c^{a (s_1 + \cdots + s_n)}.
\]
Since $s_1^2 + \cdots + s_n^2 \equiv 1 \mod 2$ we must have an odd number of $s_i$ being odd for $i = 1, \ldots, n$. This gives $s_1 + \cdots + s_n$ odd. Thus $e^{\pi i (s_1 + \cdots + s_n)} = -1$ for all $s \in S_2(n,a)$ which means $\lambda_b = k$. □

Now we would like to study the relationship between the eigenvalues associated to $X_2(n,a)$ and the eigenvalues associated to $X_{2r-1}(n,a)$. We then relate this property to lower dimensional graphs.

**Theorem 9.** For $r \geq 4$ the eigenvalue $\lambda_b^{(r)}$ for the connected Euclidean graph $X_2(n,a)$ is given by

$$
\lambda_b^{(r)} = \begin{cases} 
2^{n-1} \lambda_b^{(r-1)} / 2 & \text{if } 4|b_j \text{ for all } j = 1, \ldots, n, \\
0 & \text{if } 4 \nmid b_j \text{ for some } j = 1, \ldots, n.
\end{cases}
$$

**Proof.** Case 1: $4|b_j$ for all $j = 1, \ldots, n$. Let $s \in S_{2r-1}(n,a)$ such that $s = [s_1, \ldots, s_n]'$. Then $s_1^2 + \cdots + s_n^2 \equiv a \mod 2^{r-1}$. Let $i$ be the first index such that $s_i$ is odd. Then we know that $s' = [s_1, \ldots, s_i + 2^{r-2}, \ldots, s_n]'$ is also in $S_{2r-1}(n,a)$. Then

$$
s_1^2 + \cdots + s_n^2 \equiv s_1^2 + \cdots + (s_i + 2^{r-2})^2 + \cdots + s_n^2 \mod 2^{r-1}.
$$

This gives

$$
2^{r-1}s_i \equiv 0 \mod 2^{r-1}.
$$

Since $s_i$ is odd, $2^{r-1}s_i \not\equiv 0 \mod 2^{r-1}$ and thus only one of the pair $s, s'$ lifts to a solution modulo $2^r$ for $r \geq 4$. Given that all the $b_j$ are even, we have

$$
b' \cdot s - b' \cdot s' \equiv -2^{r-2}b_i \equiv 0 \mod 2^{r-1}.
$$

This implies that $s$ and $s'$ contribute to $\lambda_b^{(r-1)}$ in the same manner and the pairs of $s,s'$ produce the same exponential in $\lambda_b^{(r)}$. From the preceding only half of the $2^n$ solutions lift to a solution modulo $2^r$. Thus,

$$
\lambda_b^{(r)} = 2^{n-1} \lambda_b^{(r-1)} / 2 \quad \text{for } r \geq 4.
$$

**Case 2: $4 \nmid b_j$ for some $j = 1, \ldots, n$.** Let $s \in S_2(n,a)$ such that $s = [s_1, \ldots, s_n]'$. We know that replacing an odd $s_i$ with $s_i + 2^{r-1}$ gives another solution $s'$. Since there exists an odd $b_k$ in $b$, match the $s$ with $s' = [s_1, \ldots, s_k + 2^{r-1}, \ldots, s_n]'$. This gives

$$
e^{2\pi i b \cdot s' / 2^r} = e^{2\pi i b \cdot s / 2^r} e^{\pi i b_k} = -e^{2\pi i b \cdot s / 2^r}.
$$

For $s \neq s'$, we rearrange the exponential sum to obtain

$$
\lambda_b^{(r)} = \sum_{d(s,s')=a \atop s \neq s'} [e^{2\pi i b \cdot s / 2^r} + e^{2\pi i b \cdot s' / 2^r}] = 0. \quad \square
$$
Corollary 10. For \( r \geq 3 \) the eigenvalue \( \lambda_b^{(r)} \) for the connected Euclidean graph \( X_2^r(n,a) \) is given by

\[
\lambda_b^{(r)} = \begin{cases} 
2^{(n-1)(r-3)}\lambda_b^{(3)} & \text{if } 2^{r-2}|b_j \text{ for all } j = 1, \ldots, n, \\
0 & \text{otherwise}.
\end{cases}
\] (4)

We note that at \( b = 0 \), \( \lambda_b^{(r)} = k \) which is the order of the symmetric set \( S_2^r(n,a) \). This gives us

\[
|S_2^r(n,a)| = 2^{(n-1)(r-3)}|S_0(n,a)|.
\] (5)

Combining this result with Theorem 3, we conclude for \( r \geq 3 \),

\[
|S_2^r(n,a)| = 2^{(n-1)(r-3)}|S_4(n,a)| + \begin{cases} 
0 & \text{if } n \not\equiv a \mod 4, \\
2^{2n-1} & \text{if } n \equiv a \mod 8, \\
-2^{2n-1} & \text{if } n \equiv a + 4 \mod 8.
\end{cases}
\] (6)

Theorem 11. For \( r \geq 4 \) the connected Euclidean graph \( X_2^r(n,a) \) is not Ramanujan.

Proof. Let \( k^{(r)} \) be the degree of \( X_2^r(n,a) \) and \( v^{(r)} \) be the number of vertices of \( X_2^r(n,a) \). If \( A_2^r(n,a) \) is the corresponding adjacency operator on \( X_2^r(n,a) \), let

\[
\mu^{(r)} = \max\{|\lambda| \text{ such that } \lambda \text{ is in the spectrum of } A_2^r(n,a), \lambda \neq k\}.
\]

Then by the inequality at the bottom of p. 455 of Angel et al. [1] for \( r \geq 4 \),

\[
\mu^{(r)} \geq \sqrt{k^{(r)}[v^{(r)} - k^{(r)}]/v^{(r)} - 1}.
\] (7)

Using Corollary 10, we have a lower bound for \( \mu^{(r)} \):

\[
\mu^{(r)} \geq 2^{(n-1)(r-3)} \sqrt{k^{(3)}[v^{(3)} - k^{(3)}]/v^{(3)} - 1}.
\]

By Eq. (5), we have that the Ramanujan bound becomes

\[
2\sqrt{2^{(n-1)(r-3)}k^{(3)} - 1}.
\]

Thus for \( r \geq 4 \) we need to show that

\[
2^{(n-1)(r-3)} \sqrt{k^{(3)}[v^{(3)} - k^{(3)}]/v^{(3)} - 1} \geq 2\sqrt{2^{(n-1)(r-3)}k^{(3)} - 1}.
\]

It suffices to prove

\[
2^{(n-1)(r-3)} \sqrt{k^{(3)}[v^{(3)} - k^{(3)}]/v^{(3)} - 1} \geq 2\sqrt{2^{(n-1)(r-3)}k^{(3)}}.
\]
Simplifying, we obtain
\[ 2^{(n-1)(r-3)}[F^{(3)} - k^{(3)}] \geq 2^{3n+2}. \] (8)

For \( a \equiv m \mod 4 \) where \( m = 1 \) or \( 3 \), set \( K = \sum_{j=0}^{\lfloor(n-m)/4\rfloor} \binom{n}{4j+m} \). We have an upper bound on \( k^{(3)} \) by Eq. (2):
\[ k^{(3)} \leq 2^{2n-1}K + 2^{2n-1}. \]

Also, since \( \sum_{j=1}^{n} \binom{n}{j} = 2^n \), we have
\[ 2^n - K = \sum_{j=1}^{n} \binom{n}{j} - \sum_{j=0}^{\lfloor(n-m)/4\rfloor} \binom{n}{4j+m} \geq 2^{n-1}. \]

Combining these results gives \( k^{(3)} \leq 2^{3n-2} + 2^{2n-1} \). Substituting into Eq. (8) we have
\[ 2^{(n-1)(r-4)}[2^{3n} - 2^{3n-2} - 2^{2n-1}] \geq 2^{3n+2} \]
which reduces to
\[ 2^{(n-1)(r-3)} \left[ 3 - \frac{1}{2^{n-1}} \right] \geq 2^4. \]

This inequality holds for all \( n \geq 2 \) and \( r \geq 4 \) except \( r = 4 \) when \( n = 2 \) or \( 3 \) and \( r = 5 \) when \( n = 2 \). Checking these cases on computer we see that the Ramanujan bound does not hold.

We note that the inequality in Eq. (7) does not hold for \( r = 2 \) or \( 3 \). Without a bound on the second largest eigenvalue, it is difficult to prove that these graphs are not Ramanujan. Instead, we note that the only Ramanujan graphs found by computer are \( r = 2 \) when \( n = 2 \) and \( 3 \).

### 2.2. Spectra and level curves of \( X_{2r}(n,a) \)

Although the graphs are non-Ramanujan we are still interested in studying the spectra and level curves corresponding to these graphs as the spectra is analogous to the energy levels of a complex physical system.

**Example 1.** The distribution of eigenvalues (upper row) with their multiplicities listed below:
\[
\begin{pmatrix}
-384 & -128 & -90.51 & 0 & 90.51 & 128 & 384 \\
1 & 15 & 48 & 3968 & 48 & 15 & 1
\end{pmatrix}.
\]

Spectra of \( X_{16}(3,1) \)

We note that the Ramanujan bound \( 2\sqrt{\text{degree} - 1} \) for \( X_{16}(3,1) \) is \( 2\sqrt{383} \cong 39.141 \neq |128| \), the second largest eigenvalue which shows the graph is non-Ramanujan. Notice
that zero forms \( \frac{3968}{4096} \approx 96.87\% \) of eigenvalues.

\[
\begin{pmatrix}
-512 & -256 & -128 & 0 & 128 & 256 & 512 \\
1 & 4 & 32 & 4022 & 32 & 4 & 1
\end{pmatrix}
\]

Spectra of \( X_8(4, 1) \)

Also note that the Ramanujan bound for \( X_8(4, 1) \) is \( 2\% \sqrt{511} \approx 45.211 \). This shows the graph is non-Ramanujan. Notice that zero forms \( \frac{4022}{4096} \approx 98.19\% \) of eigenvalues.

It is easy to produce histograms of eigenvalues of these Euclidean graphs modulo \( 2^r \) for small \( r \) and \( n \). The distribution of the eigenvalues appears to have gaps similar to those found in Medrano et al. \[6\]. These facts lead us to believe that as \( n \) approaches infinity, the percentage of non-zero eigenvalues approaches zero.

**Theorem 12.** The percentage of non-zero eigenvalues of the adjacency operator associated to the connected Euclidean graphs \( X_{2r}(n, a) \) approaches zero as \( n \) tends to infinity.

**Proof.** By Theorem 9, we can have at most \( 2^n(r-1) \) of the \( 2^{nr} \) eigenvalues being non-zero. Thus,

\[
limit_{n \to \infty} \frac{2^n(r-1)}{2^{nr}} = \frac{1}{2^n} \to 0
\]

gives us that the percentage of non-zero eigenvalues tends to zero as \( n \) tends to infinity. \( \square \)

We note that replacing \( p = 2 \) with \( p \) an odd prime produces the same result and proves the observations in Medrano et al. \[6\]

The question regarding level curves of eigenfunctions for finite Euclidean models is interesting in that the level curves have analogues in mathematical physics. In the real case the eigenfunctions of the Laplacian in \( \mathbb{R}^2 \) relate to points on a vibrating drum. For a circular drum, the contour lines of radial eigenfunctions are circles. The simultaneous eigenfunctions of the combinatorial Laplacians on the finite Euclidean graphs \( X_{2r}(n, a) \) for fixed \( r \) and \( n \) can be seen as finite spherical functions. Thus, the analogy of finding \( x \) such that \( f(x) = \text{constant} \) is the same as \( d(x, 0) = \text{constant} \) and the level “curves” are finite analogues of circles.

For \( n = 2 \), we obtain Figs. 2 and 3 by associating point-pairs \( (x, 0) \) (where \( x = (x_1, x_2) \)) with the same distance modulo \( 2^r \) to dots of the same color. In other words, the points \( x \) where \( f(x) = c \equiv x_1^2 + x_2^2 \mod 2^r \) have the same color. We note that these patterns look similar to those in Myers \[7\] over \( \mathbb{Z}_p \) and in Medrano et al. \[5\] over \( \mathbb{F}_p \) where \( p \) is an odd prime.

To view Figs. 2 and 3 in color and to experiment with other \( r \), use the Mathematica command: \( \text{ListDensityPlot[Mod[x_1^2 + x_2^2, 2^r], \{x_1, 1, 2^r\}, \{x_2, 1, 2^r\}, Mesh->False, Color Function->Hue} \).
Fig. 2. List Density Plot for $r = 7$.

Fig. 3. List Density Plot for $r = 8$. 
References