Abstract: We give a decomposition theorem for Platonic graphs over finite fields and use this to determine the spectrum of these graphs. We also derive estimates for the isoperimetric numbers of the graphs.
THE SPECTRUM OF PLATONIC GRAPHS OVER FINITE FIELDS

MICHELLE DEDEO, DOMINIC LANPHIER, AND MARVIN MINEI

Abstract. We give a decomposition theorem for Platonic graphs over finite fields and use this to determine the spectrum of these graphs. We also derive estimates for the isoperimetric numbers of the graphs.

1. Introduction

Let \( p \) be an odd prime, let \( r \geq 1 \), and let \( \mathbb{F}_q \) be the finite field with \( q = p^r \) elements. We consider a generalization of the Platonic graphs studied in [2], [7], [8], and [9]. In particular, using the notation from [7], we define our graphs \( G^*(q) \) as follows.

**Definition 1.** Let the vertex set of \( G^*(q) \) be
\[
V(G^*(q)) = \{ (\alpha \ \beta) \mid \alpha, \beta \in \mathbb{F}_q, \ (\alpha \ \beta) \neq (0 \ 0) \} / \langle \pm 1 \rangle,
\]
and let \((\alpha \ \beta)\) be adjacent to \((\gamma \ \delta)\) if and only if \( \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1 \).

For \( q = p \), the graph \( G^*(q) \) is called the \( q \)th Platonic graph. The name comes from the fact that when \( p = 3 \) or 5, the graphs \( G^*(p) \) correspond to 1-skeletons of two of the Platonic solids, in other words, the tetrahedron and the icosahedron as in [2]. More generally, \( G^*(p) \) is the 1-skeleton of a triangulation of the modular curve \( X(p) \), see [7].

The spectrum \( \Lambda_G \) of a graph \( G \) is the set of eigenvalues of the adjacency operator, see [4]. For a finite, \( k \)-regular graph \( G \) it is known that \( k \in \Lambda_G \). Further, \( \lambda \in \Lambda_G \) satisfies \( |\lambda| \leq k \).

Let \( \{ G_m \}_{m \geq 1} \) be a family of finite, connected, \( k \)-regular graphs with \( \lim_{m \to \infty} |V(G_m)| = \ldots \)

**Key words and phrases.** Cayley graph; Ramanujan graph; isoperimetric number.
Let $\lambda_1(G_m)$ be the largest element in $\Lambda_{G_m}$ distinct from $k$. Then it is known that
\[ \liminf_{m \to \infty} \lambda_1(G_m) \geq 2\sqrt{k-1}, \]
see [1] and [5].

A $k$-regular graph $G$ is called Ramanujan if for all $\lambda \in \Lambda_G$ with $|\lambda| \neq k$ we have $|\lambda| \leq 2\sqrt{k-1}$. From the inequality for $\lambda_1$ above, we see that this estimate is optimal for families of regular graphs of fixed degree. It follows that Ramanujan graphs make the best expanders in the sense of [11].

In this paper, we determine the spectrum of $G^*(q)$ and thus generalize results of [7].

**Theorem 1.** Let $p > 2$ be prime and $q = p^r$. The spectrum of $G^*(q)$ is $q$ with multiplicity 1, $-1$ with multiplicity $q$, and $\pm \sqrt{q}$ each with multiplicity $(q+1)(q-3)/4$.

It follows that the graphs $G^*(q)$ are Ramanujan.

To do this, we demonstrate a decomposition theorem for the graphs $G^*(q)$ which we also use to estimate their isoperimetric numbers. Let $K_n$ denote the complete graph on $n$ vertices and let $K_n^m$ denote the complete multigraph on $n$ vertices with $m$ edges connecting each pair of distinct vertices. For $(0 \ a)$ in $V(G^*(q))$, let $H_a$ be the graph defined in Section 3. Thus, $H_a$ has $q+1$ vertices consisting of $p^{r-1}$ disjoint $p$-circuits, with every vertex in a given circuit adjacent to the vertex $(0 \ a)$.

**Theorem 2.** The graph $G^*(q)$ can be partitioned into $(q-1)/2$ disjoint copies of $H_a$ with 2$q$ edges joining every pair of $H_a$’s. Alternatively, $G^*(q)$ is the complete multigraph $K_{(q-1)/2}^{2q}$ where each vertex should be viewed as a copy of $H_a$. 
For an arbitrary graph $G$ and $S \subset V(G)$, the boundary of $S$, denoted $\partial S$, is the set of all edges having exactly one endpoint in $S$. The isoperimetric number [12] is defined to be

$$\text{iso}(G) = \inf_S \frac{|\partial S|}{|S|}$$

where the infimum is over all $S \subset V(G)$ so that $|S| \leq |V(G)|/2$. The isoperimetric number, also called the Cheeger constant, is a measure of the connectedness of $G$. A set $S$ where $\text{iso}(G) = |\partial S|/|S|$ is called an isoperimetric set for $G$. The isoperimetric number has applications to combinatorics and computer science. For example, if a graph $G$ is viewed as a communications network then a large $\text{iso}(G)$ means that information is transmitted easily throughout the network. A small isoperimetric number means that communication can be easily disrupted.

As a consequence of Theorems 1 and 2, we obtain the following bounds for the isoperimetric numbers $\text{iso}(G^*(q))$ of $G^*(q)$.

**Corollary 1.** We have

$$\frac{q}{2} - \sqrt{q} \leq \text{iso}(G^*(q)) \leq \begin{cases} \frac{q(q-1)}{2(q+1)} & \text{if } q \equiv 1 \pmod{4} \\ \frac{q}{2} & \text{if } q \equiv 3 \pmod{4} \end{cases}$$

In the next section, we define a Cayley graph $G(q)$ of $PSL_2(\mathbb{F}_q)$ and consider a graph $G'(q)$ obtained from the quotient of $PSL_2(\mathbb{F}_q)$ by the unipotent subgroup. We relate $G^*(q)$ to $G'(q)$ in order to prove Theorem 2. We then demonstrate bounds for the isoperimetric number of $G^*(q)$ using the methods of [8] and [9]. In the last section, we use the decomposition theorem to determine the spectrum of $G^*(q)$.

In [10], Li and Meemark determined the spectrum of Cayley graphs on $PGL_2(\mathbb{F}_q)$ modulo either the unipotent subgroup, the split torus, or the nonsplit torus of $G$. They use the
Kirillov models of the representations of $PGL_2(\mathbb{F}_q)$. This is a different approach from the one used here or in [6], as [10] relies primarily on representation theory rather than an analysis of the graph structure to determine the spectrum. Note that in this paper, character sums are not needed to evaluate the eigenvalues.

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2. A Cayley Graph of the Projective Linear Group

The Cayley graph of a group $\Gamma$ with respect to a symmetric generating set $\Omega$ is the graph with vertex set $\Gamma$ and with two vertices $\gamma_1, \gamma_2$ joined by an edge if we have $\gamma_1 = \alpha \gamma_2$ for some $\alpha \in \Omega$. Cayley graphs are vertex transitive and $|\Omega|$-regular. In this section, we define a Cayley graph of $\Gamma_q = PSL_2(\mathbb{F}_q)$ for $q = p^r$ and $p > 2$ a prime. We then study a quotient graph of the Cayley graph.

For $q \equiv 1 \pmod{4}$, there exists some $\omega \in \mathbb{F}_q$ so that $\omega^2 = -1$. From [3], for appropriately chosen $x \in \mathbb{F}_q$, the set

$$\Omega_q = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm x \\ 0 & 1 \end{pmatrix} \right\}$$

is a generating set for $\Gamma_q$. For $q \equiv 3 \pmod{4}$ and $q > 11$, from [3], $PSL_2(\mathbb{F}_q)$ is generated by

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix} \right\}$$

where $a^2 + b^2 = -1$, $\omega \in \mathbb{F}_q^\times$, and $\omega \neq \pm 1$. Since $q \equiv 3 \pmod{4}$, both $a$ and $b$ are nonzero. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a^{-1} & b \\ 0 & -a \end{pmatrix},$$

we have that $PSL_2(\mathbb{F}_q)$ is generated by

$$\Omega_q = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm x \\ 0 & 1 \end{pmatrix} \right\}$$
for appropriately chosen $x \in \mathbb{F}_q$. We take $\Omega_q$ to be the symmetric generating set for $\text{PSL}_2(\mathbb{F}_q)$ for $q \equiv 1, 3 \pmod{4}$, where $\omega^{-1} = -\omega$ in the former case. Thus, $|\Omega_q| = 4$ for $q \equiv 1 \pmod{4}$ and $|\Omega_q| = 5$ for $q \equiv 3 \pmod{4}$. Note that for $q \equiv 3 \pmod{4}$ and $q \leq 11$, $q$ is a prime. In that case, Theorems 1 and 2 follow from [7] and [8].

Let $G(q)$ be the Cayley graph of $\Gamma_q$ with respect to $\Omega_q$. Note that $|V(G(q))| = q(q^2 - 1)/2$. Let

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{F}_q \right\}.$$ 

Thus, $|N| = q$. Let $\Gamma_q' = N \setminus \Gamma_q$ and note that

$$\Gamma_q' \cong \left\{ (\alpha \beta) \mid \alpha, \beta \in \mathbb{F}_q, \ (\alpha \beta) \neq (0 0) \right\}/(\pm 1),$$

which is the vertex set of $G'(q)$.

Let $G'(q)$ denote the quotient graph $N \setminus G(q)$. That is, $G'(q)$ is the multigraph whose vertices are given by the cosets $\Gamma_q'$ with distinct cosets $N\gamma_1$ and $N\gamma_2$ joined by as many edges as there are edges in $G(q)$ of the form $(\alpha_1, \alpha_2)$ where $\alpha_j \in N\gamma_j$. Note that $\Gamma_q'$ is not a group, and thus the graphs $G'(q)$ are not themselves Cayley graphs. Also, note that the vertex set of $G'(q)$ is $\Gamma_q'$.

**Lemma 1.** Let $(\alpha \beta), (\gamma \delta) \in G'(q)$. Then $(\alpha \beta)$ is adjacent to $(\gamma \delta)$ if and only if

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1, \pm \omega, \text{ or } \pm \omega^{-1}.$$

**Proof.** Left multiplication of $g \in \Gamma_q$ by elements of $N$ preserves the bottom row of $g$. So, $g' \in G'(q)$ is adjacent to precisely those elements attained by left multiplication by $\gamma \in \Omega_q$ with $\gamma \notin N$. Left multiplication of $g$ by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ multiplies the top row of $g$ by $-1$ and switches the rows. Multiplication by $\begin{pmatrix} 0 & -1 \\ -\omega & 0 \end{pmatrix}$ multiplies the top row of $g$ by $-\omega$ and
switches the rows, and similarly multiplication by \( \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix} \) multiplies the top row of \( g \) by \(-\omega^{-1}\) and switches the rows. So, if \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_q \) with determinant equal to 1 and \( \xi g = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \) for \( \xi \in \Omega_q - N \), then \( (\gamma' \delta') = - (\alpha \beta), -\omega (\alpha \beta), \) or \(-\omega^{-1} (\alpha \beta)\). Therefore, we must have

\[
\det \begin{pmatrix} \gamma & \delta' \\ \gamma' & \delta \end{pmatrix} = \pm 1, \pm \omega, \text{ or } \pm \omega^{-1}.
\]

If \( \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1, \pm \omega, \text{ or } \pm \omega^{-1} \) then call the edge incident with these vertices a 1-edge, and otherwise call the edge an \( \omega \)-edge. Note that \( G^*(q) \) is obtained from \( G'(q) \) simply by removing all of the \( \omega \)-edges. Further, it is easy to see that the number of \( \omega \)-edges incident with a vertex in \( G'(q) \) is \(|\Omega_q| - 3\) times the number of 1-edges incident with that vertex. It follows that \( G^*(q) \) is \( q \)-regular.

Note that \(|V(G'(q))| = (q^2 - 1)/2\). Two edges incident with \( g \in G(q) \) arise by multiplication by the two elements of \( \Omega_q \) that are in \( N \). The remaining \(|\Omega_q| - 2\) edges come from cosets \( Nx, Ny \) with \( x, y \notin N \). Further, any vertex in the graph \( G'(q) \) corresponds to \( q \) vertices in \( G(q) \). It follows that the regularity of \( G'(q) \) is \(|\Omega_q| - 2\) and

\[
|E(G'(q))| = \frac{(|\Omega_q| - 2)q(q^2 - 1)}{4}.
\]

3. The Decomposition Theorem

Let \( (\alpha \beta) \) and \( (\gamma \delta) \) be adjacent vertices in \( G'(q) \) and thus \( \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1, \pm \omega \text{ or } \pm \omega^{-1} \) from Lemma 1. If the determinant is \( \pm 1 \) then call the edge incident with these vertices a 1-edge, and otherwise call the edge an \( \omega \)-edge. Note that \( G^*(q) \) is obtained from \( G'(q) \) simply by removing all of the \( \omega \)-edges. Further, it is easy to see that the number of \( \omega \)-edges incident with a vertex in \( G'(q) \) is \(|\Omega_q| - 3\) times the number of 1-edges incident with that vertex. It follows that \( G^*(q) \) is \( q \)-regular.
Let $\alpha \in \mathbb{F}_q^\times$ and define

$$V_\alpha = \{ (0, \alpha), (\alpha^{-1}, \beta) \mid \beta \in \mathbb{F}_q \}.$$ 

Note that $|V_\alpha| = q + 1$. If $\alpha' \in \mathbb{F}_q^\times$, $\alpha' \neq \pm \alpha$ then it is easy to see that $V_\alpha \cap V_{\alpha'} = \emptyset$. Otherwise, $V_\alpha = V_{\alpha'}$. It is also easy to see that any vertex in $G^*(q)$ lies in $V_\alpha$ for some $\alpha \in \mathbb{F}_q^\times$ and therefore $V(G^*(q))$ can be partitioned into disjoint $V_\alpha$’s. In particular, $V(G^*(q))$ can be partitioned into $|V(G^*(q))|/|V_\alpha| = (q - 1)/2$ many copies of $V_\alpha$. Define $H_\alpha$ to be the subgraph of $G^*(q)$ induced by $V_\alpha$. Any $(\alpha^{-1}, \beta) \in H_\alpha$ belongs to a $p$-circuit in $H_\alpha$ given by the following sequence of adjacent vertices,

$$\{(\alpha^{-1}, \beta), (\alpha^{-1}, \beta + \alpha), (\alpha^{-1}, \beta + 2\alpha), \ldots, (\alpha^{-1}, \beta + (p - 1)\alpha), (\alpha^{-1}, \beta)\}.$$ 

Further, any two such circuits share a common vertex $(\alpha^{-1}, \beta + j\alpha) = (\alpha^{-1}, \beta' + k\alpha)$ if and only if $\beta - \beta' = (k - j)\alpha$ and this occurs if and only if such circuits are identical. Thus, there are $p^{r-1}$ disjoint $p$-circuits in a given $H_\alpha$ and each vertex in such a circuit is adjacent to $(0, \alpha)$. We refer to the vertex $(0, \alpha)$ as the center of $H_\alpha$. As $(\alpha^{-1}, \beta) \in H_\alpha$ is adjacent to $(\alpha^{-1}, \beta') \in H_\alpha$ if and only if $\beta' = \pm \alpha + \beta$, then $(\alpha^{-1}, \beta)$ is adjacent to only two other vertices in $H_\alpha$, other than the center. This accounts for all the possible edges in $H_\alpha$ and so $|E(H_\alpha)| = 2q$ for any $\alpha \in \mathbb{F}_q^\times$.

Suppose $H_\alpha \neq H_{\alpha'}$ and $(\alpha^{-1}, \beta) \in H_\alpha$ is adjacent to $(\alpha'^{-1}, x) \in H_{\alpha'}$. Then

$$\det \begin{pmatrix} \alpha^{-1} & \beta \\ \alpha'^{-1} & x \end{pmatrix} = \pm 1,$$

and the number of solutions for $x$ to this equation is independent of the choice of $\alpha' \in \mathbb{F}_q^\times$.

The number of 1-edges connecting $H_\alpha$ to $H_{\alpha'}$ is therefore independent of $\alpha$ and $\alpha'$, where $H_{\alpha'} \neq H_\alpha$. Since $H_\alpha$ contains $2q$ edges and there are $(q - 1)/2$ copies of $H_\alpha$ in $G^*(q)$, we
have \( \bigcup_{\alpha \in F \times q} H_\alpha \) contains \( q(q - 1) \) edges. As \(|V(G^*(q))| = (q^2 - 1)/2 \) and \( G^*(q) \) is \( q \)-regular, \(|E(G^*(q))| = q(q^2 - 1)/4 \). Thus, there are \( q(q^2 - 1)/4 - q(q - 1) = q(q - 1)(q - 3)/4 \) edges joining different \( H_\alpha \)'s. As there are \( \left( \frac{(q - 1)/2}{2} \right) \) different ordered pairs \((\alpha, \alpha')\) that give distinct \( H_\alpha \)'s and \( H_{\alpha'} \)'s, the number of edges joining a given \( H_\alpha \) to a distinct \( H_{\alpha'} \) is

\[
\frac{q(q - 1)(q - 3)/4}{\left( \frac{(q - 1)/2}{2} \right)} = 2q.
\]

This gives the decomposition as stated in Theorem 2.

4. The Isoperimetric Number

In this section we use the decomposition theorem to estimate the isoperimetric number of \( G^*(q) \) as stated in Corollary 1. In addition, we estimate the isoperimetric numbers of the related graphs \( G'(q) \) and \( G(q) \).

According to [12], the isoperimetric number for the complete graph \( K_n \) is

\[
\text{iso}(K_n) = \begin{cases} 
\frac{n}{2} & \text{n even} \\
\frac{n + 1}{2} & \text{n odd.}
\end{cases}
\]

As a consequence of the decomposition theorem, for our set \( S \) we can take \((q - 1)/4\) copies of \( H_\alpha \) if \( q \equiv 1 \) (mod 4) and \((q + 1)/4\) copies of \( H_\alpha \) if \( q \equiv 3 \) (mod 4). In the former case, we have

\[
|S| = (q + 1)\left( \frac{q - 1}{4} \right),
\]

\[
|\partial S| = 2q\left( \frac{q - 1}{4} \right)^2,
\]

and we get

\[
\text{iso}(G^*(q)) \leq \frac{|\partial S|}{|S|} = \frac{q(q - 1)}{2(q + 1)}.
\]

The \( q \equiv 3 \) (mod 4) case follows similarly and we get \( \text{iso}(G^*(q)) \leq q/2 \).
We can follow the arguments from [2] and [9] to obtain lower bounds for \( \text{iso}(G^*(q)) \), independent of Section 5. Thus, we do not need to determine the spectrum in order to demonstrate these graphs are expanders. As this allows us to obtain lower bounds for the isoperimetric numbers of the related graphs \( G'(q) \), we include the argument. The proof for the following lemma is the same as in [2].

**Lemma 2.** If \( (\alpha \beta), (\alpha' \beta') \in G^*(q) \) with \( \det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \neq 0 \) then there are exactly two paths of length two joining \( (\alpha \beta) \) to \( (\alpha' \beta') \).

Partition the vertices of \( G^*(q) \) into sets \( S_1 \) and \( S_2 \) with \( |S_1| \leq |S_2| \). From Lemma 2, it follows that at least one edge from each of these paths must be cut for every pair \( v_1 \in S_1 \) and \( v_2 \in S_2 \) where \( v_1 \) and \( v_2 \) are not multiples of each other. Since any \( v \in G^*(q) \) has \( q - 1 \) nonzero multiples, the number of such pairs is at least \( |S_1|(|S_2| - q + 1) \). As \( G^*(q) \) is \( q \)-regular, each edge can lie in no more than \( 2(q - 1) \) different paths of length 2. Thus, we have

\[
|\partial S_1| \geq 2|S_1|(|S_2| - q + 1) \quad \frac{2(q - 1)}{q - 1}
\]

and since \( |S_2| \geq (q^2 - 1)/4 \), we get

\[
\frac{|\partial S_1|}{|S_1|} \geq \frac{(|S_2| - q + 1)}{(q - 1)} \geq \frac{q - 3}{4}.
\]

This gives the lower bound \( (q - 3)/4 \leq \text{iso}(G^*(q)) \). It follows in this elementary way that these are expander graphs in the sense of [11].

In a similar manner we can apply a version of Lemma 2 to \( G'(q) \) and get the bounds

\[
\frac{(q - 1)(q - 3)}{2q - 1} \leq \text{iso}(G'(q)) \leq \begin{cases} 
\frac{q(q - 1)}{q + 1} & \text{if } q \equiv 1 \pmod{4} \\
\frac{3q}{2} & \text{if } q \equiv 3 \pmod{4}.
\end{cases}
\]
By the construction of $G'(q)$, each vertex of $G'(q)$ corresponds to $q$ vertices of $G(q)$. Further, $S' \subset V(G'(q))$ corresponds to a set $S \subset V(G(q))$ containing $q|S'|$ vertices and it is easy to see that for such sets, $|\partial S| = |\partial S'|$. Therefore, the estimate $\text{iso}(G'(q)) \leq |\partial S'|/|S'|$ implies the estimate $\text{iso}(G(q)) \leq (q - 1)/(q + 1)$ for $q \equiv 1 \pmod{4}$ and $\text{iso}(G(q)) \leq 3/2$ for $q \equiv 3 \pmod{4}$. However, it does not follow that a lower bound for the isoperimetric number of a quotient graph implies a lower bound for the isoperimetric number of the larger graph. In general, we expect the isoperimetric numbers of the two graphs to be related, but an isoperimetric set for $G(q)$ may possibly be constructed that cuts through the subgraphs of $G(q)$ associated to the vertices in the quotient graph $G'(q)$. This may give a smaller isoperimetric number than $\text{iso}(G'(q))/q$.

Using the results of the next section, we can get an improved lower bound for $\text{iso}(G^*(q))$. For $\lambda_1 \neq k$, the second largest eigenvalue of the adjacency matrix of $G^*(q)$, we have

\[
\frac{q}{2} - \frac{\lambda_1}{2} \leq \text{iso}(G^*(q)).
\]

See [1] or [5].

From the results of Section 5 we have $\lambda_1 = \sqrt{q}$ and so obtain

\[
\frac{q}{2} - \frac{\sqrt{q}}{2} \leq \text{iso}(G^*(q)),
\]

which gives Corollary 1.

5. The Spectrum of Platonic Graphs over $\mathbb{F}_q$

Let $S_q$ be the complex vector space of functions $s : V(G^*(q)) \to \mathbb{C}$ equipped with the canonical basis $\{e_v\}_{v \in V(G^*(q))}$ where $e_v(w) = \delta_{vw}$ and $\delta_{vw}$ is the Kronecker delta function.
The adjacency operator \( A_q : S_q \to S_q \) is the linear transformation defined by

\[
(A_q s)(v) = \sum_{w \text{ is adjacent to } v} s(w).
\]

The spectrum of \( G^*(q) \), denoted \( \Lambda_q \), is the set of eigenvalues of \( A_q \). As \( A_q \) is a symmetric matrix with respect to the basis \( \{e_v\}_{v \in V(G^*(q))} \) then \( \Lambda_q \subseteq \mathbb{R} \). Further, since \( G^*(q) \) is \( q \)-regular, we know that \( \Lambda_q \subseteq [-q,q] \) and the eigenvalue \( q \) occurs with multiplicity 1.

Note that \( \Gamma_q \) acts on \( S_q \) and that this action commutes with \( A_q \). Let \( \nu \) be a generator of \( \mathbb{F}_q^\times \), let \( \mu = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \), and let \( \lambda \in \Gamma_q \) be of order \( (q + 1)/2 \). Then, the conjugacy classes of \( \Gamma_q \) are those of \( b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( b_2 = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \), \( \mu^k \) for \( 1 \leq k \leq (q - 5)/4 \), and \( \lambda^\ell \) for \( 1 \leq \ell \leq (q - 1)/4 \).

There are 5 types of representations of \( \Gamma_q \), see [13] for example. There is the trivial representation \( 1 \) and the Steinberg representation \( \psi_{\text{St}} \) of degree \( q \). For even characters \( \theta : \mathbb{F}_q^\times \to \mathbb{C}^\times \), there are the principal series representations \( \chi_\theta \) of degree \( q + 1 \). These occur with multiplicity 2. There are \( (q - 5)/4 \) isomorphism classes of this representation for \( q \equiv 1 \pmod{4} \), and \( (q - 3)/4 \) for \( q \equiv 3 \pmod{4} \). There are the \( (q - 1)/4 \) isomorphism classes of the discrete series representations \( \phi_j \) of degree \( q - 1 \). Finally, for \( q \equiv 1 \pmod{4} \), there are the two split principal series \( \xi_\pm \) of degree \( (q + 1)/2 \). Let \( S_r \) be the irreducible submodule of \( S_q \) generated by the isomorphism classes of the representation \( r \).

The values of the relevant irreducible characters of \( \Gamma_q \) on representatives of the isomorphism classes are given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>( 1 _2 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( \mu^k )</th>
<th>( \lambda^\ell )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_{\text{St}} )</td>
<td>( q )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_\theta )</td>
<td>( q + 1 )</td>
<td>1</td>
<td>1</td>
<td>( \theta(\nu^k) + \theta(\nu^{-k}) )</td>
<td>0</td>
</tr>
<tr>
<td>( \xi_\pm )</td>
<td>( (q + 1)/2 )</td>
<td>( 1 \pm \sqrt{2}/2 )</td>
<td>( 1 \mp \sqrt{2}/2 )</td>
<td>(( -1 )^k)</td>
<td>0</td>
</tr>
</tbody>
</table>
Let $\chi$ be the character of $\Gamma_q$ acting on $S_q$. Then, trivially $\chi(12) = (q^2 - 1)/2$. Since the stabilizer in $\Gamma_q$ of $(1 \ 0) \in V(G^*(q))$ is isomorphic to $N$ and any element in the stabilizer fixes only the associates of $(1 \ 0)$, it follows that $\chi(b_1) = \chi(b_2) = (q - 1)/2$. From the definition of $\Gamma_q$, no other elements of $\Gamma_q$ stabilize a vertex. Therefore, $\chi(c) = 0$ for $c$ not in the classes of $1_2$, $b_1$, and $b_2$. From these values of $\chi$ and the character table, we can easily determine that the representations $1$, $\psi_{St}$, and $\chi_{\theta}$ occur with multiplicities 1, 1, and 2, respectively. Further, for $q \equiv 1 \pmod{4}$, the representations $\xi_{\pm}$ occur with multiplicity 1. By dimension counting, these are the only occurring representations. It follows that $S_q$ decomposes into the direct sum of the submodules $S_1$, $S_{\psi_{St}}$, $S_{\chi_{\theta}}$, and $S_{\xi_{\pm}}$.

For $\epsilon \in \mathbb{F}_q^*$, let $s_v \in S_q$ be defined by $s_v(\epsilon v) = q$, and $s_v(w) = -1$ for all other vertices. For $v$ the center vertex of $H_\alpha$ and $v'$ not associate to $v$, we can easily calculate from Theorem 2 that $(A_q s_v)(v') = 1$. Therefore, we can see that for any $v \in V(G^*(q))$,

$$(A_q s_v)(v') = \begin{cases} -q & \text{for } v' = \epsilon v \\ 1 & \text{otherwise} \end{cases}$$

$$= -s_v(v').$$

Thus, $s_v$ is an eigenvector with eigenvalue $-1$. Let $M$ be the submodule of $S_q$ generated by all the $s_v$'s above. We see that $s_v = s_{\epsilon v}$ for all $v \in V(G^*(q))$ and $\epsilon \in \mathbb{F}_q^*$. Since $\sum_{v \in V(G^*(q))} s_v = 0$, we have $\dim M \leq q$. Further, since the intersection of $M$ with the submodule generated by $s_w$ with $w$ adjacent to a fixed $v$ has dimension $q$, it follows that $\dim M = q$. As the submodules $S_r$ are irreducible, we must have $M = S_{\psi_{St}}$. As $\Gamma_q$ and $A_q$ commute and $S_{\psi_{St}}$ is irreducible, then $A_q$ acts on $S_{\psi_{St}}$ by a scalar and it follow that $S_{\psi_{St}}$ is a $q$-dimensional eigenspace with eigenvalue $-1$. 

Let $s_{\theta,v}(\epsilon v') = \pm \theta(\epsilon) \sqrt{q}$ if $v' = v$, $s_{\theta,v}(w) = \theta(\epsilon)$ if $w$ is adjacent to $v$, and $0$ otherwise.

From Theorem 2, we can compute
\[
(A_q s_{\theta,v}^\pm)(\epsilon v) = \sum_{w \text{ is adjacent to } \epsilon v} s_{\theta,v}^\pm(w) = \theta(\epsilon)q
\]
\[
= \pm \sqrt{q} s_{\theta,v}^\pm(\epsilon v)
\]

and for $w \sim \epsilon v$ we have
\[
(A_q s_{\theta,v}^\pm)(w) = s_{\theta,v}^\pm(v) + \sum_{w' \text{ is adjacent to } w} s_{\theta,v}^\pm(w')
\]
\[
= \pm \theta(\epsilon) \sqrt{q} + \sum_{\epsilon' \in \mathbb{F}_q^\times} \theta(\epsilon')
\]
\[
= \pm \sqrt{q} s_{\theta,v}^\pm(w).
\]

Thus, $s_{\theta,v}^\pm$ is an eigenvector of $A_q$ with eigenvalue $\pm \sqrt{q}$.

Trivially, $s_{\theta,v}^\pm \notin S_1$ or $S_{\psi_{St}}$ as they have different eigenvalues. Therefore, we must have $s_{\theta,v}^\pm \in S_{\chi_\theta}$ or $S_{\xi^\pm}$ if $\theta$ is quadratic. For any even character $\theta : \mathbb{F}_q^\times \to \mathbb{C}^\times$, the above computations show that there are eigenvectors $v_1$ and $v_2$ so that $A_q(v_i) = (-1)^i \sqrt{q}$. Further, $\Gamma_q$ acts on $v_i$ with the character $\theta$. As $S_{\chi_\theta}$ and $S_{\xi^\pm}$ are irreducible then the orbit of $v_i$, $\Gamma_q(v_i)$, must either be a copy of $S_{\chi_\theta}$, $S_{\xi^\pm}$, or $\{0\}$. But as $v_i \in \Gamma_q(v_i)$ then $\Gamma_q(v_i) \neq \{0\}$.

As $\Gamma_q$ commutes with $A_q$ then $A_q$ acts on $\Gamma_q(v_i)$ by a scalar. As $\Gamma_q(v_1)$ and $\Gamma_q(v_2)$ are the same dimension, it follows that $\Gamma_q(v_i)$ are $(q+1)$-dimensional subspaces of $S_q$. That is, $S_{\chi_\theta}$ and $S_{\xi^\pm}$ are eigenspaces. As $\Gamma_q(v_1) \cap \Gamma_q(v_2) = \{0\}$ since they are different eigenspaces, we have two copies of $S_{\chi_\theta}$ in $S_q$ and a copy each of $S_{\xi^\pm}$ for quadratic $\theta$, as this applies to all the principal series representations. Counting dimensions of the relevant representations, we get
all of $S_q$. Thus we have completely determined the spectrum of $A_q$ and this gives Theorem 1.

REFERENCES


