Optimal Frame Size Analysis for Framed Slotted ALOHA Based RFID Networks

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Abstract

We offer an analytical solution for the optimal frame size of the non-muting version of the Basic Frame Slotted ALOHA collision resolution protocol for RFID networks. Previous investigations of RFID frame size have been empirical and have not yielded a general result. Our theoretical analysis provides a generalized result. Our solution can be used to determine the optimal frame size for any given number of RFID tags. Suboptimal selection for the frame size can result in substantially longer than minimum census delays and can unnecessarily increase energy consumption. We were able to demonstrate about 20% performance improvement in reduced census delay for a given range of values. Our results can help speed-up reader-side processing times, lower the implementation complexity of RFID readers, and increase their energy efficiency.

Index Terms—RFID, Slotted ALOHA, collision resolution

1. Introduction

Radio Frequency Identification (RFID) networks use radio signal broadcasts to automatically sense the presence of and identify items with attached RFID tags. An RFID system consists of an application, reader, and tags (see Figure 1). The application is a program installed on a proxy computer, which can control readers. The reader, also called interrogator, is a device, which can read, write and authenticate tags. A tag is used to identify an object and is located on (or in) the object itself. It consists of a control unit (microchip) and attached to it a coupling element (antenna). The tag can transmit its unique ID to the reader
when energized by radio signal transmitted from the reader. A reader is connected to a proxy computer and has a transmitter and receiver. When the reader gathers data from tags it forwards it directly to the computer application. The reader is capable of “reading” (or, identifying) the entire set of tags within its interrogation range. The process of reading in-range tags is called a census. After an initial census is completed, the reader can answer queries about the presence of specific tag(s) within its range sent to it via radio frequency signals through the application.

RFID systems have many advantages over barcode and smart card systems. RFID networks use radio frequency as a method of data transmission. Thus, unlike a barcode label, a tag does not need to be placed in a line of sight position from the reader, or even be in contact with a reader as do smart cards, in order to be successfully identified. Depending on whether they use low, high, or ultrahigh transmission frequencies, RFID tags are identifiable within about three meters distance in the case of a typical far-field reader [1]. Some commercial implementations of RFID tags can store data of 16 to 64 Kbytes [2]. RFID tags can store (or represent) roughly the same amount of data as smart cards, and much more data than barcodes. The main reason why RFID systems have not gained more popularity is their relatively high cost. The production cost of RFID chips as of early 2009 is less than ten US cents, while back in 1999; the cost was around two US dollars per RFID chip. Since tag readers have limits on their operations range imposed by the frequency of the wireless signal used, when RFID networks need to cover large spaces, multiple readers have to be used. The cost of current reader implementations is hundreds of US dollars. As a result, RFID networks may not be yet suitable to track large inventories of inexpensive items until further hardware design and communication protocol simplifications are introduced.

RFID tags can be passive, that is not requiring an internal power source or active where an internal power source (such as a battery) is needed. In this paper, we consider only RFID systems of passive tags. A reader typically charges a set of passive tags within its interrogation zone using inductive coupling; the reader broadcasts electromagnetic signals that the antenna of each in-range tag absorbs. The received signal is converted to electrical current and is stored in on-board capacitor. This technique is called load
modulation for near-field coupling and back scattering for far-field coupling. The tag antennas backscatter part of reader’s irradiated power back to the reader’s antenna in the form of a modulated signal [3]. For a deeper review of RFID systems, see [3],[4], [5].

Since tags transmit information at the same time and the reader can only parse a single received transmission at a time, a collision resolution algorithm has to be used. A major consideration in the design of RFID systems for large-scale deployment is the design simplicity and implementation cost of collision resolution. Due to its simplicity, the Framed Slotted ALOHA (FSA) collision resolution algorithm has been proposed and is currently used in existing implementations [6], [7]. The reader should note that our study focuses on the performance of anti-collision protocols for RFID systems. We do not address another interesting problem, general RFID system performance in terms of delays, cost, and security considerations.

2. Collision resolution in framed slotted ALOHA

The read cycle time for an RFID system is divided into a number of slots that constitute a frame. A slot is a discrete time interval, sufficiently long in duration to allow a tag to transmit its ID number and the 16-bit CRC code of this number. The reader synchronizes the slot boundaries to ensure that collisions are complete, that is, no partial bit collisions can occur. A read cycle is the time interval between two REQUEST commands and it can be repeated until all tags in the interrogation range are identified. Figure 2 shows the beginning of a census with three slots per frame and four tags in range. When frame size is fixed, each consecutively transmitted frame has the same number of slots. After RESET and CALIBRATION commands sent by the reader tags are activated and the reader broadcasts a REQUEST command. Tags then transmit their ID (and appended ID’s CRC code, not shown on the figure).

2.1. Classification of framed slotted ALOHA protocols

FSA can be classified into the BFSA (Basic FSA) and the DFSA (Dynamic FSA) according to whether it
uses a fixed or variable census frame size [8]. If the number of actual tags is unknown in advance, DFSA can identify tags more efficiently than BFSA, by starting a census with a tag estimation procedure [9]. In addition, BFSA and DFSA can further be classified based on whether they support muting and early-end features [8]. Muting makes tags remain silent after being identified by the reader while the early-end allows a reader close an idle slot early when no response is detected. Tags that are not muted by the reader would continue to transmit their ID numbers even after being identified, for the duration of the census. In this paper, we address only the non-muting version of BFSA protocol. For very large populations of tags, using tag muting has been proposed back in 1999 with the ISO 18000 Mode 2 standard [7]. Muting could be implemented at the expense of sending additional acknowledgement messages from the reader to identified tags. As a result, when muting is used, frame times become longer for all frames transmitted. Since computing the minimal total census delay is an optimization problem for given slot duration and given total tag number, non-muting protocols may perform better than muting and vice versa. The expression for total census delay that needs to be optimized includes as variables the total number of tags to be identified, the size of the slot used to identify a tag, and the size of a frame (round) in number of slots. When muting is used, the general performance effect is that the size of the slot (slot duration) becomes larger, while total number of rounds becomes smaller. To address this performance implication, before muting identified tags, a MODE 2 compliant reader considers a ratio of tags responding in the current round to the total tags being muted since the beginning of the census procedure. Estimating the time complexity of FSA using tag muting after successful transmission is a computationally difficult problem, involving the solution of an exponential recurrence [8]. There is an industry trend toward reducing the production cost to draw down the market price of tags. It favors the simplicity of hardware implementation of ALOHA protocols [10]. Allowing for low cost production, non-muting FSA protocols are particularly interesting in this regard.

To maximize network throughput, frame size should be chosen in accordance with the number of tags, as formally presented in the next section. For a fixed slot duration, the sum of total collisions, total idle slots, and total slots with successful transmissions during a census increases with an increase in the total
number of tags. If there are many tags to be identified (that is, read by a reader) and frame size is fixed to a small size, the probability of collisions will be relatively high and the number of identified tags in a frame will be low because tags will be competing for a lesser number of frame slots. When the total number of tags to be read in a census is low and the frame size is large, the probability of more than one tag transmitting during any given frame slot is reduced, but at the expense of the number of idle slots increasing and total census will take longer. If an optimal frame size is used, the total number of slots needed to read a given number of tags is minimized. If the number of tags is unknown in advance, the reader should estimate the number of tags in each read cycle using the number of empty slots, number of slots filled with one tag, and number of collided slots from the previous cycle.

Previous studies estimate optimal frame size for given number of tags based on empirical observations for tag sets of up to 600 tags and frame sizes of about the same cardinality. Their results appear to misestimate the actual optimal frame size by about 20% for the given range of number of tags. Existing standards such as ISO/IEC 15693 [6] and ISO 18000 (Mode 1) [7], propose a tag set size of up to 10,000 tags. Our study provides a theoretical result for optimal frame size that can be applied to any tag set size.

3. Optimal frame size analysis

We first assume that a census can be completed in \( R \) frames, where \( R \) is a positive real number.

**Notation**

\( n \): fixed (for the duration of a census) total number of tags, responding in a read round (frame)

\( N \): fixed (for the duration of a census) frame size in slots

\( \alpha \): assurance level, or the probability that no less than \( n \alpha \) tags will be identified after \( R \) rounds, where \( R \) is the total number of read cycles (frames) required to complete a census with assurance level \( \alpha \)

\( N_{opt} \): optimal frame size, is the frame size that results in minimum total census delay, measured as \( NR \)

\( (n, N, \alpha) \): a three-tuple completely describing a FSA system for the purposes of this analysis
Some studies use the concept of system efficiency as a basis to derive the optimal frame size given a fixed tag set $n$ (or derive a related quantity, such as the optimal number of tags given a fixed frame size $N$). A tag transmits in a frame slot according to a uniform probability of $\frac{1}{N}$. Let $i$ be the frame number, then, the probability $p_1(i)$ of exactly one tag transmitting in a slot for the duration of frame $i$ is distributed as [3]:

$$p_1(i) = \binom{n}{1} \left(\frac{1}{N}\right)^1 \left(1 - \frac{1}{N}\right)^{n-1}. \quad (1)$$

To simplify the analysis, in this paper we assume that no capture effect [11] exist in the system, that is a successful ID reading occurs only when exactly one tag transmits its ID during a slot. The expected number of successful tag ID transmissions is then given by:

$$Np_1(i) = n \left(1 - \frac{1}{N}\right)^{n-1} \quad (2)$$

System efficiency has been defined as [12],[13],[14]:

$$\frac{Np_1(i)}{N} = \frac{n}{N} \left(1 - \frac{1}{N}\right)^{n-1} = p_1(i) \quad (3)$$

To maximize system efficiency, the probability $p_1(i)$ of exactly one tag transmitting in a slot during frame $i$ is maximized, resulting in $n \approx N$ as an optimization relationship between $n$ and $N$. However, maximum system efficiency does not imply minimum total census delay, $\min(RN)$. More specifically, given the cardinality of a tag set $n$, maximizing the frame size $N$ (which results in maximum $p_1(i)$) does not imply $\min(RN)$. Hence, the results from [12],[13], and [14] are suboptimal in terms of minimizing total census delay. We next derive a simple expression for the optimal frame size that guarantees achieving minimum total census delay.

**Theorem 1** If $(n,N,\alpha)$ is a FSA system, where $n,N,\alpha \in \mathbb{Z}^+$ and $\alpha < 1$ then
\[ N_{opt} = \frac{n}{\ln(2)} \]  

(4)

**Proof** Total census delay can be presented as the product [3]:

\[
RN = N \frac{\log_2(1-\alpha)}{\log_2 \left( 1 - \frac{Np_1}{n} \right)}
\]

(5)

Note that we represent \( p_1(i) \) with \( p_1 \), to show that, when frame size is fixed for the duration of a census and tags are non-muting, \( p_1(i) \) is the same in each read round and hence does not vary with \( i \).

Since \( Np_1 \approx ne^{-\frac{N}{n}} \) [9] (see Appendix A) we can write:

\[
RN \approx N \frac{\log_2(1-\alpha)}{\log_2 \left( 1 - e^{-\frac{N}{n}} \right)}
\]

(6)

\( RN \) is a differentiable function with first order derivative:

\[
f'(N) = \frac{d}{dN} \left( \frac{N \log_2(1-\alpha)}{\log_2 \left( 1 - e^{-\frac{N}{n}} \right)} \right)
\]

Or,

\[
f'(N) = \log_2(1-\alpha) \left[ \log_2 \left( 1 - e^{-\frac{N}{n}} \right) - \frac{1}{N} \frac{\frac{-n}{e^N} n}{N} \frac{1}{1 - e^{-\frac{N}{n}}} \log_2 \left( 1 - e^{-\frac{N}{n}} \right) \ln 2 \right]
\]

After performing a second order derivative test, we show that \( RN \) has a minimum in \( N = \frac{n}{\ln 2} \) (see Appendix B for details).
**Corollary** If \((n,N,\alpha)\) is Non-Muting BFSA based RFID system, where \(n,N,\alpha \in \mathbb{R}^+\) and \(\alpha < 1\), the number of tags transmitting can be reduced to an optimum number, for a given frame size \(N\)

\[
n_{\text{opt}} = N \ln 2 ,
\]

resulting in minimum total census delay.

In a real life implementation, the last frame in a census is fully completed, or all of the frame slots are incurred. We hence relax now the assumption that a census can be completed in \(R\) frames by assuming that a census can be completed in an integer number of frames, \([R]\) instead.

**Notation**

\(N_{\text{impl}}\) : Implementation optimum value for \(N\) that results in \(\min([R]N)\) for given tag set \(n\) and assurance level \(\alpha\)

\(R_{\text{impl}}\) : Implementation optimum value for \(R\) , such that \(R_{\text{impl}} \in \mathbb{Z}^+\) and \(R_{\text{impl}}N_{\text{impl}} = \min([R]N)\) for given tag set size \(n\) and assurance level \(\alpha\)

\(R_{r\_opt}\) : A value of \(R\) , such that \(R_{r\_opt} = [R_{opt}]\)

\(R_{l\_opt}\) : A value of \(R\) , such that \(R_{l\_opt} = [R_{opt}] - 1\)

\(N_{r\_opt}\) : A value of \(N\) that yields \(R_{r\_opt}\), for given \(n\)

\(N_{l\_opt}\) : A value of \(N\) that yields \(R_{l\_opt}\), for given \(n\)

**Theorem 2** If \((n,N,\alpha)\) is a FSA system, where \(n,N,\alpha \in \mathbb{Z}^+\) and \(\alpha < 1\), \(N_{\text{impl}}\) can be approximated with \(N\), such that: \(N = r\) if

\[
[R_{opt}]^r = \min([R_{opt}]^r, ([R_{opt}] - 1)^r) ,
\]

or \(N = l\) if
where

\[ r = \text{round} \left( \frac{1}{1 - \left( 1 - 2 \cdot \frac{1}{R_{opt}}\right)^{\frac{1}{n-1}}} \right) \]

and

\[ l = \text{round} \left( \frac{1}{1 - \left( 1 - 2 \cdot \frac{1}{R_{opt}}\right)^{\frac{1}{n-1}}} \right) \]

Proof By the definitions of \( R_{r_{opt}} \) and \( R_{l_{opt}} \)

\[ R_{r_{opt}} = \left\lceil R_{opt} \right\rceil \]

and

\[ R_{l_{opt}} = \left\lfloor R_{opt} \right\rfloor - 1 \]

Solving (10) for \( N_{r_{opt}} \), \( N_{r_{opt}} \approx r \). Solving (11) for \( N_{l_{opt}} \), \( N_{l_{opt}} \approx l \).
Corollary 1 If \((n, N, \alpha)\) is a FSA system, where \(n, N, \alpha \in \mathbb{Z}^+\) and \(\alpha < 1\), \(N_{impl} \approx N\), where, given that (8)

\[
\text{holds, } N = s\left\lfloor \frac{r}{s} \right\rfloor, \text{ if} \\
\left| r - s\left\lfloor \frac{r}{s} \right\rfloor \right| = \min\left(\left| r - s\left\lfloor \frac{r}{s} \right\rfloor \right|, \left| r - s\left(\left\lfloor \frac{r}{s} \right\rfloor - 1\right) \right|\right),
\]

(12)

and \(N = s\left(\left\lfloor \frac{r}{s} \right\rfloor - 1\right), \text{ if} \\
\left| r - s\left(\left\lfloor \frac{r}{s} \right\rfloor - 1\right) \right| = \min\left(\left| r - s\left\lfloor \frac{r}{s} \right\rfloor \right|, \left| r - s\left(\left\lfloor \frac{r}{s} \right\rfloor - 1\right) \right|\right).
\]

(13)

Similarly, given that (9) holds, \(N = s\left\lfloor \frac{l}{s} \right\rfloor\) or \(N = s\left(\left\lfloor \frac{l}{s} \right\rfloor - 1\right), \text{ if} \) (12) and (13) hold as above, but are modified by replacing \(r\) with \(l\).

Corollary 2 If \((n, N, \alpha)\) is a FSA system, where \(n, N, \alpha \in \mathbb{Z}^+\) and \(\alpha < 1\). If \(N = 1.4n\), \(N \neq N_{impl}\).

Previous studies [3] suggest the use of \(N = 1.4n\) to minimize the delay \(NR\) in real RFID implementations. This solution is suboptimal and can result in up to about 20% (or, potentially more) longer than optimal census delays. Figure 3 shows the error in estimating minimum total census delay when using \(N = 1.4n\) for \(n\) ranging from 100 to 200 tags.

Corollary 3 If \((n, N, \alpha)\) is a FSA system, where \(n, N \in \mathbb{Z}^+\) and \(0 < \alpha < 1\), if \(N = n\), \(N \neq N_{impl}\).
Real life implementations have limitations on the maximum frame size used $N_{\text{max}}$. As presented in the beginning of this section, when $n > N_{\text{max}}$, previous studies suggest the use of $n \approx N$ to reduce the number of responding tags, when $N_{\text{max}}$ is reached. This may result in up to about 20\% (or, potentially more) longer than optimal census delay $NR$. Figure 4 shows the error in estimating minimum total census delay when using $N=n$ for $n$ ranging from 10 to 890 tags.

4. Numerical presentation of optimal frame size

The optimal census delay $R_{N_{\text{opt}}}$ for tag sets of size 10 to 100 tags is shown in Figure 5 as circled values of $RN$. In real life implementations census delay $[R]N$ largely fluctuates with increase in $N$ due to fact that $[R_{\text{opt}}]$ frames are incurred instead of $R_{\text{opt}}$ until a census is completed with some assurance level $\alpha$.

Figure 6 shows values of $[R]N$ for tag sets of sizes 10 to 140 tags and $s=5$, where graphs for different values of $n$ are depicted in different grayscale colors. Census delay for two arbitrarily selected values of tag set size $n=45$ and $n=115$ is shown in Figure 7. A frame size of $N_{\text{opt}}$ (shown in triangle) results in suboptimal census delays in actual system implementations, while approximated values of $N_{\text{impl}}$ (shown circled) result in optimal values for the minimum census delay.

As can be seen by comparing the graphs behavior in Figure 5 and Figure 6, the immediate region around the optimal value $N_{\text{opt}}$ for each graph exhibits great results fluctuation, when $[R]N$ is used instead of $RN$. Nevertheless, the approximation of $N_{\text{impl}}$ provides the exact minimum values of the graphs in Figure 6.

To understand the practical implications of our analysis, the slot length in seconds needs to be considered when computing values of frame duration and total census delay. Differences in current RFID related standards result in significant differences in slot duration. Some specifications implement low or high frequency tags and more recently ultra high frequency tags. The bit length of tag identifiers used in the
latest international Electronic Product Code standard is 64 r 96 bits. A 16-bit CRC code is appended to the ID before transmission. The reader transmits an ACK only when an ID is received correctly. After a tag transmit its ID, it waits for an ACK sent by the reader. For example, the size of an ACK message in [15] is six bits. Assuming an implementation of the high frequency standard in [6], the time to transmitted the 48-bit ID with appended 16-bit CRC in such system will be 2.5 ms when transmitted at 26 kbps and 0.6 ms at 106kbps data rate, while the time to receive an ACK for a transmitted ID is 0.23 ms when transmitted at 26 kbps and 0.06 ms at 106kbps data rate. This corresponds to a 4-fold difference in slot duration.

5. Conclusion

In this study, we presented an analytical solution for the optimal frame size of the non-muting version of the BFSA collision resolution protocol. Current standards such as ISO 18000 (Mode 1) [7] and ISO/IEC 15693 [6] recognize the need for handling up to 10,000 tags on a single census of BFSA. Or, the standards allow for up to 10,000 tags to be recognized successfully, if they enter the reader’s range. With such large cardinality for \( n \), errors in estimating total census delay may result in intolerably long actual census procedures and it is preferred that precise optimum values for frame size selection are used. We show that when suboptimal frame size is used, the resulting total census delay may be up to about 20% or more longer than the optimal, which not only results in increased delays, but in more battery consumption. Our results are particularly interesting for another reason, too. Since we present a simple expression for the frame size of choice, the implementation complexity of RFID readers can be reduced substantially, as well. Although we discuss our findings in the context of RFID networks, our results apply to any application of this collision resolution protocol.

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References


[7] ISO18000-6, Air Interface, International Association for Standardization. URL:


Appendix A

What follows is an elaboration on the approximation \( Np_1 \approx ne^{-\frac{n}{N}} \) used in [9]. From the definition of \( e \),

\[
\lim_{x \to 0} \left(1 + x\right)^{\frac{1}{x}} = e.
\]

Let \( x = -\frac{1}{N} \). If we substitute for \( x \) in the definition expression and if \( N \to \infty \), we obtain

\[
\lim_{N \to \infty} \left(1 - \frac{1}{N}\right)^{\frac{-n}{N}} = \lim_{N \to \infty} \left(1 - \frac{1}{N}\right)^{-N} = e^{-\frac{n}{N}}.
\]

Note that \( n > 0 \) and for large values of \( n \), we can approximate \( \left(1 - \frac{1}{N}\right)^{-1} \) with \( \left(1 - \frac{1}{N}\right)^n \) (by dropping the “\(-1\)” in the power expression). Hence, we can write \( Np_1 = n \left(1 - \frac{1}{N}\right)^{-1} \approx ne^{-\frac{n}{N}} \).
Appendix B

To find the minimum of $RN$, a second order derivative test is performed as follows.

$$f'(N) = d\left( \frac{N \log_2 (1-\alpha)}{\log_2 \left( \frac{-n}{1-e^{\frac{-n}{N}}} \right)} \right) = \log_2 (1-\alpha) \left[ \log_2 \left( 1-\frac{n}{1-e^{\frac{-n}{N}}} \right) - \frac{1}{\left( 1-e^{\frac{-n}{N}} \right) \ln 2} \right] \left( \frac{-n}{1-e^{\frac{-n}{N}}} \right)^n = 0$$

Let $x = e^{\frac{-n}{N}}$, then $N = \frac{-n}{\ln x}$. Since $\log_2 (1-\alpha) \neq 0$, $f'(N) = 0$ if $\log_2 (1-x) - \frac{1}{(1-x) \ln 2} \left( \frac{-x}{N} \right)^n = 0$.

Solving $f'(N) = 0$ for $x$ and $N$, we conclude that $x = 0.5$ and $N = \frac{n}{\ln 2}$.

We next find the second order derivative of $RN$,

$$f''(N) = \left. \frac{d}{dN} \right| \left. \frac{d}{dN} \right| \left[ d \left( c \frac{1}{\log_2 \left( 1-\frac{n}{1-e^{\frac{-n}{N}}} \right)} - c \frac{1}{1-e^{\frac{-n}{N}} \log_2 \left( \frac{-n}{1-e^{\frac{-n}{N}}} \right)} \right) \right]$$

where $c = \log_2 (1-\alpha) < 0$.

$$f''(N) = \left. \frac{d}{dN} \right| \left. \frac{d}{dN} \right| \left[ d \left( c \frac{1}{\log_2 \left( 1-\frac{n}{1-e^{\frac{-n}{N}}} \right)} - c \frac{1}{1-e^{\frac{-n}{N}} \log_2 \left( \frac{-n}{1-e^{\frac{-n}{N}}} \right)} \right) \right] = c \left. \frac{d}{dN} \right| \left. \frac{d}{dN} \right| \left[ \frac{1}{\log_2 \left( 1-\frac{n}{1-e^{\frac{-n}{N}}} \right)} - \frac{1}{1-e^{\frac{-n}{N}} \log_2 \left( \frac{-n}{1-e^{\frac{-n}{N}}} \right)} \right] = A + B$$
\[ d \frac{1}{\log_2 \left( 1 - e^{-n} \right)} \left( \frac{1}{1 - e^{-n}} \log_2 e^{-n} \right) \]

\[ A = c \frac{d}{dN} \left( \frac{1 - e^{-n}}{N^2 \ln 2 \log^2 \left( 1 - e^{-n} \right)} \right) = c \frac{\ln 2}{n} = \frac{n}{\ln 2} n^2 \ln 2 \]

\[ B = -c \frac{d}{dN} \left( \frac{1 - e^{-n}}{1 - e^{-N}} \log_2 \left( 1 - e^{-n} \right) \right) \]

\[ = -c \frac{\ln 2}{n} = \frac{n}{\ln 2} n^2 \ln 2 \]

\[ B = -c \frac{\ln 2}{n} \log^4 \left( 1 - e^{-n} \right) \]
We next make a substitution $N = \frac{n}{\ln 2}$ as follows.

$$
B = -c \frac{\frac{1}{n} \left( \frac{n}{\ln 2} \right)^2}{(-1)(-1)(-1)(-1)} = -c \left( \frac{(\ln 2)^4 - 2(\ln 2)^3}{2n^3} \right)
$$

Since $c < 0$ and $n \geq 1$, $f'''(N) = A + B = -c \frac{\ln 2}{n} - c \left( \frac{(\ln 2)^4 - 2(\ln 2)^3}{2n^3} \right) > 0$.  