

Lights Out!: A Survey of Parity Domination in Grid Graphs

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Abstract

A non-empty set of vertices is called an *even dominating set* if each vertex in the graph is adjacent to an even number of vertices in the set (adjacency is reflexive). An odd dominating set is defined analogously. Results on parity dominating sets in grid graphs are surveyed and related results on “Lights Out!” games on grids and graphs are discussed.

1 Introduction

Domination is one of the most widely studied topics in graph theory: the 1998 book by Haynes et al. [17] contains a bibliography with over 1200 papers on the subject. In this survey, we shall be concerned with *even dominating sets*. An even dominating set of a graph is a non-empty subset, D , of the vertices such that each vertex has an even number of neighbors in D , where the “neighbor” relation is reflexive, i.e., a vertex is considered its own neighbor. Likewise, an *odd dominating set* is a subset, O , of the vertices such that each vertex has an odd number of neighbors in O . *Parity domination* has been previously studied in [1, 2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 16, 17, 18]. The initial result on this subject was Sutner’s theorem, which states that every graph contains an odd dominating set [18]. However, not every graph contains an even dominating set: consider the path on three vertices, for example. Therefore we are interested in characterizing graphs that have even dominating sets.

In [13], Fibonacci polynomials over the binary field $GF(2)$ were used to obtain an algebraic condition for determining which $m \times n$ grid graphs

have even dominating sets: the $m \times n$ grid graph has an even dominating set if and only if the $(n + 1)^{st}$ Fibonacci polynomial evaluated at $x + 1$ and the $(m + 1)^{st}$ Fibonacci polynomial evaluated at x are not relatively prime over $GF(2)$. This is perhaps the fundamental result on even dominating sets in grid graphs. Definitions and relevant results related to the Fibonacci polynomials are reviewed in section 2 of this paper. Section 3 is devoted to a survey of structural and algorithmic results on parity domination in grid graphs. Open questions and future directions are stated in section 4.

2 Background

2.1 Nullspace Matrices

Unless otherwise stated, in this paper all polynomials are over the binary field $GF(2)$. When no confusion results, we denote the all-zero n -vector simply by 0. A non-zero $m \times n$ $(0, 1)$ -matrix A is called a *nullspace matrix* if entry (i, j) of A has an even number of 1's in the set of entries consisting of (i, j) and its rectilinear neighbors – this terminology is justified by (9) of Theorem 1, below. It is easy to see that the set of 1's in an $m \times n$ nullspace matrix corresponds to an even dominating set in an $m \times n$ grid graph. See Table 1 for an example of a nullspace matrix.

1	0	0	0
1	1	0	0
1	0	1	0
0	1	1	1

Table 1: A 4×4 Nullspace Matrix

If we choose a non-zero vector $w \in F^n$ where F^n is the binary n -tuple space and let w be the first row of a matrix A , for each $i > 1$ there is a unique way to choose the i^{th} row to make the number of 1's in the closed neighborhood of each entry in the $(i - 1)^{st}$ row even. If r_i is the i^{th} row, the unique way of doing this is given by

$$r_i = Br_{i-1} + r_{i-2} \quad i \geq 2, \quad r_0 = 0, \quad r_1 = w \quad (1)$$

where $B = [b_{ij}]$ is the $n \times n$ tridiagonal $(0, 1)$ -matrix with $b_{ij} = 1$ if and only if $|i - j| \leq 1$ (and the r_i 's in (1) are written as column vectors). In other words we can compute the entries, such as in Table 1, as

$$r_i[j] = r_{i-1}[j] + r_{i-1}[j-1] + r_{i-1}[j+1] + r_{i-2}[j] \pmod{2} \quad (2)$$

where undefined entries are taken to be zero.

If $r_{m+1} = 0$ for some positive integer m , then r_1, r_2, \dots, r_m are the rows of an $m \times n$ nullspace matrix. In [12, 13] it was shown that for each n and for each $w \in F^n$ there exists an m such that $r_{m+1} = 0$. This result and many others, were obtained by using the sequence of Fibonacci polynomials over $GF(2)$ which are defined by

$$f_i = x f_{i-1} + f_{i-2} \quad i \geq 2 \quad f_0 = 0 \quad f_1 = 1 \quad (3)$$

If $w \in F^n$ and $\{r_0, r_1, r_2, \dots\}$ is the sequence of vectors generated by equation (1), let $L(w)$ be the smallest positive integer i such that $r_i = 0$ and let $L_n = \max\{L(w) | w \in F^n\}$. Denote the vector $(1, 0, 0, \dots, 0)$ by e_1 . The following results were proved in [12, 13].

Theorem 1 *Let n be a positive integer and $w \in F^n$.*

- (1) *The sequence $\{r_0, r_1, r_2, \dots\}$ is periodic with period length at most 2^{2^n} .*
- (2) *$L(w)$ exists for each $w \in F^n$ (and $L(w) \leq 2^{2^n}$).*
- (3) *If $r_1 = w$ and $r_i = v$ for some positive integer i , then $L(v)$ divides $L(w)$.*
- (4) *$L_n = L(e_1)$.*
- (5) *$L(w) | L_n$ for each $w \in F^n$.*
- (6) *$r_i = f_i(B)w$ for $i = 0, 1, 2, \dots$ where f_i is the i^{th} Fibonacci polynomial over $GF(2)$ and B is the order n tridiagonal $(0, 1)$ -matrix with $b_{ij} = 1$ if and only if $|i - j| \leq 1$.*
- (7) *L_n is equal to the smallest positive integer t such that $f_t(B) = 0$. So L_n is also the smallest integer t such that for every $w \in F^n$ there exists a $(t - 1) \times n$ nullspace matrix with first row w .*
- (8) *L_n is equal to the smallest positive integer t such that $f_{n+1}(x+1)$ divides $f_t(x)$.*
- (9) *The set of all vectors w which can be the first row of an $m \times n$ nullspace matrix is equal to the nullspace N_{m+1} of $f_{m+1}(B)$. If $d_{m+1}(x)$ is the greatest common divisor of $f_{n+1}(x+1)$ and $f_{m+1}(x)$, then the nullspace of $d_{m+1}(B)$ is equal to N_{m+1} and has dimension equal to the degree of d_{m+1} .*
- (10) *The $m \times n$ grid graph has no even dominating set if and only if the greatest common divisor of $f_{n+1}(x+1)$ and $f_{m+1}(x)$ is 1.*

Experimental evidence¹ for small values of n suggested that the bound

¹A large list of Fibonacci polynomials in factored form and related information can be viewed at the author's web site.

on the period length given in (1) of Theorem 1 could be tightened. If r_0, r_1, r_2, \dots is the sequence produced by equation (1) and if $k = L_n$ (so $r_k = 0$), it is easy to show that $r_{k+j} = r_{k-j}$ for all $j \in \{1, 2, \dots, k\}$. So the period length is equal to L_n or to twice L_n . In [13], it was conjectured that $L_n \leq 3 \times 2^{\frac{n}{2}}$, for $n \neq 5$, and the conjecture was verified in the special case where $f_{n+1}(x)$ is the square of an irreducible polynomial. This conjecture was proved in [15]:

Theorem 2 *If $n \neq 5$ then $L_n < 3 \cdot 2^{\frac{n}{2}}$.*

A key to proving Theorem 2 is an analysis of the *Fibonacci index* of a polynomial. If $p(x)$ is an irreducible polynomial, we say that the Fibonacci index of $p(x)$ is t if t is the smallest positive integer such that $p(x)$ divides $f_t(x)$. Fibonacci indices also play a crucial role in determining which grids have even dominating sets.

2.2 Fibonacci Polynomials

The recurrence relation defined in equation (3) can also be used to generate the sequence of Fibonacci polynomials over the real numbers: $\{g_0 = 0, g_1 = 1, g_2 = x, g_3 = x^2 + 1, g_4 = x^3 + 2x, \dots\}$. Letting $x = 1$ gives the Fibonacci numbers, while reducing the coefficients of the polynomials mod 2 gives the sequence $\{f_0, f_1, f_2, \dots\}$ of Fibonacci polynomials over $GF(2)$. Some of the divisibility properties of the Fibonacci numbers $\{h_0, h_1, h_2, \dots\}$ are known to have analogues in the sequence $\{g_0, g_1, g_2, \dots\}$ of Fibonacci polynomials over the reals. For example $h_i | h_j$ if and only if $i | j$ and $g_i | g_j$ if and only if $i | j$. If h_n is prime, then n is prime; if g_n is irreducible, then n is prime. The converse is false for the Fibonacci numbers ($h_{19} = 4181 = 13 \cdot 37$) but true for the Fibonacci polynomials over the reals [19]. Other results on the divisibility properties of the Fibonacci polynomials may be found in [5, 19].

Most of the results in [13] were obtained using the following properties of the Fibonacci polynomials over $GF(2)$ (Lemma 4 of [13]).

Lemma 3 *Let $f_0, f_1, f_2 \dots$ be the sequence of Fibonacci polynomials over $GF(2)$. Then*

(a) f_i is an odd function for i even and an even function not divisible by x for i odd.

(b) $f_{n-t} + f_{n+t} = x f_n f_t$ for $0 \leq t \leq n$.

(c) $f_{2n} = x f_n^2$ $n \geq 0$.

(d) $f_{2n+1} = f_n^2 + f_{n+1}^2$ $n \geq 0$.

- (e) $f_{mn}(x) = f_m(x)f_n(xf_m(x))$ for $m, n \geq 0$.
- (f) $f_{2mn-p} = xf_{mn}f_{mn-p} + f_p$ for $0 \leq p \leq mn$.
- (g) $f_{2mn+p} = xf_{mn}f_{mn+p} + f_p$.
- (h) $f_t|f_r$, over $GF(2)$, if and only $t|r$.
- (i) If $2^k|n$ then $x^{2^k-1}|f_n(x)$.

2.3 Factorization of Fibonacci Polynomials over $GF(2)$

A fundamental lemma from [15] is now stated and proved, showing how Lemma 3 can be applied.

Lemma 4 [15] *Let k be a positive integer. Then $f_{2^{k+1}}(x) \cdot f_{2^{k-1}}(x)$ is equal to the square of the product of all irreducible polynomials with degrees dividing k , except x .*

Proof: Repeated application of (c) of Lemma 3 shows that $f_{2^m}(x) = x^{2^m-1}$ for every positive integer m . We let $n = 2^k + 1$ and $t = 2^k - 1$ in (b) of Lemma 3 to get

$$xf_{2^{k-1}}(x) \cdot f_{2^{k+1}}(x) = f_{2^{k+1}} + f_2 = x^{2^{k+1}-1} + x = x(x^{2^k-1} + 1)^2$$

But it is well-known that $x^{2^k-1} + 1$ is equal to the product of all irreducible polynomials with degrees dividing k , except x . \square

It is shown in [15] that if $f_n(x)$ is the square of an irreducible polynomial, then n is prime. The converse is not true, as with the Fibonacci numbers and unlike the Fibonacci polynomials over the reals [19]. For example, $f_{17} = (x^4 + x + 1)^2(x^4 + x^3 + x^2 + x + 1)^2$ and $f_{43} = (x^7 + x^6 + x^5 + x^4 + x^2 + x + 1)^2(x^7 + x^5 + x^4 + x^3 + x^2 + x + 1)^2(x^7 + x + 1)^2$.

A fundamental factorization result was proved in [15]. It provides a criterion for determining the values of t for which $f_t(x)$ is irreducible.

Theorem 5 [15] *Let n be an odd integer greater than 1 and let d be the smallest positive integer such that 2^d is congruent to $\pm 1 \pmod{n}$. Then every irreducible polynomial that has Fibonacci index equal to n has degree d . The number of such irreducible polynomials is $\frac{\phi(n)}{2d}$, where $\phi(n)$ is the number of positive integers less than n that are relatively prime to n . Furthermore, the degree of every irreducible factor of $f_n(x)$ divides d .*

This result is then used by Goldwasser et al. to prove further results on the factorization of Fibonacci polynomials; we refer the reader to [15].

3 Results

Consider an $m \times n$ rectangular *grid* of vertices, each of which can be in the *off* or *on* state. We have an *activation* operation that, when applied to vertex v , changes the state of each vertex in v 's closed neighborhood). We say the pair (m, n) is *completely solvable* if every initial configuration of on/off states of the $m \times n$ grid can be transformed to the all-off state by a sequence of activations. The electronic game “Lights Out!” is an example of this using a 5×5 grid. As stated in the introduction, if G is a graph, we say that a non-empty subset D of the vertices of G is an *even (odd) dominating set* if each vertex of G has an even (odd) number of vertices of D in its closed neighborhood. It can be shown that (m, n) is completely solvable if and only if the $m \times n$ grid graph has no even dominating set [12, 13, 14]; equivalently if there is no $m \times n$ nullspace matrix [1, 2, 4, 6, 12, 13, 14].

3.1 Square Nullspace Matrices

We say two polynomials $p_1(x)$ and $p_2(x)$ are *conjugates* if $p_1(x+1) = p_2(x)$ (so also $p_2(x+1) = p_1(x)$), and that $p(x)$ is *self-conjugate* if $p(x+1) = p(x)$. Recall from Theorem 1(10) that there exists an $n \times n$ nullspace matrix if and only if $f_{n+1}(x)$ and $f_{n+1}(x+1)$ are not relatively prime.

For example, there exists a 16×16 nullspace matrix because $f_{17}(x)$ has the self-conjugate irreducible factor $x^4 + x + 1$ and there exists a 32×32 nullspace matrix because $f_{33}(x)$ has the conjugate pair of irreducible factors $x^5 + x^4 + x^3 + x + 1$ and $x^5 + x^3 + x^2 + x + 1$.

Theorem 6 [15] *If $n = 2^k$ or if $n = 2^k - 2$, where k is an integer greater than three, then there exists an $n \times n$ nullspace matrix.*

It was also shown in [15] that $f_0 = 0$, $f_1 = 1$ and $f_5 = (x^2 + x + 1)^2$ are the only self-conjugate Fibonacci polynomials. Among other things, this implies that, except in the case of 4×4 grids, more than $\frac{1}{2^n}$ of the initial configurations of the an $n \times n$ grid can be transformed to the all-off state by a sequence of activations, by a result of [13].

3.2 Algorithmic Problems

The algorithm to determine if (m, n) is completely solvable is now presented. Assume without loss of generality that $n \geq m$.

1. input m and n
2. compute the polynomial $f_{n+1}(x+1)$
3. compute the polynomial $f_{m+1}(x)$
4. if $GCD(f_{n+1}(x+1), f_{m+1}(x)) = 1$ then
 output ‘‘completely solvable’’
 else output ‘‘not completely solvable’’

Algorithm 1. Determines if $m \times n$ grid is completely solvable.

The running time of Algorithm 1 is $O(n \log^2 n)$ [13].

Let $G = (V, E)$ be a simple graph with $|V| = n$. We formulate a decision problem as follows. ‘‘Using a sequence of activation operations, can G (with some initial on/off configuration) be transformed to a graph with at least k switches in the off state.’’ Certain restricted cases are solvable in polynomial time, for example, when G is a series-parallel graph, or when $k = n$ [1, 2, 3]. We call this the *Maximizing Off Switches (MOS)* problem.

Theorem 7 [14] *Maximizing Off Switches is NP-complete.*

A trivial approximation algorithm exists with performance ratio two for MOS [14].

Corollary 8 [14] *There exists a fixed $\epsilon > 0$ such that no polynomial time algorithm for Maximizing Off Switches can have a performance ratio of less than $1 + \epsilon$ unless $P = NP$.*

Using techniques from coding theory, it was shown in [14] that for a constant c , we can determine in polynomial time whether every starting configuration of a graph can be reduced to one with at most c on vertices by a series of activations.

Turning our attention back to grid graphs, Goldwasser and Klostermeyer showed the following.

Fact 9 [14] *In an $n \times m$ grid graph, $m \leq n$, at least $mn - \lceil \frac{m}{2} \rceil$ vertices can be turned off using the activate operation.*

Theorem 10 [13] *For each positive integer t and each real number ϵ , there exist positive integers m and n with $t < m \leq n$ such that some starting configuration of an $n \times m$ grid cannot be changed to a configuration with fewer than $(1 - \epsilon) \frac{m}{\log m}$ switches on.*

4 Future Directions

Some open problems and future directions are now stated. The first problem is key to understanding the complete solvability of grid graphs.

Problem 1 *Let $p(x)$ be factor of one of $\{f_{2^{k+1}}(x), f_{2^{k-1}}(x)\}$, say it is a factor of $f_{2^{k+1}}(x)$. Can one characterize when is $p(x+1)$ a factor of $f_{2^{k+1}}(x)$ and when is it a factor of $f_{2^{k-1}}(x)$?*

A more concrete question about grid graphs is the following.

Question 2 *If $G(n)$ is the fraction of the integers $t \in \{1, 2, \dots, n\}$ such that there exists a $t \times t$ nullspace matrix, what is $\lim_{n \rightarrow \infty} G(n)$?*

It is reported in [15] that $G(10,000) = 0.423$.

Conjecture 3 *Let f_{n+1} be a Fibonacci polynomial over $GF(2)$ which is equal to the square of an irreducible polynomial and let m be the smallest positive integer such that $f_{m+1}(B)e_1 = 0$. Then all initial configurations of the $m \times n$ grid graph can be made to have at most two vertices on by a sequence of activations.*

The main algorithmic question that remains is the following.

Problem 4 *Devise a polynomial time approximation algorithm for MOS with performance ratio less than $2 - \lfloor \frac{c}{n} \rfloor$, where c is a constant.*

Since every graph has at least one odd dominating set, Caro et al. considered the sizes of these odd dominating sets for certain classes of graphs [7, 8]. The sizes of the smallest odd dominating sets for square grids with at most 25 rows is given in [8]. Further study seems warranted for grids and other classes of graphs.

A particularly intriguing question posed by Caro is now stated, which has been verified in the special case of trees [9]. Let G be a graph and $k > 1$ be an integer. A set of vertices D in G is called *non-zero (mod k) dominating set* of G if for every vertex $v \in V(G)$ the following holds: $N[v] \cap D \not\equiv 0 \pmod{k}$. $N[v]$ is the closed neighborhood of v in G . Sutner's theorem states that that every graph G has a non-zero (mod 2) dominating set [18] (and hence a non-zero (mod k) dominating set for k even).

Question 5 *Is it true that every graph G has a non-zero (mod k) dominating set for every integer $k > 1$? What about the special case when $k = 3$?*

Brendan McKay verified by computer for $k = 3$ that this is true for all graphs with up to eleven vertices and for several other special families of graphs with as many as twenty vertices.

References

- [1] A. Amin, L. Clark, and P. Slater (1998), Parity Dimension for Graphs, *Discrete Mathematics*, vol. 187, pp. 1-17
- [2] A. Amin and P. Slater (1992), Neighborhood Domination with Parity Restriction in Graphs, *Congressus Numerantium*, vol. 91, pp. 19-30
- [3] A. Amin and P. Slater (1996), All Parity Realizable Trees, *J. Comb. Math. and Comb. Comput.*, vol. 20, pp. 53-63
- [4] M. Anderson and T. Feil (1998), Turning Lights Out with Linear Algebra, *Math Magazine*, vol. 71, October 1998, pp. 300-303
- [5] M. Bicknell (1970), A Primer for the Fibonacci Numbers part VII: Introduction to Fibonacci Polynomials and their Divisibility Properties, *Fibonacci Quarterly*, vol. 8. no. 5, pp. 407-420
- [6] Y. Caro (1996), Simple Proofs to Three Parity Theorems, *Ars Combinatoria*, 42, pp. 175-180
- [7] Y. Caro and W. Klostermeyer (2002), The Odd Domination Number of a Graph, *J. Comb. Math and Comb. Comput.*, to appear
- [8] Y. Caro, W. Klostermeyer, and J. Goldwasser (2001), Odd and Residue Domination Numbers of a Graph, *Discuss. Math. Graph Theory*, vol. 21, pp. 119-136
- [9] Y. Caro and M. Jacobson (2001), On non-zero(mod k) dominating sets, manuscript
- [10] M. Conlon, M. Falidas, M. Forde, J. Kennedy, S. McIlwaine, and J. Stern (1999), Inversion Numbers of Graphs, *Graph Theory Notes*, vol. XXXVII, pp. 43-49

- [11] R. Cowen, S. Hechler, J. Kennedy, and A. Ryba (1999), Inversion and Neighborhood Inversion in Graphs, *Graph Theory Notes*, vol. XXXVII, pp. 38-42
- [12] J. Goldwasser, W. Klostermeyer, G. Trapp, and C.-Q. Zhang (1995), Setting Switches on a Grid, *Technical Report TR-95-20*, Dept. of Statistics and Computer Science, West Virginia University,
- [13] J. Goldwasser, W. Klostermeyer, and G. Trapp, (1997), Characterizing Switch-Setting Problems, *Linear and Multilinear Algebra*, vol. 43, pp. 121-135
- [14] J. Goldwasser and W. Klostermeyer (1997), Maximization Versions of “Lights Out” Games in Grids and Graphs, *Congressus Numerantium*, vol. 126, pp. 99-111
- [15] J. Goldwasser, W. Klostermeyer, and H. Ware (2002), Fibonacci Polynomials and Parity Domination in Grid Graphs, *Graphs and Combinatorics*, to appear
- [16] M. Halldorsson, J. Kratochvil, and J. Telle (1999), Mod-2 Independence and Domination in Graphs, *Proc. Workshop on Graph-Theoretic Concepts in Computer Science '99*, Bergen, Norway
- [17] T. Haynes, S. Hedetniemi, and P. Slater (1998), **Fundamentals of Domination in Graphs**, Marcel Dekker, New York
- [18] K. Sutner (1989), Linear Cellular Automata and the Garden-of-Eden, *The Mathematical Intelligencer*, vol. 11, no. 2, pp. 49-53
- [19] W. Webb and E. Parberry (1969), Divisibility Properties of Fibonacci Polynomials, *Fibonacci Quarterly*, vol. 7, no. 5, pp. 457-463