Graphs and Combinatorics (2002) 18:119–132

Graphs and Combinatorics

n-Tuple Coloring of Planar Graphs with Large Odd Girth

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Abstract. The main result of the papzer is that any planar graph with odd girth at least $10k - 7$ has a homomorphism to the Kneser graph G_k^{2k+1}, i.e. each vertex can be colored with k colors from the set $\{1, 2, \ldots, 2k + 1\}$ so that adjacent vertices have no colors in common. Thus, for example, if the odd girth of a planar graph is at least 13, then the graph has a homomorphism to G_2^5, also known as the Petersen graph. Other similar results for planar graphs are also obtained with better bounds and additional restrictions.

1. Introduction

Let $G = (V, E)$ be a simple, undirected graph. The *odd girth* of a graph is the length of its shortest odd circuit. An (n, k)-*coloring* of G is a mapping $c : V(G) \mapsto \mathscr{P}_k(Z_n)$ where $\mathscr{P}_k(Z_n)$ is the collection of all k-subsets of $Z_n = \{0, 1, \ldots, n-1\}$ such that $c(u) \cap c(v) = \emptyset$ if $uv \in E(G)$. The concept of (n, k)-coloring is a generalization of the conventional vertex coloring problem, in which case $k = 1$. For example, Fulkerson's Conjecture [4], one of the well-known conjectures in graph theory, is the edge version of the $(6, 2)$-coloring problem for bridgeless cubic graphs.

The main results of the paper are about $(2k + 1, k)$-colorings of planar graphs with large odd girth. It was proved by Stahl that that the chromatic number of an (n, k)-colorable graph is at most $n - 2(k - 1)$ [12]. Thus $(2k + 1, k)$-coloring is in a sense "stronger" than 3-coloring – it is well-known that C_r (the circuit on r vertices) cannot be $(2k + 1, k)$-colored when r is odd and $r < 2k + 1$, since the odd girth of G_k^{2k+1} is $2k + 1$ [11], [12]. It is known from Grötzsch's Theorem [5] (see also [6], [1], [14]) that every planar graph with odd girth at least five is 3-colorable. Of course, every bipartite graph (i.e., a graph with odd girth ∞) is 2-colorable. The property of $(2k + 1, k)$-colorability is naturally expected for planar graphs with sufficiently large odd girth since

$$\text{2-colorability} \underset{\not\Leftarrow}{\Rightarrow} (2k+1, k)\text{-colorability} \underset{\not\Leftarrow}{\Rightarrow} \text{3-colorability}.$$

* This research was partially supported by the National Security Agency under Grant MDA-904-96-1-0014

However, our results cannot be generalized to arbitrary graphs without the restriction of the planarity: non-planar graphs with high girth and high chromatic number were constructed by Erdős [3] (see also [15], page 408).

An (n, k)-coloring of a graph can also be viewed as a homomorphism of a graph to a Kneser graph. The *Kneser graph*, denoted by G_k^n, is defined to be the graph in which vertices represent subsets of cardinality k taken from $Z_n = \{0, \ldots, n-1\}$ and two vertices are adjacent if the intersection of their labels is empty, see for example [13]. Lovász proved the chromatic number of the Kneser graph G_k^n is precisely $n - 2(k-1)$ [11]. Note that G_2^5 is the famous Petersen graph. A *graph homomorphism*, ϕ, is a mapping from G to H such that if $uv \in E(G)$ then $\phi(u)\phi(v) \in E(H)$. The property of $(2k+1, k)$-colorability of a graph G is equivalent to the property that G has a graph homomorphism into the Kneser graph G_k^{2k+1}. Graph homomorphism is a subject that has been extensively studied (see survey articles: [7], [16]).

The following is one of the main results of the paper.

Theorem 1.1. *Let G be a planar graph with odd girth at least* $10k - 7$ $(k \geq 2)$. *Then G is* $(2k+1, k)$*-colorable, or equivalently, G has a homomorphism to the Kneser graph* G_k^{2k+1}.

Note that the case of $k = 1$ is simply the 3-coloring problem and the theorem is false in this case. Thus, we only consider the case $k \geq 2$ in this paper.

2. Reducible Configurations

We begin with two definitions.

Definition 2.1. *Let H be a graph and* $S \subseteq V(H)$. *A coloring (called a precoloring)* $c : S \mapsto V(G_k^{2k+1})$ *of S is extendible to the entire graph H if there is a* $(2k+1, k)$*-coloring* $c' : V(H) \mapsto V(G_k^{2k+1})$ of *H such that* $c(v) = c'(v)$ *for each* $v \in S$. *The ordered pair* (H, S) *is a reducible configuration with respect to* $(2k+1, k)$*-coloring if every precoloring of S is extendible to the entire graph H.*

The strategy and an outline of the proof of our main result are as follows. We produce a list of reducible configurations with respect to $(2k+1, k)$-coloring. That is, we show that a smallest counterexample to our main result cannot contain any of those reducible configurations. Then a calculation of Euler contributions of an alleged counterexample leads to a contradiction to the fact that the graph is planar.

2.1. Lemmas

Before producing a list of reducible configurations, we present a few observations regarding extending a $(2k+1, k)$-coloring from one end of an induced path to the another end of the path. These lemmas will be applied in generating the list of reducible configurations.

For a $(2k+1, k)$-coloring, each color is a k-subset of the set $\{1, 2, \ldots, 2k+1\}$, and is also considered to be a vertex in the Kneser graph G_k^{2k+1}. For a vertex v of a graph G, $N(v)$ denotes the set of neighbors of v in G, while for a color a $(\in V(G_k^{2k+1}))$, $N(a)$ denotes the set of neighbors of a in the graph G_k^{2k+1}.

Results of [12] and [11] imply the odd girth of G_k^{2k+1} is $2k+1$, hence we cannot precolor both endpoints of a path of length $2k-1$ with the same color and then extend that to a proper coloring of the path.

Definition 2.2. *Let $Q = v_0 \dots v_t$ be a path. Let c be a precoloring at the vertex v_0. A color b $(\in V(G_k^{2k+1}))$ is called a legal color at v_t extended from $c(v_0)$ if the coloring c at v_0 can be extended to entire Q such that $c(v_t) = b$. The set of all legal colors at v_t extended from a precoloring c at v_0 is denoted by $L(v_t, v_0, Q)$.*

Later, we shall use the term "extended" to also refer to the situation when a set of vertices is precolored and it is possible to properly color the (sub)graph from that precoloring.

Lemma 2.3. *Let $P = v_0 \dots v_t$ be a path and let v_0 be precolored with the color a $(\in V(G_k^{2k+1}))$. Then the precoloring at v_0 can be extended to the entire path P with the set of legal colors at the last vertex v_t as follows:*

$$L(v_t, v_0, P) = \begin{cases} N(a) & \text{if } t = 1 \\ V(G_k^{2k+1}) \backslash N(a) & \text{if } t = 2k-2 \\ V(G_k^{2k+1}) \backslash \{a\} & \text{if } t = 2k-1 \\ V(G_k^{2k+1}) & \text{if } t \geq 2k. \end{cases} \tag{1}$$

In general,

$$L(v_t, v_0, P) = \begin{cases} \{b : |a \cap b| \geq k - \mu\} & \text{if } t = 2\mu \\ \{b : |a \cap b| \leq \mu\} & \text{if } t = 2\mu + 1. \end{cases} \tag{2}$$

Proof. Since (1) consists of only a few of special cases of (2), we prove only (2) by induction on t. The lemma is true for $t = 1$. Assume that the lemma is true for t.

Case 1. When $t + 1 = 2\mu + 1$ is odd. Let $b \in V(G_k^{2k+1})$. For the $(2k+1, k)$-coloring, b is a legal color at v_{t+1} if and only if there is a legal color b' at the vertex v_t with $|b \cap b'| = 0$. By the inductive hypothesis, b' is a legal color at v_t if and only if $|b' \cap a| \geq k - \mu$. Therefore, b is a legal color at v_{t+1} if and only if $|b \cap a| \leq \mu$.

Case 2. When $t + 1 = 2\mu$ is even. Similar to Case 1. □

Let us refine Definition 2.2 and restate Lemma 2.3 in terms that will be more useful.

Definition 2.4. *Let H be a graph and $S \subseteq V(H)$. For a vertex $v \in V(H) \backslash S$,* let *$\ell(v, S, H)$ be the least number of legal colors that v can receive under any possible $(2k+1, k)$-precoloring of S.*

Obviously, if H is a path $v_0 \cdots v_t$ and $S = \{v_0\}$, then $\ell(v_i, S, H) = |L(v_i, v_0, H)|$ where $L(v_i, v_0, H)$ is defined in Definition 2.2 and was used only for color extension of paths.

The following lemma is a corollary of Lemma 2.3.

Lemma 2.5. *Let $P = v_0 \dots v_t$. Then*

$$\ell(v_t, v_0, P) = \begin{cases} k+1 & \text{if } t = 1 \\ \binom{2k+1}{k} - (k+1) & \text{if } t = 2k-2 \\ \binom{2k+1}{k} - 1 & \text{if } t = 2k-1 \\ \binom{2k+1}{k} & \text{if } t \geq 2k. \end{cases}$$

The next simple observation will also be applied in the construction of some reducible configurations.

Lemma 2.6. *Let $a, b \in V(G_k^{2k+1})$ and $a \neq b$. Then*

$$|N(a) \cap N(b)| \leq 1$$

with equality only if $|a \cup b| = k+1$. (Note that a, b are considered as k-subsets of a $(2k+1)$-set.)

The following lemmas will be applied in the next subsection regarding reducible configurations. To remind the reader, we assume that $k \geq 2$ throughout the paper.

Lemma 2.7. *Let $k \geq 2$ and $P = v_0 \dots v_t$ be a path and c be a precoloring of $\{v_0, v_t\}$.*

1. *If $t \geq 2k$, then c is extendible to P and*

$$\ell(v_1, \{v_0, v_t\}, P) \geq k \geq 2;$$

2. *If $t = 2k-1$ and $c(v_0) \neq c(v_{2k-1})$, then c is extendible to P and*

$$\ell(v_1, \{v_0, v_{2k-1}\}, P) \geq k \geq 2;$$

Proof. The extendibility of c is an obvious corollary of Lemma 2.3 for both cases. Now we need to prove only the remainder of the lemma.

Notation. A subpath of a path $P = v_0 \dots v_t$ between the vertices v_i and v_j ($i \leq j$) is denoted by $P[v_i \dots v_j]$.

By Lemma 2.3

$$L(v_1, \{v_0\}, P[v_0 v_1]) = N(c(v_0))$$

and

$$L(v_1, \{v_t\}, P[v_1 \cdots v_t]) \supseteq \begin{cases} V(G_k^{2k+1}) \backslash \{c(v_t)\} & \text{if } t \geq 2k \\ V(G_k^{2k+1}) \backslash N(c(v_t)) & \text{if } t = 2k-1 \end{cases}$$

Note that

$$L(v_1, \{v_0, v_t\}, P) = L(v_1, \{v_0\}, P[v_0 v_1]) \cap L(v_1, \{v_t\}, P[v_1 \cdots v_t]).$$

Since

$$L(v_1, \{v_0, v_t\}, P) \supseteq \begin{cases} N(c(v_0)) \backslash \{c(v_t)\} & \text{if } t \geq 2k \\ N(c(v_0)) \backslash N(c(v_t)) & \text{if } t = 2k-1 \end{cases}$$

and $|N(c(v_0))| = k+1$, by Lemma 2.6,

$$\ell(v_1, \{v_0, v_t\}, P) \geq \begin{cases} k \geq 2 & \text{if } t \geq 2k \\ k \geq 2 & \text{if } t = 2k-1 \text{ and } c(v_0) \neq c(v_t) \end{cases} \qquad \square$$

2.2. Reducible Configurations

We now list some reducible configurations with respect to $(2k+1, k)$-coloring. Some examples are shown below.

Let $\mathscr{T}(k) = \{T_1^k, T_2^k, \cdots\}$ be a family of graphs constructed as follows.

- T_1^k is a path $P_1 = v_0^1 v_1^1 \cdots v_{2k}^1$;
- The graph T_i^k (for $i \geq 2$) is the union of paths $P_1 = v_0^1 v_1^1 \cdots v_{2k}^1$, and $P_\mu = v_0^\mu v_1^\mu \cdots v_{2k-2}^\mu v_1^{\mu-1}$, for each $\mu \in \{2, \ldots, i\}$. Denote the set of degree one vertices of T_i^k by $S(T_i^k)$. That is, $S(T_i^k) = \{v_0^1, v_{2k}^1, v_0^2, v_0^3, \ldots, v_0^i\}$.

Lemma 2.8. *Let i and k be a pair of positive integers.* (1). *Each ordered pair* $(T_i^k, S(T_i^k))$ *is a reducible configuration with respect to* $(2k+1, k)$*-coloring. That is, any precoloring of* $S(T_i^k)$ *can be extended to the entire graph* T_i^k. (2).

$$\ell(v_1^i, S(T_i^k), T_i^k) \geq k \geq 2.$$

Proof. Induction on *i*. The lemma is true for $i = 1$ (by Lemma 2.7 (1)). Assume that the lemma is true for *i*. Note that

$$T_{i+1}^k = T_i^k \cup P_{i+1}$$

and

$$S(T_{i+1}^k) = S(T_i^k) \cup \{v_0^i\}.$$

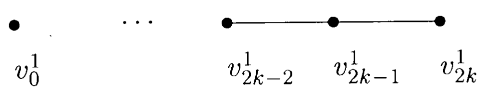

Fig. 1. An example of T_i^k: T_1^k

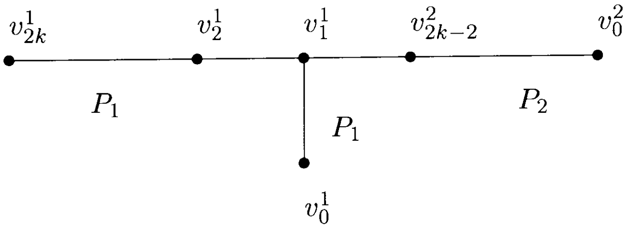

Fig. 2. An example of T_i^k: T_2^k

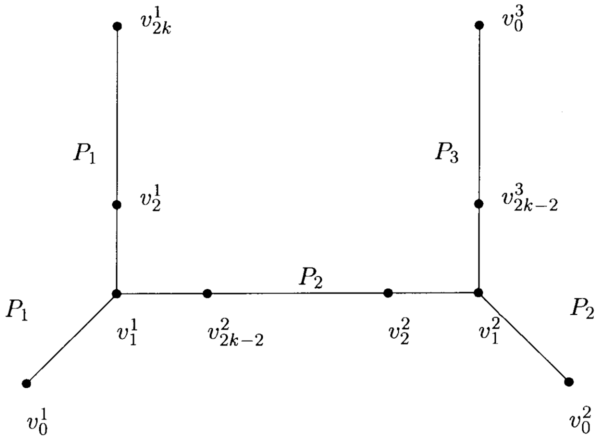

Fig. 3. An example of T_i^k: T_3^k

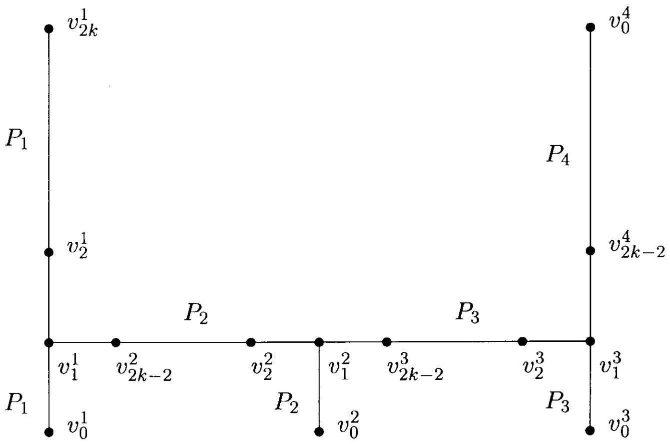

Fig. 4. An example of T_i^k: T_4^k

By inductive hypothesis, we can extend the precoloring c to T_i^k and

$$\ell(v_1^i, S(T_i^k), T_i^k) \geq k \geq 2.$$

We choose a coloring of T_i^k such that $c(v_1^i) \neq c(v_0^{i+1})$. Thus, by Lemma 2.7 (2), the coloring c is further extendible to T_{i+1}^k and

$$\ell(v_1^{i+1}, S(T_{i+1}^k), T_{i+1}^k) \geq k \geq 2. \qquad \square$$

Let $\mathscr{R}(k) = \{R_2^k, R_3^k, \ldots\}$ be a family of graphs constructed as follows (an example is shown in Figure 5). For each $T_i^k \in \mathscr{T}(k)$ $(i \geq 2)$, add a path $Q = v_{2k-1}^1 u_1 u_2 \cdots u_{2k-2} v_1^i$ where $u_1, \ldots, u_{2k-2}$ are new vertices. Denote the set of degree one vertices of R_i^k by $S(R_i^k)$. That is, $S(R_i^k) = \{v_{2k}^1, v_0^1, v_0^2, \ldots, v_0^i\}$.

Lemma 2.9. *Each ordered pair $(R_i^k, S(R_i^k))$ is a reducible configuration with respect to $(2k+1, k)$-coloring. That is, any precoloring of $S(R_i^k)$ can be extended to the entire graph R_i^k.*

Proof. By Lemma 2.8, a precoloring c of $S(R_i^k) = S(T_i^k)$ can be extended to T_i^k and

$$\ell(v_1^i, S(T_i^k), T_i^k) \geq k \geq 2.$$

We may choose an extension of c in T_i^k such that

$$c(v_1^i) \neq c(v_{2k-1}^1).$$

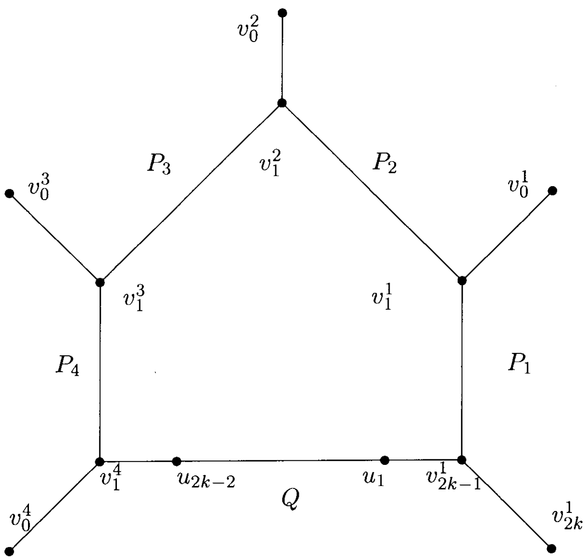

Fig. 5. An example of R_i^k: R_4^k

By Lemma 2.7 (2), the coloring c at $\{v_1^i, v_{2k-1}^1\}$ can be further extended to the entire path $Q = v_{2k-1}^1 u_1 u_2 \cdots u_{2k-2} v_1^i$. That is, any precoloring of $S(R_i^k)$ can be extended to the entire graph R_i^k. □

3. Euler Contributions

Let G be a planar graph (without loops or bridges) embedded in the plane with the vertex set $V(G)$, the edge set $E(G)$, and the face set $F(G)$.

Let the degree of a vertex v be $d(v)$ and the degree of the face f (i.e., the length of the boundary of f) be $d(f)$. For a vertex v of graph G, the set of all edges of G incident with v is denoted by $E(v)$.

Definition 3.1. *At a vertex $v \in V(G)$, let $\{e_1, \ldots, e_{d(v)}\} = E(v)$ where e_i, e_{i+1} $(\mathrm{mod}\, d(v))$ are on the boundary of a face. An angle α (at v) of G is a pair of edges $\{e_i, e_{i+1}\}$.*

Denote the set of all angles of G by $\Lambda(G)$. For an angle $\alpha \in \Lambda(G)$ at a vertex v and at a corner of a face f, denote the vertex v by v_α and the face f by f_α. Note that there are $d(v)$ angles at a vertex v and there are $d(f)$ angles at the corners of a face f and each edge appears in four angles and each angle consists of two edges. It is obvious that

$$|V(G)| = \sum_{\alpha \in \Lambda(G)} \frac{1}{d(v_\alpha)},$$

$$|E(G)| = \sum_{\alpha \in \Lambda(G)} \frac{1}{2},$$

$$|F(G)| = \sum_{\alpha \in \Lambda(G)} \frac{1}{d(f_\alpha)}.$$

By Euler's formula,

$$|F(G)| + |V(G)| = |E(G)| + 2,$$

we have the following *Lebesgue's formula*

$$\sum_{\alpha \in \Lambda(G)} \left(\frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2} \right) = 2. \tag{3}$$

For each angle α, the general term of equation (3)

$$\Phi(\alpha) = \frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2} \tag{4}$$

is called *the Euler contribution* of the angle α.

Let f be a face of G. When one sums the Euler contributions of all angles at all corners of a face f, one obtains the *Euler contribution* of the face f

$$\Phi(f) = 1 - \frac{d(f)}{2} + \sum \frac{1}{d(v)} \tag{5}$$

where the sum is over all the vertices on the boundary of f.

For a vertex v, when one sums the Euler contributions of all angles at v, one obtains the *Euler contribution* of the vertex v

$$\Phi(v) = 1 - \frac{d(v)}{2} + \sum \frac{1}{d(f)} \tag{6}$$

where the sum is over all the faces having v on their boundaries.

For an edge $e = v_1v_2$, let f_1, f_2 be two faces incident with e. Note that e appears in four angles and each angle consists of two edges. When one sums a half of the Euler contributions of all angles containing e, one obtains the *Euler contribution* of the edge e

$$\Phi(e) = \frac{1}{d(v_1)} + \frac{1}{d(v_2)} + \frac{1}{d(f_1)} + \frac{1}{d(f_2)} - 1. \tag{7}$$

According to Lebesgue's formula (3), we have the total Euler contributions of angles, vertices, faces and edges as

$$\sum_{\alpha \in \Lambda(G)} \Phi(\alpha) = \sum_{v \in V(G)} \Phi(v) = \sum_{f \in F(G)} \Phi(f) = \sum_{e \in E(G)} \Phi(e) = 2. \tag{8}$$

Since the total Euler contributions of a planar graph is two, we have the following lemma.

Lemma 3.2 (Lebesgue [10]). *Let G be a planar graph without loops and bridges. There must be an angle, a vertex, a face and an edge such that each of their Euler contributions is positive.*

4. Eliminating Even Circuits and Long Odd Circuits

A facial circuit of a planar graph is a circuit formed by edges of the boundary of a face.

The following lemma was proved in [8], which will enable us to eliminate even circuits and long odd circuits in the proofs of the main results.

Lemma 4.1. *Let g be an odd integer and G be a planar graph with odd girth at least g. If $C = v_0 \ldots v_{r-1}v_0$ is a facial circuit of G with $r \neq g$, then there is an integer $i \in \{0, \ldots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (mod r) remains planar and of odd girth at least g.*

5. *n*-Tuple Coloring of Planar Graphs – Part I.

We can now prove the main result of the paper.

Theorem 1.1. *Let G be a planar graph with odd girth at least* $10k - 7$, *where* $k \geq 2$. *Then* G is $(2k + 1, k)$*-colorable.*

Proof. Let G be a counterexample to the theorem with the least number of vertices.

For vertex-cut $S \subseteq V(G)$, let N_1 and N_2 be the two parts of $G \backslash S$. Let H_i be the subgraph of G induced by the vertex set $V(N_i) \cup S$ for each $i = 1, 2$. Note that H_2 is a proper subgraph of G, therefore, H_2 is $(2k + 1, k)$-colorable. By Lemmas 2.8 and 2.9, the ordered pair (H_1, S) cannot be any configuration listed in those lemmas, for otherwise, any coloring of H_2 can be extended to H_1 since $V(H_1) \cap V(H_2) = S$.

Then G does not contain any reducible configuration listed in Section 2.2.

I. By Lemma 4.1, we have that every facial circuit of G is of equal length $10k - 7$.

By Lemma 2.8 (1) (for $i = 1$), no induced path of G is of length $\geq 2k$. Thus, each facial circuit of G contains at least three degree ≥ 3 vertices since $k \geq 2$ and each facial circuit is of length precisely $10k - 7$.

II. Let $\hat{G}$ be the underlying graph of G. That is, $\hat{G}$ is obtained from G by replacing each maximal induced path with an edge. Since $k \geq 2$, we have that $\lceil \frac{10k-7}{2} \rceil \geq 2k + 1$. Since G does not contain any of the reducible configurations listed in Section 2.2, in particular an induced path on $2k + 1$ vertices, we also have that any facial circuit of G has at least three vertices, each of degree at least three. Hence $\hat{G}$ is in fact a simple graph.

By Lemma 3.2, let $\hat{C}$ be a facial circuit of $\hat{G}$ with a positive Euler contribution. Note that the Euler contribution of a facial circuit of length at least six is at most zero, since the minimum degree $\delta(\hat{G}) \geq 3$. Hence, $\hat{C}$ is of length at most five. Let C be the circuit of G corresponding to $\hat{C}$ in the underlying graph $\hat{G}$.

III. Let $\{x_1, \ldots, x_h\}$ be the set of all vertices of C with degree at least 3 in G. Let $Q_i = x_i \ldots x_{i+1}$ be the segment (a maximal induced path) of C between two degree ≥ 3 vertices ($i = 1, \ldots, h$, mod(h)). (See Figure 6.)

IV. We claim that *each segment* Q_i *is of length at most* $2k - 1$. Since a segment Q_i of length $\geq 2k$ is $T_1^k \in \mathscr{T}(k)$, by Lemma 2.8, it would be a reducible configuration.

V. We claim that C has precisely five segments. That is, C has precisely five degree ≥ 3 vertices $\{x_1, \ldots, x_5\}$. Since $k \geq 2$, we have that

$$|E(C)| = 10k - 7 \geq 8k - 3.$$

If $h \leq 4$, then at least one segment of C is of length at least $2k$. This contradicts IV.

Since $|E(C)| = 10k - 7$ and C has five segments each of which is of length at most $2k - 1$, *the number of segments of length* $2k - 1$ *in* C *is either three or four.*

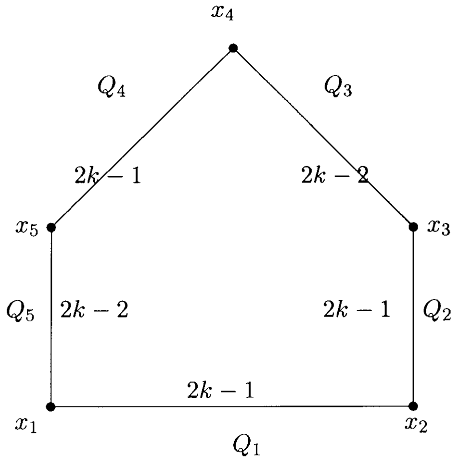

Fig. 6. The facial circuit C in G and its segments

VI. We claim that *if C has two consecutive segments, say, Q_i and Q_{i+1}, of length $2k-1$, then the common vertex x_{i+1} of Q_i and Q_{i+1} must be of degree at least four.* For otherwise, we have a reducible configuration $T_2^k \in \mathscr{T}(k)$ (by Lemma 2.8) with $T_2^k = G[Q_i \cup Q_{i+1} \cup N(x_{i+1})]$ and $S(T_2^k) = \{x_i, x_{i+2}, z_{i+1}\}$ where z_{i+1} is the neighbor of x_{i+1} not contained in C.

VII. We claim that *C cannot have three consecutive segments of length $2k-1$.* Assume that C has three consecutive segments, say, Q_1, Q_2, Q_3, of length $2k-1$, then, by VI, we have that $d(x_2), d(x_3) \geq 4$. Thus, the Euler contribution of the facial circuit $\hat{C}$ in $\hat{G}$ is non-positive, contradicting the choice of $\hat{C}$.

VIII. *C cannot have four segments of length $2k-1$*, since otherwise they are all adjacent to each other around the circuit C and this contradicts VII. By V, the circuit C has precisely three segments of length $2k-1$, and by VII again, they cannot be consecutively around the circuit.

IX. By VIII, the facial circuit C has precisely three segments of length $2k-1$: Q_1, Q_2 and Q_4. Since $|E(C)| = 10k-7$, Q_3 and Q_5 are of length $2k-2$. (See Figure 6.) Similar to VI, by avoiding the reducible configuration $T_2^k \in \mathscr{T}(k)$ (by Lemma 2.8), we have that $d(x_2) \geq 4$. By avoiding the reducible configuration $T_3^k \in \mathscr{T}(k)$ (by Lemma 2.8), $d(x_i) \geq 4$ for either $i=3$ or $i=4$, and $d(x_j) \geq 4$ for either $j=5$ or $j=1$. Again, the Euler contribution of the facial circuit $\hat{C}$ in $\hat{G}$ is non-positive and contradicts the choice of $\hat{C}$. □

6. *n*-Tuple Coloring of Planar Graphs – Part II.

In this section, we restrict the problem to planar graphs of maximum degree three. However, in this case we cannot utilize Lemma 4.1 as that operation may increase a vertex's degree.

Theorem 6.1. *If G is a planar graph with girth at least $10k-9$ ($k \geq 2$) and the maximum degree $\Delta(G) \leq 3$, then G can be $(2k+1,k)$-colored.*

Proof. I. Let G be a counterexample to the theorem with the least number of vertices. Similar to the proof Theorem 1.1, we have the following structure for G:

(i) G does not contain any reducible configuration listed in Section 2.2 ($\mathscr{T}(k)$ and $\mathscr{R}(k)$).
(ii) By Lemma 3.2, the underlying graph $\hat{G}$ contains a facial circuit $\hat{C}$ with a positive Euler contribution. Let C be the circuit of G corresponding to $\hat{C}$ in the underlying graph $\hat{G}$.
(iii) The length of $\hat{C}$ is at most five since $\hat{G}$ is 3- regular and the Euler contribution of a facial circuit of length at least six is at most zero.
(iv) Let $\{x_1,\ldots,x_h\}$ be the set of all vertices of C with degree 3 in G. Let $Q_i = x_i \cdots x_{i+1}$ be the segment (a maximal induced path) of C between two degree 3 vertices ($i = 1,\ldots,h$, mod(h)). Note that a segment of length $\geq 2k$ is a reducible configuration $T_1^k \in \mathscr{T}(k)$ (by Lemma 2.8). Thus, each segment Q_i is of length at most $2k-1$.

Let z_i be the neighbor of x_i not contained in C for each $i = 1,\ldots,h$.

II. We claim that *any two segments of length $2k-1$ in C cannot be adjacent to each other around the facial circuit C*. For otherwise, let Q_1 and Q_2 be two segments of length $2k-1$. Then we have a reducible configuration $T_2^k \in \mathscr{T}(k)$ (by Lemma 2.8) with $T_2^k = G[Q_1 \cup Q_2 \cup N(x_2)]$ and $S(T_2^k) = \{x_1, z_2, x_3\}$.

III. We claim that *C has precisely five segments*. That is, C has precisely five degree 3 vertices $\{x_1,\ldots,x_5\}$. Assume to the contrary that $h \leq 4$. Since $k \geq 2$, we have that

$$|E(C)| \geq 10k-9 \geq 8k-5.$$

Hence, by I-(iv), at least three segments of C are of length precisely $2k-1$. Two of them must be adjacent to each other around the circuit C since $h \leq 4$. This contradicts II and therefore, $h = 5$.

IV. We claim that *C has either one or two segments of length $2k-1$*. By III, C has precisely five segments. Since the length of C is at least $10k-9$ and each segment of C is of length at most $2k-1$ (by II), the number of segments of C with length $2k-1$ is at least one. Furthermore, C cannot have more than two segments of length $2k-1$, for otherwise, two of them must be adjacent to each other around the circuit C and this contradicts II.

V. *Case 1. C has precisely two $(2k-1)$-segments*. By II, without loss of generality, let them be Q_1 and Q_3. If Q_2 is a segment of length $2k-2$, then we have a reducible configuration $T_3^k \in \mathscr{T}(k)$ (by Lemma 2.8) with $T_3^k = G[Q_1 \cup Q_2 \cup Q_3 \cup \{z_2, z_3\}]$ and $S(T_3^k) = \{x_1, z_2, z_3, x_4\}$.

So, Q_2 is of length at most $2k-3$. Then Q_4 and Q_5 are of length precisely $2k-2$ since the length of C is at least $10k-9$ and any pair of segments of length $2k-1$ cannot be adjacent to each other (by II). But, we have a reducible configuration $T_4^k \in \mathscr{T}(k)$ (by Lemma 2.8) with $T_4^k = G[Q_3 \cup Q_4 \cup Q_5 \cup Q_1 \cup \{z_4, z_5, z_1\}]$ and $S(T_4^k) = \{x_3, z_4, z_5, z_1, x_2\}$.

VI. *Case 2. C has precisely one* $2k-1$*-segment*, say, Q_1. Since $|V(C)| \geq 10k-9$, each of $Q_2, \ldots, Q_5$ is of length precisely $2k-2$. Then, we have a reducible configuration $R_4^k \in \mathscr{R}(k)$ (by Lemma 2.9 with $R_4^k = G[C \cup \{z_1, z_2, z_3, z_4, z_5\}]$ and $S(R_4^k) = \{z_1, z_2, z_3, z_4, z_5\}$. □

7. Lower Bounds and Future Work

We believe that the bound in Theorem 1.1 can be improved, perhaps to odd girth seven in the case when $k=2$. A lemma is presented before we prove a lower bound in Proposition 7.2.

Lemma 7.1. *Let H be a graph consisting of a circuit* $v_1 \cdots v_{2k+1} v_1$ *and an extra vertex x and two extra edges* v_1x, xv_3. *Then, for any* $(2k+1, k)$*- coloring c of H,*

$$c(v_2) = c(x).$$

Proof. Let $c(v_1) = a, c(v_2) = b, c(v_3) = c$ and $c(x) = d \in V(G_k^{2k+1})$. By Lemma 2.3,

$$b, d \in N(a) \cap N(c)$$

since $v_1v_2, v_2v_3, v_1x, xv_3 \in E(H)$. By Lemma 2.3 again, $a \neq c$ since $v_3v_4 \cdots v_{2k+1}v_1$ is a path of length $2k-1$. Since $a \neq c$ and $b, d \in N(a) \cap N(c)$, by Lemma 2.6,

$$|N(a) \cap N(c)| = 1 \quad \text{and} \quad |a \cup c| = k+1.$$

That is, $b = d$. □

Proposition 7.2. *There exists a planar graph with odd girth* $2k+1$ *that cannot be* $(2k+1, k)$*-colored.*

Proof. Let H be the complete graph with four vertices $\{u, v, w, x\}$. Let G be the graph obtained from H by replacing the edge uw with a path P_{uw} of length $2k-1$ and replacing the edge vx with a path P_{vx} of length $2k-1$. Note that $P_{uw} \cap P_{vx} = \emptyset$.

Assume that c is a $(2k+1, k)$-coloring of G. By Lemma 7.1, $c(u) = c(w)$ since u is adjacent to v and x in the $(2k+1)$-circuit $vwxP_{vx}v$. But, that $c(u) = c(w)$ contradicts Lemma 2.3 since u and w are joined by the path P_{uw} of length $2k-1$. □

Whether there exist planar graphs with odd girth $2k+1$ and high girth that cannot be $(2k+1,\ k)$-colored is an open question. However, there do exist cubic graphs of girth seven that cannot be (5, 2)-colored [9].

We also note that it is easy to prove by induction that outerplanar graphs with odd girth $2k+1$ can be $(2k+1, k)$-colored, since outerplanar graphs either have a cut-vertex or a chord on the exterior circuit (in the case the graph is 2-connected).

Acknowledgments. The authors wish to thank Gary MacGillivray for pointing out some of the references and for some useful discussions. They also wish to thank the anonymous referees for their valuable comments.

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Received: June 14, 1999
Final version received: July 5, 2000