Odd and Even Dominating Sets with Open Neighborhoods

John L. Goldwasser† and William F. Klostermeyer‡

† Dept. of Mathematics
West Virginia University
Morgantown, WV 26506

‡ Dept. of Computer and Information Sciences
University of North Florida
Jacksonville, FL 32224-2669
E-mail: jgoldwas@math.wvu.edu, wkloster@unf.edu

Abstract

A subset D of the vertex set V of a graph is called an *open oddd dominating set* if each vertex in V is adjacent to an odd number of vertices in D (adjacency is irreflexive). In this paper we solve the existence and enumeration problems for odd open dominating sets (and analogously defined *even open dominating sets*) in the $m \times n$ grid graph and prove some structural results for those that do exist. We use a combination of combinatorial and linear algebraic methods, with particular reliance on the sequence of Fibonacci polynomials over $GF(2)$.

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1 Introduction

An even dominating set of a graph G is a non-empty subset, D, of the vertices such that each vertex in G has an even number of neighbors in D, where the "neighbor" relation is reflexive, i.e., a vertex is its own neighbor. Likewise, an *odd dominating set* is a subset, D, of the vertices such that each vertex has an odd number of neighbors in D. *Parity domination* has been studied in, for example [1, 2, 4, 5, 6] and several other papers. A fundamental result on the subject is Sutner's theorem that every graph contains an odd dominating set, see for example [7].

A *closed dominating set* (usually just called a dominating set) in a graph G is a subset D of the vertex set V such that $N[v] \cap D$ is non-empty for each $v \in V$, where $N[v]$ is the closed neighborhood of v. An *open dominating set* has a similar definition, except we use the open neighborhood, $N(v)$, instead. Open dominating sets are a well-studied variant of dominating sets, see, for example, Chapter 6 of [7]. In this paper, we consider open dominating sets with parity constraints, which we now define. Specifically for a graph $G = (V, E)$:

Odd Open Dominating Set is a set D such that $|N(v) \cap D|$ is odd for all $v \in V(G)$.

Even Open Dominating Set is a set D such that $|N(v) \cap D|$ is even for all $v \in V(G)$.

We remark that whereas an odd open (closed) dominating set is an open (closed) dominating set, by our definitions an even open (closed) dominating set is not necessarily an open (closed) dominating set, because a vertex might have zero neighbors in the even open (closed) dominating set; that is, zero is an even number.

As in [2, 3, 4, 5, 6], our attention will be focused on grid graphs. We shall review some of the previous results and methodology on the subject of open (and closed) even and odd dominating sets (and give new proofs in some cases) and present new results on the problem and related matrices and recurrences. We note that the special case in which we require that $|N(v) \cap D| = 1$ for all $v \in V(G)$ is discussed in [3, 8].

2 Background

Let $G = (V, E)$ be a simple, undirected graph with adjacency matrix $A = A(G) = [a_{ij}]$ with respect to the ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertex set V (so $a_{ij} = 1$ if $v_i v_j$ is an edge of G and $a_{ij} = 0$ otherwise). We can represent a subset S of V by its characteristic vector $x(S) = [x_1, x_2, \ldots, x_n]^t$ where $x_i = 1$ if and only if $v_i \in S$. Let J_n denote the row n-vector with all entries equal to 1. Let I_n denote the $n \times n$ identity matrix. Sutner's theorem can then be re-stated as:

Theorem 1 *If G is a graph with $A = A(G)$ then the system $(A+I_n)x = J_n^t$ has a solution for x over the binary field.*

For notational simplicity, we shall sometimes omit the superscript indicating transpose when it is obvious from the context that a vector transpose is necessary.

Since $A + I_n$ is a symmetric matrix, its nullspace and rangespace are orthogonal complements (using the standard inner product operation over the binary field). So J_n is in the rangespace of $A + I_n$ if and only if each vector in the nullspace of $A + I_n$ has an even number of 1's (and that gives the standard proof of Sutner's theorem, because a simple parity argument shows this to be the case).

Clearly, S is an even closed dominating set for G if and only if $x(S)$ is in the nullspace of $A(G) + I_n$. So G has a non-empty even closed dominating set if and only if $A(G) + I_n$ is not invertible over $GF(2)$. In one sense, this gives a characterization (with a polynomial time algorithm) of which graphs have even closed dominating sets, but it would be nicer to have another (even faster) method. While this has not been done in general, a method exists for grid graphs [4]. Let $G_{m,n}$ denote the $m \times n$ grid graph, i.e., $P_m \times P_n$.

Let f_i be the i^{th} Fibonacci polynomial defined over $GF(2)$ by

$$f_n = xf_{n-1} + f_{n-2} \quad n \geq 2,\ f_0 = 0, f_1 = 1$$

(so $f_2 = x, f_3 = x^2+1, f_4 = x^3, f_5 = x^4+x^2+1$). Goldwasser, Klostermeyer, and Trapp proved the following connection.

Theorem 2 *[4] The number of even closed dominating sets in $G_{m,n}$ is 2^d where d is the degree of the greatest common divisor of $f_{m+1}(x)$ and $f_{n+1}(x+1)$ where $f_0, f_1, \ldots$ is the sequence of Fibonacci polynomials over $GF(2)$.*

Thus there is no non-empty even closed dominating set of $G_{m,n}$ if and only if $f_{m+1}(x)$ and $f_{n+1}(x+1)$ are relatively prime (which happens if and only if $A(G_{m,n}) + I_n$ is invertible). So an understanding of the Fibonacci polynomials sheds some light on this problem; in [4, 6], Goldwasser and Klostermeyer prove some of the relevant properties.

The story is somewhat different (and generally less complicated) for odd open and even open dominating sets. For one thing, there is no analogue of Sutner's theorem: the $m \times n$ grid graph does not have an odd open dominating set for infinitely many pairs (m, n) (the parity argument used to prove Sutner's theorem requires closed neighborhoods). Cowen et al. [3], have solved the existence problem for odd open dominating sets in grid graphs. In this paper we give a complete solution for both the existence and enumeration problems for odd open and even open dominating sets in grid graphs, as well as present results on a related recurrence. Where our results overlap with those in [3], we often present our own proofs for completeness and because of their greater reliance on the Fibonacci polynomials. We see the main value of this work as the completeness of the solutions and the interplay between linear algebra and combinatorics used in the proofs.

3 Results

Our main results are the following, whose proofs appear in subsequent sections. When clear, we use 0 to denote the all-zeroes vector of some prescribed length.

Theorem 3 *The following are equivalent:*
(i) $m+1$ *and* $n+1$ *are relatively prime.*
(ii) The adjacency matrix $A(G_{m,n})$ *is invertible.*
(iii) There does not exist a non-empty even open dominating set of $G_{m,n}$*.*
(iv) There exists a unique odd open dominating set of $G_{m,n}$*.*

Cowen et al. [3] provided a complete characterization of which grid graphs have odd open dominating sets. We extend their results in part (ii) of the next theorem by also giving the number of these sets.

Theorem 4 *Let* m, n *be positive integers and let* $d+1$ *bet the gcd of* $m+1$ *and* $n+1$*. Then*
(i) The number of even open dominating sets of $G_{m,n}$ *is* 2^d*.*
(ii) The number of odd open dominating sets of $G_{m,n}$ *is* 2^d *if there does not exist a positive integer* t *such that* $\frac{m+1}{2^t}$ *and* $\frac{n+1}{2^t}$ *are both odd integers (which is the case if* m *or* n *is even) and 0 if there does exist such a positive integer* t*.*

Note that Theorem 4 gives very fast algorithms for determining if the $m \times n$ grid graph contains a non-empty even open or an odd open dominating set. The former simply requires a gcd calculation, the latter requires finding if a suitable t exists, which can be done by checking if the binary representations of $m+1$ and $n+1$ each end with the same number of 0's (which can of course be done in linear time).

We say that a binary $m \times n$ matrix D is an odd (even) open dominating set matrix for $G_{m,n}$ if the positions of the 1's in D correspond to an odd (even) open dominating set in $G_{m,n}$. Denote the rows in such a matrix by $r_1, r_2, \ldots, r_m$. For example, the 2×4 matrix with $r_1 = 1001$ and $r_2 = 1001$ is an odd open dominating set matrix.

A basic observation is that once r_1 is fixed, the remaining rows of an even open dominating set matrix can be computed by the following recurrence:

$$r_{i,j} = r_{i-1,j-1} + r_{i-1,j+1} + r_{i-2,j} \pmod 2 \tag{1}$$

where undefined entries are taken to be zero. In the case of an odd open dominating set matrix, simply modify the equation as follows:

$$r_{i,j} = 1 + r_{i-1,j-1} + r_{i-1,j+1} + r_{i-2,j} \ \ (mod\ 2) \tag{2}$$

Then $r_1, r_2, \ldots, r_m$ are the rows of an $m \times n$ even/odd dominating set matrix if and only if $r_{m+1} = 0$ using Equation (1)/(2).

Corollary 5 *If n is odd, there does not exist an odd open dominating set in $G_{n,n}$. If n is even, then for each binary n-vector w, there exists a unique $n \times n$ odd open dominating set matrix with first row w.*

Theorem 6 *Let n be an even integer. For each $i \in \{1, 2, \ldots, n\}$ there exists an $n \times n$ odd open dominating set matrix with i^{th} row all 0's, but there does not exist one with more than one row of 0's.*

The following Theorem reformulates many of the results from Cowen et al. [3].

Theorem 7 *Let n be an odd integer.*
(i) For each binary n-vector w there exists a unique $(2n+1) \times n$ odd open dominating set matrix with first row w.
(ii) For each $j \in \{1, 2, \ldots, 2n+1\}$ such that there does not exist a positive integer t such that $\frac{j+1}{2^t}$ and $\frac{n+1}{2^t}$ are both odd integers, there exists a $(2n+1) \times n$ odd open dominating set matrix with j^{th} row all 0's, but there does not exist one with more than one row of 0's.
(iii) Let k_n be the n-vector with alternating 1's and 0's with first and last entry 1. The $(n+1)^{st}$ row of every $(2n+1) \times n$ odd open dominating set matrix is k_n. If $j \in \{1, 2, \ldots, n\}$, then the j^{th} row is all 0's if and only if the $(j+n+1)^{st}$ row is k_n, while the j^{th} row is k_n if and only if the $(j+n+1)^{st}$ row is all 0's.

If w is any binary n-vector, then we can generate an infinite odd open dominating set matrix with n columns and rows $r_1, r_2, \ldots$ by setting $r_1 = w$ and applying the recurrence in Equation (2). That is, the number of columns is fixed and there are infinitely many rows. If n is even, then by Corollary 5, no matter what vector we choose for r_1, r_{n+1} will be 0. If n is odd, then by Theorem 7, r_{n+1} will be k_n and r_{2n+2} will be 0. Since each row in the sequence $r_1, r_2, r_3, \ldots$ is determined by the previous rows (with $r_0 = 0$), the sequence must be periodic.

We say that an n-vector $x = (x_1, x_2, \ldots, x_n)$ is symmetrical if $x_i = x_{n+1-i}$ for $i = 1, 2, \ldots, n$.

Theorem 8 *Let M be an infinite odd open dominating set matrix with n columns and row vectors $r_1, r_2, \ldots$. Then the row sequence has period $n+1$ if n is even and r_1 is symmetrical and has period $2n+2$ otherwise.*

$$A_{m,n} = \begin{bmatrix} P_n & I_n & 0 & 0 & \dots & 0 & 0 \\ I_n & P_n & I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & P_n & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_n & P_n & I_n \\ 0 & 0 & \dots & 0 & 0 & I_n & P_n \end{bmatrix}$$

Figure 1: Matrix A

So, for example, by Theorem 7 and Theorem 8, there exists an infinite odd open dominating set matrix with 13 columns and row sequence $r_1, r_2, \dots$ where $r_i = 0$ if and only if $i \equiv 0 (\bmod\ 28)$ or $i \equiv 11 (\bmod\ 28)$ (because 4 divides 12, but does not divide 14) and $r_i = 1010101010101$ if and only if $i \equiv 14 (\bmod\ 28)$ or $i \equiv 25 (\bmod\ 28)$.

4 Construction of Odd and Even Open Dominating Set Matrices

In this section, we review the basics. Number the vertices in the $m \times n$ grid graph $G_{m,n}$ from 1 to $k = mn$ from top left to bottom right increasing across each row. Let $A_{m,n}$ to be the $mn \times mn$ binary matrix shown in block form in Figure 1, where I is the $n \times n$ identity matrix and let $P_n = [p_{ij}]$ be the $n \times n$ matrix defined by $b_{ij} = 1$ if $|i - j| = 1$ and $b_{ij} = 0$ otherwise. Note that P_n is the adjacency matrix for the path on n vertices, with the usual vertex ordering (i.e., from left to right).

Let $\mathcal{E}$ ($\mathcal{O}$) be the set of all ordered pairs (m, n) such that there exists a non-empty even (odd) open dominating set in $G_{m,n}$. Clearly, $(m, n) \in \mathcal{E}(\mathcal{O})$ if and only if $(n, m) \in \mathcal{E}(\mathcal{O})$.

The following is analogous to the result for closed even dominating sets; and follows because a set S is an even open dominating set if and only if the characteristic vector of S is in the nullspace of $A_{m,n}$.

Proposition 9 *$(m, n) \in \mathcal{E}$ if and only if $A_{m,n}$ is singular.*

Proposition 10 *Let J_{mn} be the mn-vector with all entries equal to 1. The following are equivalent:*
(i) $(m, n) \in (\mathcal{O})$.
(ii) J_{mn} is in the rangespace of $A_{m,n}$.
(iii) Every vector in the nullspace of $A_{m,n}$ has an even number of 1's.

Proof: The equivalence of (i) and (ii) is obvious because $A_{m,n}x = J_{mn}$ if and only if x is the characteristic vector for an odd open dominating set. The equivalence of (ii) and (iii) follows because the rangespace and nullspace of the symmetric matrix $A_{m,n}$ are orthogonal complements under the standard inner product operation over $GF(2)$. □

Corollary 11 *If n is odd, then there does not exist an $n \times n$ even open dominating set matrix.*

Proof: If n is odd then I_n is an odd open dominating set matrix with an odd number of 1's. □

Corollary 11 is just a part of Corollary 5, but we will need it for a proof before we actually prove Corollary 5.

Proposition 12 *(m, n) is either in $\mathcal{E}$, $\mathcal{O}$ (or both).*

Proof: If $(m, n) \notin \mathcal{E}$, then, by Proposition 9, $A_{m,n}$ is invertible. Hence $A_{m,n}x = J_{mn}$ has a (unique) solution and, by Proposition 10, $(m, n) \in \mathcal{O}$. □

Proposition 13 *Let w be any binary n-vector. Define vector sequence $z_1(w), z_2(w), \ldots$ by $z_i(w) = f_i(P_n)w, i = 1, 2, \ldots$ where $f_1, f_2, \ldots$ is the sequence of Fibonacci polynomials over $GF(2)$. If $z_{m+1}(w) = 0$, the $m \times n$ matrix with i^{th} row $z_i(w), i = 1, 2, \ldots, m$ is an even open dominating set matrix.*

Proof: To satisfy equation (1), with $w = (r_{1,1}, r_{1,2}, \ldots, r_{1,n})$, we need to have

$$z_0(w) = 0 \quad z_1(w) = w \quad z_i(w) = P_n z_{i-1}(w) + z_{i-2}(w) \quad i \geq 2 \tag{3}$$

It is easy to check by induction that the solution to the vector recurrence relation (3) is $z_i(w) = f_i(P_n)w, i = 0, 1, 2, \ldots$. So, if $z_{m+1}(w) = 0$, we do get an $m \times n$ even open dominating set matrix. □

Proposition 14 *Let w be any binary n-vector. Define vector sequence $x_1(w), x_2(w), \ldots$ by $x_i(w) = f_i(P_n)w, + \sum_{j=1}^{i-1} f_j(P_n)J_n, i = 1, 2, \ldots$ where $f_1, f_2, \ldots$ is the sequence of Fibonacci polynomials over $GF(2)$, J_n is the all 1's vector and P_n is the usual adjacency matrix for the path on n vertices. If $x_{m+1}(w) = 0$, the $m \times n$ matrix with i^{th} row $x_i(w), i = 1, 2, \ldots, m$ is an odd open dominating set matrix.*

Proof: To satisfy equation (2), with $w = (r_{1,1}, r_{1,2}, \ldots, r_{1,n})$, we need to have

$$x_0(w) = 0 \;\; x_1(w) = w \;\; x_i(w) = P_n x_{i-1}(w) + x_{i-2}(w) + J_n \;\; i \geq 2 \qquad (4)$$

We shall show by induction that the solution to the vector recurrence relation (4) is as given. If $i \geq 2$ then

$$P_n x_{i-1}(w) + x_{i-2}(w) + J_n$$

$$= P_n[f_{i-1}(P_n)w + \sum_{j=1}^{i-2} f_j(P_n)J_n] + f_{i-2}(P_n)w + \sum_{j=1}^{i-3} f_j(P_n)J_n + J_n$$

$$= [P_n f_{i-1}(P_n)w + f_{i-2}(P_n)w] + P_n \sum_{j=2}^{i-1} f_{j-1}(P_n)J_n + \sum_{j=3}^{i-1} f_{j-2}(P_n)J_n + J_n$$

$$= f_i(P_n)w + \sum_{j=2}^{i-1}[P_n f_{j-1}(P_n) + f_{j-2}(P_n)]J_n + J_n$$

$$= f_i(P_n)w + \sum_{j=1}^{i-1} f_j(P_n)J_n = x_i(w).$$

□

5 Preparatory Results and Fibonacci Polynomials

The usefulness of the Fibonacci polynomials for finding dominating sets in grid graphs is because of Propositions 13 and 14 and the following relationship between Fibonacci polynomials and the characteristic and minimal polynomials of the matrix P_n.

Proposition 15 *For each positive integer n, let $\mathcal{X}_n$ and m_n be the characteristic and minimal polynomials for P_n, respectively. Then $\mathcal{X}_n = m_n = f_{n+1}$ and $f_{n+1}(P_n) = 0$..*

Proof: Laplace expansion on the first row gives us

$$det[xI_n + P_n] = xdet[xI_{n-1} + P_{n-1}] + det[xI_{n-2} + P_{n-2}],$$

so the sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ satisfies the Fibonacci recurrence relation. Since $\mathcal{X}_1 = x = f_2$ and $\mathcal{X}_2 = x^2 + 1 = f_3$, we get $\mathcal{X}_n = f_{n+1}$ for all positive integers n.

To show that $\mathcal{X}_n = m_n$, we generate the $n \times n$ even open dominating set matrix with first row $e_1 = (1, 0, 0, 0, \ldots, 0)$. Clearly, we get the identity matrix I_n with i^{th} row e_i (all 0's except for a 1 in the i^{th} position), which is equal to $f_i(P_n)e_1$, by Proposition 14. Thus $f_1(P_n)e_1, f_2(P_n)e_1, \ldots, f_n(P_n)e_1$ is a linearly independent set of vectors. Since the degree of f_i is $i-1$, the polynomials $f_1, f_2, \ldots, f_n$ are a basis for the space of all polynomials over $GF(2)$ with degree at most $n-1$. So if m_n has degree less than n, then $m_n = \sum_{i=1}^n c_i f_i$ for some constants $c_1, c_2, \ldots, c_n$, not all equal to 0. But then $0 = m_n(P_n) = \sum_{i=1}^n c_i f_i(P_n)$, contradicting the independence of $f_1(P_n)e_1, \ldots, f_n(P_n)e_1$. Hence the degree of m_n is equal to n and thus $m_n = \mathcal{X}_n$. By the Cayley-Hamilton theorem, cf. [9], $f_{n+1}(P_n) = 0$. □

Proposition 16 *If g is any polynomial over $GF(2)$, then the dimension of the nullspace of $g(P_n)$ is the degree of the gcd of g and f_{n+1}.*

Proof: This is a direct consequence of the Primary Decomposition Theorem, cf. [9], and the fact that the minimal and characteristic polynomials of P_n are both equal to f_{n+1}, as shown in Proposition 15. □

We remark that if we are looking for closed dominating sets then we would be interested in the dimension of the nullspace of $g(P_n + I_n)$, and it is equal to the degree of the gcd of $g(x+1)$ and $f_{n+1}(x)$, making subsequent calculations much more difficult, see [4, 6].

We will need the following properties of Fibonacci polynomials, all proved in [4].

Proposition 17 *[4] Let $f_0, f_1, f_2 \ldots$ be the sequence of Fibonacci polynomials over $GF(2)$. Then*
(i) If $n \geq 0$, then f_{n+1} has degree n and is an odd function for n odd and an even function with constant term 1 for n even.
(ii) For each positive integer k, $f_{2k-1} = (f_{k-1} + f_k)^2$.
(iii) For each positive integer k, $f_{2k} = x f_k^2$.
(iv) If $d+1$ is the gcd of $m+1$ and $n+1$, then f_{d+1} is the gcd of f_{m+1} and f_{n+1}.

Propositions 9, 10, 12 give conditions for the existence of even/odd open dominating set matrices in terms of the nullspace of $A_{m,n}$, an $mn \times mn$ matrix. With the Fibonacci polynomials, we can now derive conditions in terms of $m \times n$ matrices with nice algebraic properties.

Proposition 18 *Let $A_{m,n}$ be the adjacency matrix for $G_{m,n}$ (with the usual vertex ordering) and let $d+1$ equal the gcd of $m+1$ and $n+1$. Then*
(i) $x \in$ nullspace$(A_{m,n})$ if and only if x is the characteristic vector for an

even open dominating set of $G_{m,m}$.
(ii) $w \in$ nullspace$(f_{m+1}(P_n))$ if and only if w is the first row of an $m \times n$ even open dominating set matrix.
(iii) $y \in$ nullspace$(f_{n+1}(P_m))$ if and only if y is the first column of an $m \times n$ even open dominating set matrix.
(iv) nullspace$(f_{m+1}(P_n))$, nullspace$(A_{m,n})$, and nullspace$(f_{n+1}(P_m))$ all have dimension d.
(v) $G_{m,n}$ has 2^d even open dominating sets and either 2^d or 0 odd open dominating sets.

Proof: We have discussed (i) earlier.

By Proposition 13, if we generate an even open dominating set matrix with first row w, then the$(m+1)^{st}$ row is $f_{m+1}(P_n)w$. So the first m rows form and $m \times n$ even open dominating set matrix if and only if $f_{m+1}(P_n)w = 0$, proving (ii). The statement (iii) is the same as (ii) working with transposes.

The three nullspaces in (iv) must have the same dimension because it is just counting the same thing in three ways. By Proposition 16, the dimension of nullspace$(f_{m+1}(P_n))$ is the degree of the gcd of f_{m+1} and f_{n+1}, and this is equal to d by Proposition 17 part (iv). By (iv), $G_{m,n}$ obviously has 2^d even open dominating sets. If B is an odd open dominating set matrix, then the set of all odd open dominating set matrices is clearly equal to $\{B + E : E$ is an even open dominating set matrix$\}$, so there are either 2^d or 0 of them. □

6 Proofs

We have actually already proved Theorem 3 piece by piece, but we present a proof here to have it all in one place.

Proof of Theorem 3
(i) $\Rightarrow$ (iii) If $m+1$ and $n+1$ are relatively prime and $d+1$ is the gcd of $m+1$ and $n+1$, then $d = 0$, so by Proposition 18(v), there is no even open dominating set of $G_{m,n}$.
(iii) $\Rightarrow$ (ii) By (iii), the nullspace of $A_{m,n}$ contains only 0, so it is invertible.
(ii) $\Rightarrow$ (iv) If (ii) holds, then the system $A_{m,n}x = J_{mn}$ has a unique solution.
(iv) $\Rightarrow$ (i) If the system $A_{m,n}x = J_{mn}$ has a unique solution, then $A_{m,n}$ is invertible and the dimension of nullspace$(A_{m,n})$ is 0. If $m+1$ and $n+1$ are not relatively prime, then their gcd, $d+1$, is such that $d \geq 1$. Thus by Proposition 18, $G_{m,n}$ has at least two even open dominating sets, which is impossible. □

If $x = (x_1, x_2, \ldots, x_n)$ is an n-vector, let x^R be the n-vector with the elements of x in reverse order.

Lemma 19 *If A is an $n \times n$ even open dominating set matrix with rows $r_1, r_2, \ldots, r_n$, then $r_i = r^R_{n+1-i}$ for $i = 1, 2, \ldots, n$.*

0	0	1	0	0	0	0
0	1	0	1	0	0	0
1	0	1	0	1	0	0
0	1	0	1	0	1	0
0	0	1	0	1	0	1
0	0	0	1	0	1	0
0	0	0	0	1	0	0

Figure 2. An Even Open Dominating Set Matrix

Proof: As shown in Figure 2, for $n = 7$, a first row with just one 1, in the k^{th} column, generates an $n \times n$ even open dominating set matrix with a "rectangular" array of 1's: the "corners" of the rectangle being the 1's in positions $(1, k), (k, 1), (n + 1 - k, n)$, and $(n, n + 1 - k)$. If the rows of the matrix are $s_1, s_2, \ldots, s_n$, it is clear that $s_i = s^R_{n+1-i}$ for $i = 1, 2, \ldots, n$. Since the first row of A is the sum of vectors with just one 1, we get A by adding the matrices (over $GF(2)$) which each have the property that $s_i = s^R_{n+1-i}$ for $i = 1, 2, \ldots, n$, so A has that property as well. $\Box$

Note that a square even open dominating set matrix must be symmetric about both diagonals.

We remark that since the n^{th} row of an $n \times n$ even open dominating set matrix with first row w is $f_n(P_n)w$ (Proposition 13) and it is also w^R (Lemma 19), we have proved that $f_n(P_n)$ must be equal to the matrix obtained from I_n by reversing the order of the rows. We were unable to find a linear algebra proof for this which does not rely on the "rectangular" picture argument.

The next result is an immediate consequence of Lemma 19.

Corollary 20 *If A is an $n \times n$ even open dominating set matrix with rows $r_1, r_2, \ldots, r_n$ and is r_1 symmetric, then $r_i = r_{n+i-1}$ for $i = 1, 2, \ldots, n$.*

To use the formula in Proposition 14 to find the i^{th} row of an odd open dominating set matrix with first row w, we need to find the sum of the first $i - 1$ rows of the even open dominating set matrix with first row J_n, the all 1's vector.

Lemma 21 *If n is even then $\sum_{i=1}^{n} f_i(P_n)J_n = 0$. If n is odd, then $\sum_{i=1}^{n} f_i(P_n)J_n = k_n$, the n-vector with alternating 1's and 0's starting and ending with 1.*

Proof: Let $r_1, r_2, \ldots, r_n$ be the rows of the even open dominating set matrix with $r_1 = J_n$. Since $r_i = f_i(P_n)J_n$ (by Proposition 13) and $r_i = r_{n+1-i}$ for $i = 1, 2, \ldots, n$ (by Corollary 20), the result for even n follows immediately and follows for odd n if we show $r_{\frac{n+1}{2}} = k_n$. Since $r_{\frac{n-1}{2}} = r_{\frac{n+3}{2}}$, each vertex in row number $\frac{n+1}{2}$ has an even number of neighbors on these two adjacent rows that lie in the odd open dominating set. That means row $\frac{n+1}{2}$ is itself a $1 \times n$ even open dominating set matrix, i.e., either 0 or k_n. But by Proposition 14, $x_{n+1}(J_n) = f_{n+1}(P_n)J_n + \sum_{i=1}^{n} f_i(P_n)J_n = r_{\frac{n+1}{2}}$ since $f_{n+1}(P_n) = 0$ (Proposition 15) and $r_i = r_{n+1-i}$ for $i = 1, 2, \ldots, n$. If $r_{\frac{n+1}{2}} = 0$, then $x_{n+1}(J_n) = 0$ which implies there exists an $n \times n$ odd open dominating set matrix for odd n, contradicting Corollary 11. Hence $r_{\frac{n+1}{2}} = k_n$. $\square$

We next give a linear algebra proof of Lemma 21 which does not use Lemma 19 (which we proved using the "rectangular pattern" argument).

Proof 2: Let $r_1, r_2, \ldots, r_n$ be the rows of the even open dominating set matrix with $r_1 = J_n$. First suppose n is even. By Proposition 17(ii), $(f_{\frac{n}{2}} + f_{\frac{n}{2}+1})^2 = f_{n+1}$. So $(f_{\frac{n}{2}}(P_n) + f_{\frac{n}{2}+1}(P_n))^2 = f_{n+1}(P_n) = 0$, since f_{n+1} is the characteristic polynomial for P_n. Hence $C = f_{\frac{n}{2}}(P_n) + f_{\frac{n}{2}+1}(P_n)$ is a symmetric matrix whose square is 0. That means each if its rows has an even number of 1's, so $CJ_n = 0$ and $r_{\frac{n}{2}} = f_{\frac{n}{2}}(P_n)J_n = f_{\frac{n}{2}+1}(P_n)J_n = r_{\frac{n}{2}+1}$. Since these two consecutive rows are equal, when we generate further rows in both directions, we get the same thing, i.e., $r_i = r_{n+1-i}$ for $i = 1, 2, \ldots, n$.

If n is odd then, by the Fibonacci recurrence relation, $f_{\frac{n-1}{2}} + f_{\frac{n+3}{2}} = xf_{\frac{n+1}{2}}$. By Proposition 17(iii), $(xf_{\frac{n+1}{2}})^2 = xf_{n+1}$. So

$$(f_{\frac{n-1}{2}}(P_n) + f_{\frac{n+3}{2}}(P_n))^2 = P_n f_{n+1}(P_n) = 0.$$

As in the case when n is even, $f_{\frac{n-1}{2}}(P_n) + f_{\frac{n+3}{2}}(P_n)$ is a symmetric matrix whose square is 0, so each if its rows has an even number of 1's and $r_{\frac{n-1}{2}} = f_{\frac{n-1}{2}}(P_n)J_n = f_{\frac{n+3}{2}}(P_n)J_n = r_{\frac{n+3}{2}}$.

The only way these two rows can be equal if is the row between them is itself a $1 \times n$ even open dominating set matrix, i.e., either 0 or k_n. As shown in the first proof of the Lemma, this means $r_{\frac{n+1}{2}} = k_n$. Now

$$r_{\frac{n-3}{2}} = P_n r_{\frac{n-1}{2}} + r_{\frac{n+1}{2}} = P_n r_{\frac{n+3}{2}} + r_{\frac{n+1}{2}} = r_{\frac{n+5}{2}}$$

and so on, showing $r_i = r_{n+1-i}$ for $i = 1, 2, \ldots, n$. $\square$

Proposition 22 *Let n be an even integer.*
(i) There exists a unique $(n-1)\times n$ odd open dominating set matrix. Its first and last rows are all 1's. If we add a row of 0's at the top and bottom of this matrix, we get the unique $(n+1)\times n$ odd open dominating set matrix. If we add a row of 0's at either the top or bottom of this matrix, we get an $n\times n$ odd open dominating set matrix.
(ii) For each n-vector w, there exists an $n\times n$ odd open dominating set matrix with first row w.

Proof: (i) If the first row is J_n, by Proposition 14, the i^{th} row is $x_i(J_n) = \sum_{j=1}^{i} f_j(P_n)J_n$. By Lemma 21, $x_n(J_n) = 0$ and

$$x_{n-1}(J_n) = \sum_{j=1}^{n-1} f_j(P_n)J_n = f_n(P_n)J_n.$$

Since $f_n(P_n)J_n$ is the n^{th} row of the even open dominating set matrix with first row $f_1(P_n)J_n = J_n$, by Corollary 20, $f_n(P_n)J_n = J_n$. Since $x_n(J_n) = 0$ and $x_{n-1}(J_n) = J_n$, we in fact get an $(n-1)\times n$ odd open dominating set matrix with first and last rows equal to J_n. It is easy to see that we still have an odd open dominating set matrix if we add a row of zeroes to the top and/or bottom of this matrix. Since gcd$(n, n+1)$=gcd$(n+2, n+1)$= 1, by Theorem 3, these $(n-1)\times n$ and $(n+1)\times n$ odd open dominating set matrices are unique.
(ii) Since gcd$(n+1, n+1)$= $n+1$, by Proposition 18(v), there are 2^n $n\times n$ odd open dominating set matrices, so there must be one with first row w for any n vector w. □

Proposition 23 *Let n be an odd integer.*
(i)There exists a unique $2n\times n$ odd open dominating set matrix. Its first and last rows are all 1's. If we add a row of 0's at the top and bottom of this matrix, we get the unique $(2n+2)\times n$ odd open dominating set matrix. If we add a row of 0's at either the top or bottom of this matrix, we get a $(2n+1)\times n$ odd open dominating set matrix.
(ii) For each n-vector w, there exists an $(2n+1)\times n$ odd open dominating set matrix with first row w. Its $(n+1)^{st}$ row is k_n.

Proof: (i) If the first row is J_n, by Proposition 14, the i^{th} row is $x_i(J_n) = \sum_{j=1}^{n} f_j(P_n)J_n$. So

$$x_{2n+1}(J_n) = \sum_{j=1}^{n} f_j(P_n)J_n + f_{n+1}(P_n)J_n + \sum_{j=1}^{n} f_{n+1+j}(P_n)J_n.$$

If we generate the rows $z_1(J_n), z_2(J_n), \ldots$ of an even open dominating set matrix with first row J_n, by Corollary 20, $f_n(P_n)J_n = z_n(J_n) = z_1(J_n) =$

J_n. Since $z_{n+1}(J_n) = 0$, we must have that $z_{n+2}(J_n) = z_n(J_n) = z_1(J_n) = J_n$. Then clearly, $z_{n+1+j}(J_n) = z_j(J_n)$ for $j = 1, 2, \ldots, n$, so $x_{2n+1}(J_n) = f_{n+1}(P_n)J_n = 0$. Furthermore, $x_{2n}(J_n) = x_{2n+1}(J_n) + f_{2n+1}(P_n)J_n = 0 + z_{2n+1}(J_n) = z_n(J_n) = J_n$, so we do get a $2n \times n$ odd open dominating set matrix with first and last rows equal to J_n. We obviously get a $(2n+2) \times n$ odd open dominating set matrix if we add a row of 0's to the top and bottom. Since gcd$(2n+1, n+1)$=gcd$(2n+3, n+1)$= 1, by Theorem 3, these $2n \times n$ and $(2n+2) \times n$ odd open dominating set matrices are unique.
(ii) Since gcd$(2n+2, n+1)$= $n+1$, by Proposition 18(v), there are 2^n $(2n+1) \times n$ odd open dominating set matrices, so there must be one with first row w for any n vector w. If the first row is w, by Proposition 14, the $(n+1)^{st}$ row is $x_{n+1}(w) = f_{n+1}(P_n)w + \sum_{j=1}^{n} f_j(P_n)J_n$. But $f_{n+1}(P_n) = 0$ and $\sum_{j=1}^{n} f_j(P_n)J_n = k_n$ by Lemma 21. □

We now state two lemmas needed to prove Theorem 4. Recall that $\mathcal{O}$ is the set of all ordered pairs (m, n) such that there exists an odd open dominating set in $G_{m,n}$.

Lemma 24 *(i)* $(m, 2m), (m, 2m+1), (m, 2m+2) \in \mathcal{O}$ *for each positive integer* m.
(ii) $(m, m) \in \mathcal{O}$ *if and only if* m *is even.*
(iii) If $m > 2n+2$, *then* $(m-2n-2, n) \in \mathcal{O}$ *if and only if* $(m, n) \in \mathcal{O}$.

Proof: Parts (i) and (ii) are just restatements of results proved in Corollary 11, Proposition 22, and Proposition 23. To prove (iii), suppose $m > 2n+2$ and A is an $m \times n$ odd open dominating set matrix. By Proposition 23, the $(2n+2)^{nd}$ row of A is all 0's. So if we delete the first $2n+2$ rows of A we have an $(m-2n-2) \times n$ odd open dominating set matrix. Conversely, if B is an $(m-n-2) \times n$ odd open dominating set matrix with last row w, we next generate the $(2n+1) \times n$ odd open dominating set matrix D with first row w^c where w^c is the binary complement of w. It is easy to see that if we put B and D together with a row of 0's in between, we get an $m \times n$ odd open dominating set matrix. □

For any non-negative integer t, let

$$S_t = \{k2^t - 1 : k \text{ is an odd positive integer}\},$$

except that we exclude 0 from S_0. Note that S_0 is the set of even positive integers and the sets $S_0, S_1, \ldots$ form a partition of the positive integers. The proof of the following lemma is simple arithmetic and is omitted.

Lemma 25 *(i) If* $m > 2n+2$ *then* m *and* n *are both in* S_t *for some non-negative integer* t *if and only if* $m-2n-2$ *and* n *are both in* S_t.
(ii) If $n < m < 2n$ *then* m *and* n *are both in* S_t *for some non-negative integer* t *if and only if* $2n-m$ *and* n *are both in* S_t.

Proof of Theorem 4: Part (i) was already proved in Proposition 18.

First we find all m such that $(m,1)$ (and $(1,m)$) are in $\mathcal{O}$. Obviously $(1,1) \notin \mathcal{O}$, but the odd open dominating set matrices $11, 011$, and 0110 show that $(2,1), (3,1)$ and $(4,1)$ are all in $\mathcal{O}$. It follows from Lemma 24(iii) with $n=1$ that $(m,1) \in \mathcal{O}$ if and only if m is not congruent to 1 (mod 4). Since $1 \in S_1$ and $m \in S_1$ if and only if $m \equiv 1 (\text{mod } 5)$, the result holds for $n = 1$.

We now do simultaneous induction on m and n. Assume $m > 1, n > 1$ and that our result is correct for all ordered pairs (m', n) and (m, n') where $m' < m$ and $n' < n$. Consider the ordered pair (m, n) and assume without loss of generality that $m \geq n$.

If $m > 2n + 2$ then, by Lemma 24(iii), $(m, n) \in \mathcal{O}$ if and only if $(m - 2n - 2, n) \in \mathcal{O}$ and by Lemma 25(i), m and n are both in S_t for some non-negative integer t if and only if $m - 2n - 2$ and n are both in S_t. So by the inductive hypothesis, $(m, n) \in \mathcal{O}$ if and only if m and n are not both in S_t for some positive integer t. Similarly, if $n < m < 2n$, then by the inductive hypothesis, $(m, n) \in \mathcal{O}$ if and only if m and n are not both in S_t for some positive integer t. This reduction works unless $m \in \{n, 2n, 2n+1, 2n+2\}$. If $m \in \{2n, 2n+1, 2n+2\}$, then by Lemma 24(i), $(m, n) \in \mathcal{O}$ and it is easy to verify that m and n are not both in S_t for some positive integer t. If $m = n$, the result is also correct because $(m, m) \in \mathcal{O}$ if and only if m is even (and if m is even, then $m \in S_0$). □

The ideas behind Lemma 24 and Theorem 4 can be used constructively. For example, to generate a 77×47 odd open dominating set matrix, we use the following reduction:

$$(77, 47) \to (2 * 47 - 77, 47) = (17, 47)$$

$$(47, 17) \to (47 - 2 * 17 - 2, 17) = (11, 17)$$

$$(17, 11) \to (2 * 11 - 17, 11) = (5, 11)$$

Since $11 = 2 * 5 + 1$, the reduction stops. We know how to produce an odd open dominating set of size 11×5: take the 10×5 one with first and last row J_5 and add a row of 0's at the top. The transpose of this is a 5×11 matrix; let the sixth row be all 0's, then iterate equation (2) (or Proposition 14) until we get a 23×11 odd open dominating set matrix (which will occur due to Proposition 23). Delete the first six rows and we have a 17×11 odd open dominating set matrix. The transpose is 11×17. Add rows using equation (2) until we get a 47×17 matrix (the twelfth row is all 0's, so rows 13 through 47 are that of a 35×17 odd open dominating set matrix). The transpose is a 17×47 odd open dominating set matrix. Add rows using equation (2) until we get a 95×47 odd open dominating set matrix (we will since $95 = 2 * 47 + 1$). Deleting the first 18 rows gives the desired 77×47 matrix.

If we try the reduction on a pair not in $\mathcal{O}$, we end up with an odd square, which is the only obstruction to a successful construction. For example, 19 and 27 are both in S_2:

$$(27, 19) \to (2 * 19 - 27, 19) = (11, 19)$$

$$(19, 11) \to (2 * 11 - 19, 11) = (3, 11)$$

$$(11, 3) \to (11 - 2 * 3 - 2, 3) = (3, 3)$$

Now we show that if n is even no $n \times n$ and if n is odd no $(2n+1) \times n$ odd open dominating set matrix can have more than one row of 0's. The following simple lemma provides the main idea.

Lemma 26 *In an odd open dominating set matrix, there cannot be a row and a column each of which is itself an even open dominating set matrix.*

Proof: If there were, then the element in both that row and column would not have an odd open neighborhood. □

So if a row is 0 or k_n, then no column can be 0^t or k_n^t. For convenience, we restate parts of Proposition 22 and Proposition 23 in a lemma.

Lemma 27 *If $m > n$ then the $(n+1)^{st}$ row of an $m \times n$ odd open dominating set matrix is 0 if n is even and k_n if n is odd.*

Proof of Theorem 6: If n is even, by Theorem 4, for each $j \in \{2, 3, \ldots, n\}$ there exists a $(j-1) \times n$ odd open dominating set matrix. If we generate more rows using equation (2), by Proposition 22(ii), we will get an $n \times n$ odd open dominating set matrix with j^{th} row all 0's. If we start with the first row all 0's, we will also get an $n \times n$ odd open dominating set matrix.

Next we show there cannot be two rows of 0's. Suppose the rows are $x_1, x_2, \ldots, x_n$ and that $x_i = x_j = 0$, where $i + 1 \leq j$. Taking rows $x_1, x_2, \ldots, x_{j-1}$ gives is a $(j-1) \times n$ odd open dominating set matrix A with a row of 0's. But by applying Lemma 27 to A^t we see that the j^{th} column of A is an even open dominating set (0 if $j-1$ is even and k_{j-1}^t if $j-1$ is odd), contradicting Lemma 26. □

By Proposition 18, the number of $n \times n$ odd open dominating set matrices with j^{th} row 0 is 2^d where $d + 1 = \gcd(j, n+1)$ (and the same number with j^{th} column 0). So the number with no row or column equal to 0 is $2^n - 2\sum_{j=1}^{n} 2^{d(j)}$ where $d(j) = \gcd(j, n+1) - 1$. It is not hard to show this is positive if $n > 2$ (and close to 2^n if n is large).

Proof of Theorem 7: We have already proved part (i) (Proposition 23) and the existence of the $(2n+1) \times n$ matrix with j^{th} row 0 for j satisfying the given conditions follows from Theorem 4. We now show there cannot be two rows of 0's. Suppose the rows of such a matrix D are $x_1, x_2, \ldots, x_{2n+1}$ and that $x_i = x_j = 0$, where $i < j$. If $j < n+1$, then the first $j-1$ rows form a $(j-1) \times n$ odd open dominating set matrix with a row of 0's and we get a contradiction just as in the proof of Theorem 6. The same argument applies if $i > n+1$. By Proposition 23, $x_{n+1} = k_n$. If $i < n+1 < j$, but $j - i < n+1$, then the $(j-i-1) \times n$ matrix C formed by taking rows $x_{i+1}, x_{i+2}, \ldots, x_{j-1}$ is an odd open dominating set matrix with a row equal to k_n. By Lemma 27 applied to its transpose, the $(j-1)^{st}$ column of C is either 0 or k_{j-i-1}, contradicting Lemma 26.

The remaining possibility is $i < n+1 < j$ and $j - i \geq n+1$. In fact, $j - i \neq n+1$ because, by an obvious extension of Proposition 23, $x_{j-n-1} = x_{i+n+1} = k_n$. Let B be the $(2n+1) \times n$ matrix obtained from D by adding k_n to each of its rows (modulo 2). Each entry of D not in the first or last row still has an odd open neighborhood (elements in the first and last rows in the odd numbered columns have even open neighborhoods). Let the rows of B be $y_1, y_2, \ldots, y_{2n+1}$. Then $y_{j-n-1} = y_{n+1} = y_{i+n+1} = 0$. Now $0 < i - j + 2n + 1 < n$ and the $(i - j + 2n + 1) \times n$ matrix formed by taking rows $y_{j-n}, y_{j-n+1}, \ldots, y_{i+n}$ is an odd open dominating set matrix with a row of 0's. But by Lemma 27, its $(i - j + 2n + 2)^{nd}$ column is either 0 or $k_{i-j+2n+1}$, contradicting Lemma 26.

We have already shown that the $(n+1)^{st}$ row is k_n and that if $x_j = 0$ and $j \in \{1, 2, \ldots, n\}$, then $x_{j+n+1} = k_n$ while if $j \in \{n+2, \ldots, 2n\}$, then $x_{j-n-1} = k_n$. To show that if $x_j = k_n$ then $x_{j+n+1} = 0$ if $j < n+1$ while $x_{j-n-1} = 0$ if $j > n+1$, we just repeat the argument used above where we add k_n to each row (to convert 0 to k_n and vice versa while keeping the odd open property). □

By Proposition 18 and Theorem 3, we can find the number of $(2n+1) \times n$ odd open dominating set matrices with j^{th} row 0 for odd n and then can express the number that have no row of 0's as a sum, as we did for $n \times n$ odd open dominating set matrices when n is even. Again, this number is positive when $n > 3$.

Proof of Theorem 8: If B is an infinite even open dominating set matrix with rows $z_1 = w, z_2, z_3, \ldots$, then by Lemma 19 $z_n = w^R$ where w^R is the reverse of w. Since $z_{n+1} = f_{n+1}(P_n)w = 0$, we must have $z_{n+2} = z_n$. Since $z_{2n+2} = f_{2n+2}(P_n)w = 0$, the matrix with rows $z_{n+2}, z_{n+3}, \ldots, z_{2n+1}$ is an even open dominating set matrix. Hence by Lemma 19, $z_{2n+1} = z_{n+2}^R = w$. And since $z_{2n+2} = 0$, we must have $z_{2n+3} = w$ as well. It follows that

$z_{i+2n+2} = z_i$ for $i = 1, 2, \ldots$.

Returning to the infinite odd open dominating set matrix A with rows $x_1 = w, x_2, x_3, \ldots$, we know $x_{2n+2} = 0$ (Propositions 22 and 23). By Proposition 14, $x_{2n+3} = f_{2n+3}(P_n)w + \sum_{j=1}^{2n+1} f_j(P_n)J_n + f_{2n+2}(P_n)J_n = z_{2n+3} + z_{2n+2} + f_{2n+2}(P_n)J_n = w + 0 + 0$. Since $x_{2n+2} = 0$ and $x_{2n+3} = w$, it follows that $x_{i+2n+2} = x_i$ for $i = 1, 2, \ldots$. This means the period of this matrix recurrence divides $2n + 2$. If n is odd, then the pattern of the rows of 0's described in Theorem 7 implies that the period cannot be less than $2n + 2$. If n is even, then the pattern of the rows of 0's described in Theorem 6 implies that the period is either $2n+2$ or $n+1$.For even n, $x_{2n+2} = f_{2n+2}(P_n)w + \sum_{j=1}^{n+1} f_j(P_n)J_n = z_{n+2} + \sum_{j=1}^{n} f_j(P_n)J_n + f_{n+1}(P_n)J_n = z_n + 0 + 0 = w^R$, so the period will be $n + 1$ if and only of w is symmetric. □

7 Discussion

We have completely solved the existence and enumeration problems for odd/even open dominating sets in grid graphs, using the Fibonacci polynomials. Certainly these techniques could be applied to grid graphs with "wrap-around" (a cylinder instead of a rectangle). It would be of interest to find other graph classes for which one can determine the existence of odd/even open dominating sets by means faster than testing the adjacency matrices of the graphs for singularity.

We make one further comment. If G is a bipartite graph with vertex bipartition $V_0 \cup V_1$ and if there exists a subset $D_i \subseteq V_i$ $(i = 1, 2)$ such that $|N(v) \cap D_i|$ is odd for each $v \in V_j$ $(j \in \{1, 2\}, j \neq i)$ then we say D_i is a bipartite odd open dominating set for V_j. Clearly a bipartite graph G has an odd open dominating set if and only if both V_0 and V_1 have bipartite odd open dominating sets. In terms of the row by row generation of an odd open dominating set matrix on a chessboard, the entries in the white squares of the second row are determined by the entries in the white squares in the first row (and similarly for the black squares). This separation of the search for an odd open dominating set in a bipartite graph into two distinct problems does not work for odd closed dominating sets (the entries in the white squares of the second row depend on the entries in both the white and black squares in the first row). Let m and n be odd positive integers, and let v_{ij} be the vertex of $G_{m,n}$ in the i^{th} row and j^{th} column. Let $V_0 = \{v_{ij} | i + j \text{ is even}\}$ and $V_1 = \{v_{ij} | i + j \text{ is odd}\}$. Then $V_0 \cup V_1$ is the bipartition of the vertex set for the bipartite graph $G_{m,n}$, and it is not hard to check that $D_0 = \{v_{ij} | i \text{ and } j \text{ are odd and } i \equiv j (\text{mod } 4)\}$ is a bipartite odd open dominating set for V_1. If $G_{m,n}$ does not have an odd open dominating set (so there exists a positive integer t such that $(m+1)/2^t$

and $(n+1)/2^t$ are both odd integers) that means there must not exist a bipartite odd open dominating set for V_0. The set D_0 actually has the property that size of $|D_0 \cap N(v)| = 1$ for each $v \in V_1$, a topic which is explored in [8].

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