

The Number of Non-Zero (mod k) Dominating Sets of Some Graphs

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Abstract

A set of vertices D in a graph $G = (V, E)$ is a non-zero (mod k) dominating set if $|N[v] \cap D| \not\equiv 0 \pmod{k}$ for all vertices $v \in V$. We consider the number of such sets that some elementary graphs contain.

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1 Introduction

A set of vertices D in a graph $G = (V, E)$ is a non-zero (mod k) dominating set, a.k.a. an nz_k -set, if $|N[v] \cap D| \not\equiv 0 \pmod{k}$ for all vertices $v \in V$, where $N[v]$ denotes the closed neighborhood of vertex v . Sutner proved that every graph contains at least one such set when $k = 2$ (a so-called *odd dominating set*) [6]. The general problem for all values of k was first described by Caro et al [2]. Caro and Jacobson proved that every tree has such a set for all k and they conjecture that every graph contains a non-zero (mod k) dominating set for all k [3]. This seems to be a difficult problem.

It is rather easy to prove constructively that all $n \times m$ grid graphs contain an nz_k -set for all k . More generally, Černý, Dvořák, Jelínek, and Podbrdský

discovered, via a simple application of the Lovász Local Lemma, that all graphs having each vertex degree between $c * \Delta(G)$ and $\Delta(G)$ contain at least one nz_k -set (where the constant c depends on k) [4].

In this paper, we examine the number of non-zero (mod k) dominating sets of certain graphs, including some grid graphs. Two of these values satisfy the recurrence associated with integer sequence A007040.

2 Paths and Cycles

Consider the number of non-zero (mod k) dominating sets for the path on n vertices, P_n , where $n \geq 1$. Denote this number as T_n . We focus on the case when $k \geq 3$. (If $k < 3$, P_n has two nz_2 -sets if $n \equiv 2 \pmod{3}$, one otherwise). One can verify that the first few terms in the sequence of T_n 's are 1, 3, 4, 6, 11. We shall prove that this sequence satisfies the Fibonacci-like recurrence $T_n = T_{n-1} + T_{n-2}$ ($n \equiv 0 \pmod{3}$); $T_n = T_{n-1} + T_{n-2} - 1$ ($n \equiv 1 \pmod{3}$); $T_n = T_{n-1} + T_{n-2} + 1$ ($n \equiv 2 \pmod{3}$). It is easy to show using algebraic manipulation that this recurrence is equivalent to the following:

$$T_n = T_{n-2} + 2T_{n-3} + T_{n-4} \quad (1)$$

It is the latter recurrence that we will deal with. This recurrence appears in conjunction with A007040 in the Encyclopedia of Integer Sequences (credited to Agur et al. [1], see below), though the particular sequence associated with it has different initial terms (2, 2, 6, 6, 10, 20, ...).

Theorem 1 T_n satisfies equation (1).

Proof: Denote the vertices of P_n by v_1, v_2, \dots, v_n numbered from left to right. Let T_n^1 denote the number of nz_k -sets containing v_1 and v_n ; let T_n^2 denote the number of nz_k -sets containing v_1 but not v_n ; let T_n^3 denote the number of nz_k -sets containing v_n but not v_1 ; and let T_n^4 denote the number of nz_k -sets containing neither v_1 nor v_n . One can verify that the first five terms in the sequence T_n^1 are 1, 1, 1, 3, 4; the first five terms in the sequence T_n^2 are 0, 1, 1, 1, 3; the first five terms in the sequence T_n^3 are 0, 1, 1, 1, 3; and the first five terms in the sequence T_n^4 are 0, 0, 1, 1, 1. We claim that each of these sequences obeys the recurrence specified in equation (1), which ensures the truth of the theorem.

We detail the case of T_n^1 ; the remaining cases are nearly identical. Let D be a nz_k -set of P_n , $k \geq 3$. In this case, we are supposing that $v_1, v_n \in D$. Note that we may assume $k = 3$ since the maximum degree of any vertex P_n is two if $n \geq 3$.

Consider an nz_k -set D_{n-2} of P_{n-2} vertices having both v_1 and v_{n-2} in the set. Each such set maps to an nz_k -set of P_n by letting $D = D_{n-2} \cup \{v_n\}$, i.e., vertex v_{n-1} is not contained in this set for P_n . Consider an nz_k -set D_{n-3} set of P_{n-3} having both v_1 and v_{n-3} in the set. Each such set maps to two different nz_k -sets for P_n since we must exclude those D_{n-3} containing vertex v_{n-2} (we counted those already), and then we can either include or exclude vertex v_{n-1} in order to form an nz_k -set for P_n . Now consider an nz_k -set D_{n-4} of P_{n-4} vertices containing both v_1 and v_{n-4} . Each such set maps to one nz_k -set D for P_n : we must exclude from D vertices v_{n-2} and v_{n-3} (we counted all such possibilities already) and thus must include vertex v_{n-1} in order to form an nz_k -set for P_n . Hence the proof. \square

The number of nz_k sets that the cycle C_n has also satisfies equation (1), though with different initial terms: 6, 6, 10, 20 (starting counting with the case of C_3). Agur et al. [1] proved this result. Therefore, the number of nz_3 -sets for the path or cycle is $O(\phi^n)$, where ϕ is the golden ratio [1].

3 Grid Graphs

3.1 Odd Dominating Sets of Grids

Consider the number of non-zero (mod 2) dominating sets of $m \times n$ grid graph when $m > 1$. The sequence giving the number of non-zero (mod 2) dominating sets of $2 \times n$ grid graphs, $n \geq 1$, is 2, 1, 4, 1, 2, 1, 4, 1, ... For $3 \times n$ grid graphs, the sequence is 1, 4, 1, 1, 8, 1, 1, 4, 1, 1, 8, 1, ... For $4 \times n$ grid graphs, the sequence repeats the pattern 1, 1, 1, 16, 1.

Let f_i be the i^{th} Fibonacci polynomial over $GF(2)$:

$$f_i = x f_{i-1} + f_{i-2} \quad i \geq 2 \quad f_0 = 0 \quad f_1 = 1 \quad (2)$$

The following is an application of Theorem 3 of Goldwasser et al. [5].

Corollary 2 *Let r be the degree of the greatest common divisor of $f_{n+1}(x+1)$ and $f_{m+1}(x)$. Then the $m \times n$ grid graph has 2^r non-zero (mod 2) dominating sets.*

3.2 NZ_3 -sets of $2 \times n$ grids

Recording the number of non-zero (mod 3) dominating sets of $2 \times n$ grid graphs generates the following sequence starting with $n = 1$: 3, 6, 14, 27, 64, 133, 295, 633, 1384, 2992, 6515, ... Let T_n represent the number of nz_3 -sets

in the $2 \times n$ grid graph. While we are unable to completely characterize T_n , we give some lower bounds.

Number the vertices on the top row of the $2 \times n$ grid graph as u_1, u_2, \dots, u_n and the vertices on the bottom row as v_1, v_2, \dots, v_n . Of course, each v_i is adjacent to u_i . Think of this numbering as going from left to right.

Fact 3 (i) $T_n \geq 4T_{n-3}$
(ii) $T_n \geq 3T_{n-2}$.

Proof: The idea is simple; we detail one case of (ii). Let G be the $2 \times n$ grid graph. Consider an nz_3 -set, D , of the $2 \times (n - 2)$ grid graph. We show there are at least three nz_3 -sets of G that are supersets of D . Suppose $v_{n-2} \in D$ and $u_{n-2} \notin D$ (the most involved of the cases). Two nz_3 -sets of G are $D \cup \{v_n, u_n\}$ and $D \cup \{u_n\}$. To find a third set, first suppose $u_{n-3}, v_{n-3} \notin D$. Then $D \cup \{u_n, u_{n-1}\}$ is an nz_3 -set of G . If $u_{n-3} \in D$, $v_{n-3} \notin D$, then $D \cup \{v_{n-1}, u_n\}$ is an nz_3 -set of G . If $u_{n-3} \notin D$, $v_{n-3} \in D$, then $D \cup \{v_n, u_{n-1}\}$ is an nz_3 -set of G . Finally, if $u_{n-3}, v_{n-3} \in D$. Then $D \cup \{v_n\}$ is an nz_3 -set of G . Since all these additions to D are distinct, we have the desired number of nz_3 -sets. \square

In an attempt to uncover a pattern, we examined the number of nz_3 -sets of $2 \times n$ grids when the rightmost two columns have specified relationships with the nz_3 -set. This data is in the table below. In the table and ensuing discussion, an “x” indicates a vertex is in the set D and an “o” indicates a vertex is not in D . A four symbol sequence such as $ooxx$ refers to two consecutive columns in the grid, specifically, vertices $u_{n-1}, v_{n-1}, u_n, v_n$. So for example, $xoox$ means $u_{n-1}, v_n \in D, u_n, v_{n-1} \notin D$. We have omitted columns with all entries equal to zero from the table. The table begins with the row for $n = 2$.

n	$xxoo$	$ooox$	$oxox$	$oxxo$	$xxox$	$ooxo$	$xoox$	$xoxo$	$xxxo$	$ooxx$
2	1	0	1	1	0	0	1	1	0	1
3	1	2	1	1	1	2	1	1	1	3
4	3	3	3	3	0	3	3	3	0	6
5	6	8	6	6	2	8	6	6	2	14
6	14	15	14	14	3	15	14	14	3	27
7	27	36	29	29	8	36	29	29	8	64
8	64	73	65	65	15	73	65	65	15	133
9	133	166	138	138	36	166	138	138	36	295
10	295	351	304	304	73	351	304	304	73	633
11	633	773	655	655	166	773	655	655	166	1384
12	1384	1665	1428	1428	351	1665	1428	1428	351	2992
13	2992	3633	3093	3093	773	3633	3093	3093	773	6515

We next demonstrate several patterns.

- Lemma 4** (i) The n^{th} value in column $xxoo$ is equal to T_{n-3} .
(ii) The n^{th} value in column $ooxx$ is equal to T_{n-2} .
(iii) The values in columns $xxox$ and $xxxo$ are equal to the value two rows previous in columns $ooox$ and $ooxo$, respectively.
(iv) The values in each of six columns whose values in the last row of the table are either 3633 or 3093 are as large as column $xxoo$ for all $n \geq 4$.

Proof: For (ii), suppose D' is an nz_3 -set of the $2 \times (n-2)$ grid graph. Then $D' \cup \{u_n, v_n\}$ is an nz_3 -set of the $2 \times n$ grid graph. Likewise, if D is an nz_3 -set of the $2 \times n$ grid graph containing u_n, v_n , then $D \setminus \{u_n, v_n\}$ is an nz_3 -set of the $2 \times (n-2)$ grid graph. Part (i) follows in a similar fashion. Part (iii) can be shown in a similar manner, noting that, for example, if $u_{n-1}, v_{n-1}, v_n \in D$, then $v_{n-2} \in D$ and also $u_{n-3}, v_{n-3} \notin D$.

For part (iv), first consider column $oxox$. We show these are at least as many as T_{n-3} . Let D' be an nz_3 -set of the $2 \times (n-3)$ grid graph. Add v_{n-1}, v_n to D' forming D'' . If $u_{n-3} \in D'$, then D'' is an nz_3 -set of the $2 \times n$ grid graph. So suppose $u_{n-3} \notin D'$. Then $D'' \cup \{u_{n-2}\}$ is an nz_3 -set of the $2 \times n$ grid graph. The same argument applies to columns $xoxo$, $oxxo$, and $xoox$.

Now consider column $ooox$ (which is symmetric to column $ooxo$). We show that the value in this column is at least as large as column $oxox$, which by the preceding paragraph implies column $ooox$ is at least as large as T_{n-3} . From the table above, we have that the claim is true for $3 \leq n \leq 13$ and we have also verified the claim using a computer for all n , $3 \leq n < 20$. Consider the case when $n \geq 20$.

Using a computer, we have verified that for all k , $13 \leq k \leq 15$, that there are more nz_3 -sets of the $2 \times k$ grid graph having their rightmost two columns equal to $ooox$ than $oxox$ for each of the 16 possible choices of which vertices from the leftmost two columns are in the nz_3 -set (a caveat is explained below). That is, fix whether each vertex in the leftmost two columns is in the nz_3 -set or not. Then extend this to an nz_3 -set of the $2 \times k$ grid graph so that the rightmost two columns are as required. The caveat is that we do not care if the vertices in the leftmost column are not dominated by any vertex, as in G_n they can be dominated by a vertex in a column to their left when we append these k columns to another grid graph. Note that it was not necessary to consider the case when all four vertices in the leftmost two columns are in the nz_3 -set, since two such adjacent columns cannot occur in any nz_3 -set.

In other words, we think of constructing an nz_3 -set of G_n by first choosing an “almost nz_3 -set” D_1 of the leftmost $n - 11$ columns, whereby the vertices in columns $1, 2, \dots, n - 12$ each have a non-zero (mod 3) number of neighbors in D_1 and the vertices in column $n - 11$ may or may not have the proper number of neighbors in D_1 . Then extend D_1 to be a valid nz_3 -set of the entire graph by “filling in” the remaining columns. We have verified that there are more ways to perform this extension so as to end in $ooox$ than $oxox$. \square

Note: the reason we chose the interval $13 \leq k \leq 15$ in part (iv) was that there are more $ooox$ type nz_3 -sets than $oxox$ type nz_3 -sets for each possible choice of the two left-hand columns for each k in this interval. This is not true for values of k less than 13. When $k < 13$, a small number (usually one) of the choices for each k for the two left-hand columns yield more $xoxo$ type nz_3 -sets than $ooox$ type nz_3 -sets. For the proof, it sufficed to consider just $k = 13$, but we computed the details for $k = 14, 15$ out of curiosity. This issue suggests that a traditional case-by-case hand-proof of part (iv) may be difficult, or require an alternative approach.

The lemma yields the following.

Proposition 5 $T_n > T_{n-2} + 7T_{n-3} + 2T_{n-5}$.

We expanded these calculations by considering a “lookback” of as many as the seven rightmost columns. In none of these calculations did we glean any additional insight as to the pattern for T_n , leading us to wonder if an exact recurrence exists.

4 Trees and Future Directions

We have computed the number of nz_k -sets for several other types of graphs and none of the associated integer sequences appear in the Online Encyclopedia. These sequences are listed below; finding formulae or recurrences to describe them is a potential future direction for this work.

A *full* binary tree is a binary tree with $2^k - 1$ vertices, for some $k \geq 0$, and a specified root vertex (which has degree two) in which all leaves are the same distance from the root. A *complete binary tree* is a binary tree with n vertices and a specified *root* vertex that has degree two. Further, if $n = 2^k - 1$, then a complete binary tree is a full binary tree. Otherwise, the degree one vertices all are either distance h or $h - 1$ from the root where $h = \lceil \log_2 n \rceil$ and the leaves that are distance h from the root are filled in from left to right. That is, for each $i, 2 \leq i \leq \lceil \log_2 n + 1 \rceil$, there exists at most one subtree of height i that is not a full binary tree.

Define the *height* of a complete binary tree to be $\lceil \log_2 n + 1 \rceil$, where n is the number of vertices.

Fact 6 *A complete binary tree with n vertices has two nz_2 -sets if n is even and $\lceil \log_2 n + 1 \rceil \equiv 0 \pmod{2}$ and one nz_2 -set otherwise.*

Proof: If the number of vertices is odd, the root vertex together with each vertex of even distance from the root constitutes an nz_2 -set. To see it is the only such set, note that each non-leaf vertex has exactly two children. We claim that it is not possible that a parent and child are both excluded from the nz_2 -set when the number of vertices is odd. Otherwise, exactly one of the child's two children would have to be in the nz_2 -set; let w be the one of those two children not in the nz_2 -set. Then exactly one of w 's children must be excluded from the nz_2 -set and so on down the tree. The claim follows and from that it follows that the specified nz_2 -set is unique.

If the number of vertices is even, we claim by induction that if the height of the tree, T , is odd there is a unique nz_2 -set (and that set does not include the root) and if the height is even there are two nz_2 sets, one that includes the root and one that does not. The base cases are trivial. Suppose the height is even. Consider the two subtrees rooted at the children of the root. One of these subtrees is full and thus has an odd number of vertices and one has an even number of vertices and is of odd height. If we do not include the root in the nz_2 -set of T , then we must include exactly one of its children (the child that roots the full subtree) and not include the other child (that roots the subtree with an even number of vertices). By induction, this yields

exactly one nz_2 -set of T . If we include the root of T in the nz_2 -set, we cannot include both of the root's children in the nz_2 -set (as this forces us to include every vertex along a path from the root of T to a leaf in the nz_2 -set, which does not yield a valid nz_2 -set). We then consider the four subtrees rooted at the grandchildren of the root: three are full and thus have a unique nz_2 -set, meaning we must include the roots of these three subtrees in our nz_2 -set. Parity then forces us to include the root of the fourth subtree in the nz_2 -set. Induction tells us that the nz_2 -sets of these four subtrees are unique, hence the case when the height is even is resolved.

Now suppose the height of the tree is odd and the tree has an even number of vertices. If we chose to include the root in an nz_2 -set, we cannot include both children of the root, u, v , in the nz_2 -set. Doing so would force us to include every vertex along a path from the root of T to a leaf in the nz_2 -set, which does not yield a valid nz_2 -set. Observe that one subtree rooted at a grandchild of the root, w , has an even number of vertices and is of odd height. Hence, by induction, there is no nz_2 -set of the subtree rooted at w that contains w and thus no nz_2 -set of T that contains the root.

Finally, suppose we chose not to include the root in the nz_2 -set. Then we must include exactly one of its children. In fact, we must include the child that roots a full subtree, since there is a unique way to form an nz_2 -set of that subtree. Likewise, the other child roots a subtree with an even number of vertices but which is of even height, hence there is a unique nz_2 -set of that subtree not containing the root of the subtree. \square

The sequence starting with $n = 1$ of the number of nz_3 -sets for complete binary trees is 1, 3, 4, 6, 6, 13, 15, 23, 22, 53, 59, 70, 55, 110, 105. The sequence in the case of nz_4 -sets is 1, 3, 5, 9, 13, 26, 40, 70, 100, 200, 304, 512, 720, 1326.

The number of nz_4 -sets in $2 \times n$ grids is 3, 11, 34, 110, 359, 1165, for $n = 1, 2, \dots, 6$. The number of nz_5 -sets in $2 \times n$ grids is 3, 11, 41, 149, 547, 2007, for $n = 1, 2, \dots, 6$. Bounding these sequences is a direction for future study.

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