COT 3100 Exam 2. More Practice

Answer all questions. This is a closed book exam. Each question worth 10 points.

1. Define function $f(n)$ as follows. $f(1) = 4$ and $f(n) = n * f(n-1)$ when $n > 1$. Use induction to prove that $f(n) > 2^n$ for all $n \geq 1$.

2. A perfect number is a positive integer n such that the sum of the factors of n is equal to $2n$ (1 and n are considered factors of n). So 6 is a perfect number since $1 + 2 + 3 + 6 = 12 = 2 * 6$. Prove that a prime number cannot be a perfect number. Hint: Assume $n > 2$ is prime, and show it cannot be perfect by computing the sum of the factors of n.

For the remaining questions, consider the universe to be the set of positive integers, $Z^+ = \{1, 2, 3, \ldots\}$.

3. Let $Q(x, y)$ be the statement $y = 2x + 1$. What are the values of the following:
   a) $\forall x \exists y Q(x, y)$

   b) $\exists x \forall y Q(x, y)$

4. Let $A = \{x | (x < 10) \textit{ and } (\text{x is even})\}$ and $B = \{1, 2, 3, 5\}$.

   a) What is $A \cup B$?

   b) What is $A \cap B$?

6. Let $f(n) = n \bmod 2$, where $n \in Z^+$ and $f : Z^+ \rightarrow Z$.
   a) Is $f(n)$ one-to-one?

   b) Is $f(n)$ onto?

7. Describe a smallest set $A \subset Z^+$ (or explain why none exists) such that the following statement is true for universe A.

$$\forall x \in A\ \exists y \in A(x \neq y)$$

8. Compute the value of $f(5)$ where:
f(n) = 2 if n $\leq$ 2; and f(n) = 2f(n-1) + 2f(n-2) if n > 2.

9. a. What is the prime factorization of 190?

b) What is gcd of 27 and 57?

10. Let $f(x)$ and $g(x)$ be functions. Prove, using contradiction method, that if $f(g(x))$ is one-to-one, then $g(x)$ is one-to-one. That is, suppose that $g(x)$ were not one-to-one and derive that $f(g(x))$ cannot be one-to-one.