

Title: Self-adjointness of Schrödinger-type operators with locally integrable potentials on manifolds of bounded geometry.

Proposed running head: Schrödinger-type operators on manifolds.

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ABSTRACT

We consider a Schrödinger-type differential expression $H_V = \nabla^* \nabla + V$, where ∇ is a C^∞ -bounded Hermitian connection on a Hermitian vector bundle E of bounded geometry over a manifold of bounded geometry (M, g) with metric g and positive C^∞ -bounded measure $d\mu$, and $V \in L^1_{\text{loc}}(\text{End } E)$ is a linear self-adjoint bundle map. We define the maximal operator $H_{V, \text{max}}$ associated to H_V as an operator in $L^2(E)$ given by $H_{V, \text{max}} u = H_V u$ for all $u \in \text{Dom}(H_{V, \text{max}}) = \{u \in L^2(E) : Vu \in L^1_{\text{loc}}(E), H_V u \in L^2(E)\}$, where $\nabla^* \nabla u$ in $H_V u = \nabla^* \nabla u + Vu$ is understood in distributional sense. We give a sufficient condition for the self-adjointness of $H_{V, \text{max}}$. The proof adopts Kato's technique to our setting, but it requires a more general version of Kato's inequality for Bochner Laplacian operator as well as a result on the positivity of $u \in L^2(M)$ satisfying the equation $(\Delta_M + b)u = \nu$, where Δ_M is the scalar Laplacian on M , $b > 0$ is a constant and $\nu \geq 0$ is a positive distribution on M . For local estimates, we use a family of cut-off functions constructed with the help of regularized distance on manifolds of bounded geometry.

Key words: bounded geometry, manifold, operator, Schrödinger, self-adjoint.

1. INTRODUCTION AND THE MAIN RESULT

1.1. **The setting.** Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric g , $\dim M = n$. We will assume that M is connected. We will also assume that (M, g) has bounded geometry, i.e. we assume that

- (i) $\inf_{x \in M} r_{\text{inj}}(x) > 0$, where $r_{\text{inj}}(x)$ denotes the injectivity radius of (M, g) at $x \in M$, and
- (ii) all covariant derivatives $\nabla^k R$ of the Riemann curvature tensor R are bounded:

$$|\nabla^k R| \leq C_k, \quad k = 0, 1, 2, \dots,$$

where C_k are constants.

In what follows, we will say that a function $f: M \rightarrow \mathbb{C}$ is C^k -bounded if $f \in C^k(M)$ and $|\partial_y^\alpha f(y)| \leq C_\alpha$ for every multiindex α with $|\alpha| \leq k$ and for any choice of canonical coordinates. We will say that a function $f: M \rightarrow \mathbb{C}$ is C^∞ -bounded if $f \in C^\infty(M)$ and f is C^k -bounded for every $k = 0, 1, 2, \dots$

We will assume that we are given a positive C^∞ -bounded measure $d\mu$ on M , i.e. in any local coordinates x^1, x^2, \dots, x^n there exists a strictly positive C^∞ -bounded density $\rho(x)$ such that $d\mu = \rho(x) dx^1 dx^2 \dots dx^n$.

For $x, y \in M$, let $d(x, y)$ denote the distance function on M induced by g . Let us fix a point $x_0 \in M$, and denote $d(x) = d(x, x_0)$.

Let E be a Hermitian vector bundle over M . We will assume that E is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of E on every canonical coordinate neighborhood U such that the corresponding matrix transition functions $h_{U, U'}$ on all intersections $U \cap U'$ of such neighborhoods are C^∞ -bounded, i.e. all derivatives $\partial_y^\alpha h_{U, U'}(y)$, where α is a multiindex, with respect to canonical coordinates are bounded with bounds C_α which do not depend on the chosen pair U, U').

We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u, v) = \int_M \langle u(x), v(x) \rangle_{E_x} d\mu(x). \tag{1.1}$$

Here $\langle \cdot, \cdot \rangle_{E_x}$ denotes the fiberwise inner product.

In what follows, $C^\infty(E)$ denotes smooth sections of E , and $C_c^\infty(E)$ denotes smooth compactly supported sections of E .

Let

$$\nabla: C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$$

be a Hermitian connection on E which is C^∞ -bounded as a linear differential operator, i.e. in any canonical coordinate system U (with the chosen trivializations of $E|_U$ and $(T^*M \otimes E)|_U$),

∇ is written in the form

$$\nabla = \sum_{|\alpha| \leq 1} a_\alpha(y) \partial_y^\alpha,$$

where α is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial_y^\beta a_\alpha(y)$ for any multiindex β are bounded by a constant C_β which does not depend on the chosen canonical neighborhood.

For more on manifolds of bounded geometry, bundles of bounded geometry and C^∞ -bounded linear differential operators, see, for example, Appendix 1 of [7].

We will consider a Schrödinger-type differential expression of the form

$$H_V = \nabla^* \nabla + V,$$

where V is a linear self-adjoint bundle map $V \in L_{\text{loc}}^1(\text{End } E)$. Here

$$\nabla^* : C^\infty(T^*M \otimes E) \rightarrow C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1).

If we take $\nabla = d$, where $d : C^\infty(M) \rightarrow \Omega^1(M)$, then $d^*d : C^\infty(M) \rightarrow C^\infty(M)$ is called the scalar Laplacian and will be denoted by Δ_M .

1.2. The maximal operator. We define the maximal operator $H_{V,\text{max}}$ associated to H_V as an operator in $L^2(E)$ given by $H_{V,\text{max}}u = H_Vu$ with the domain

$$\text{Dom}(H_{V,\text{max}}) = \{u \in L^2(E) : Vu \in L_{\text{loc}}^1(E), H_Vu \in L^2(E)\}. \quad (1.2)$$

Here $\nabla^* \nabla u$ in $H_Vu = \nabla^* \nabla u + Vu$ is understood in distributional sense.

We want to give a sufficient condition for self-adjointness of $H_{V,\text{max}}$. We make the following assumptions on V .

Assumption A.

- (i) $V \in L_{\text{loc}}^1(\text{End } E)$.
- (ii) For all $x \in M$,

$$V(x) \geq -q(d(x)), \quad (1.3)$$

where $q : (0, +\infty) \rightarrow \mathbb{R}$ is a positive nondecreasing function of r and $q(r) = O(r^2)$, as $r \rightarrow +\infty$. The inequality (1.3) is understood in the sense of operators $E_x \rightarrow E_x$.

We now state the main result.

Theorem 1.3. *Assume that (M, g) is a manifold of bounded geometry with positive C^∞ -bounded measure $d\mu$, E is a Hermitian vector bundle of bounded geometry over M , and ∇ is a C^∞ -bounded Hermitian connection on E . Suppose that Assumption A holds. Then $H_{V,\text{max}}$ is a self-adjoint operator.*

Remark 1.4. In [5, Main Theorem] T. Kato proved the self-adjointness of the maximal operator corresponding to $-\Delta + V$, where Δ is the standard Laplacian on \mathbb{R}^n with the standard metric and measure, and $V = V_1 + V_2$, where $V_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies (ii) of the Assumption A above, and V_2 belongs to what is now called Kato's class (in the case $n \geq 5$, another option is to require $V_2 \in L^p(\mathbb{R}^n)$, where $p = \frac{n}{2}$).

1.5. Sobolev space $W^{1,2}(E)$. By $W^{1,2}(E)$ we will denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u, v)_1 := (u, v) + (\nabla u, \nabla v) \quad u, v \in C_c^\infty(E).$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

In what follows we will use the following facts and notations from differential geometry.

Let ∇^1 be the connection on $T^*M \otimes E$ induced by ∇ and Levi-Civita connection ∇^{LC} on T^*M . Then

$$\nabla^1: C^\infty(T^*M \otimes E) \rightarrow C^\infty(T^*M \otimes T^*M \otimes E). \quad (1.4)$$

Define an operator $A: C^\infty(E) \rightarrow C^\infty(E)$ as

$$A := -(g \otimes 1) \circ \nabla^1 \circ \nabla.$$

By proposition 2.1 in Appendix C of [8], we have $A = \nabla^* \nabla$.

Moreover, the following holds: $\Delta_M = -g \circ \nabla^{LC} \circ d$.

2. PROOF OF THEOREM 1.3 IN THE CASE WHEN $q=\text{CONST}$

We adopt the arguments from Lemma 1 in [5] to our setting with the help of more general version of Kato's inequality (2.1).

Throughout this section we assume that q is a positive constant.

2.1. Kato's inequality. We begin with the following variant of Kato's inequality for Bochner Laplacian (for the proof see Theorem 5.7 in [2]).

Lemma 2.2. *Assume that (M, g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E . Assume that $w \in L^1_{\text{loc}}(E)$ and $\nabla^* \nabla w \in L^1_{\text{loc}}(E)$. Then*

$$\Delta_M |w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign } w \rangle, \quad (2.1)$$

where

$$\text{sign } w(x) = \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3. The original version of Kato's inequality was proven in Kato [4].

2.4. Positivity. In what follows, we will use the following Lemma whose proof is given in Appendix B of [2].

Lemma 2.5. *Assume that (M, g) is a manifold of bounded geometry with a smooth positive measure $d\mu$. Assume that*

$$(b + \Delta_M)u = \nu \geq 0, \quad u \in L^2(M),$$

where $b > 0$, $\Delta_M = d^*d$ is the scalar Laplacian on M , and the inequality $\nu \geq 0$ means that ν is a positive distribution on M , i.e. $(\nu, \phi) \geq 0$ for any $0 \leq \phi \in C_c^\infty(M)$.

Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).

Remark 2.6. It is not known whether Lemma 2.5 holds if M is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

2.7. Quadratic forms. In what follows, all quadratic forms are considered in the Hilbert space $L^2(E)$.

1. By h_0 we denote the quadratic form

$$h_0(u) = \int |\nabla u|^2 d\mu \tag{2.2}$$

with the domain $D(h_0) = W^{1,2}(E) \subset L^2(E)$. Clearly, h_0 is a non-negative, densely defined and closed form.

2. By h_1 we denote the quadratic form

$$h_1(u) = \int \langle Vu, u \rangle d\mu \tag{2.3}$$

with the domain

$$D(h_1) = \{u \in L^2(E) : \int \langle (V + q)u, u \rangle d\mu < +\infty\}. \tag{2.4}$$

The quadratic form h_1 is semi-bounded below, densely defined and closed (see Example VI.1.15 in [6]).

3. Define the quadratic form

$$h(u) = h_0(u) + h_1(u) \tag{2.5}$$

with the domain

$$D(h) = D(h_0) \cap D(h_1). \tag{2.6}$$

Since $C_c^\infty(E) \subset D(h)$, it follows that h is densely defined. Since h_0 is non-negative and closed and h_1 is semi-bounded below and closed, it follows by Theorem VI.1.31 from [6] that $h = h_0 + h_1$ is semi-bounded below and closed. In particular, note that for all $u \in D(h)$,

$$h(u) \geq \|\nabla u\|^2 - q\|u\|^2. \tag{2.7}$$

From (2.7) we obtain

$$h(u) \geq \|u\|_1^2 - (q+1)\|u\|^2, \quad \text{for all } u \in D(h), \quad (2.8)$$

where $\|u\|_1 = (\|\nabla u\|^2 + \|u\|^2)^{1/2}$ is the norm in $W^{1,2}(E)$.

In what follows, $h(\cdot, \cdot)$ will denote the corresponding sesquilinear form obtained from h via polarization identity.

We will use the following well-known lemma.

Lemma 2.8. *Assume that $0 \leq T \in L_{\text{loc}}^1(\text{End } E)$ is a linear self-adjoint bundle map. Assume also that $u \in Q(T)$, where $Q(T) = \{u \in L^2(E) : \langle Tu, u \rangle \in L^1(M)\}$.*

Then $Tu \in L_{\text{loc}}^1(E)$.

Proof. By adding a constant we can assume that $T \geq 1$ (in operator sense).

Assume that $u \in Q(T)$. We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $1 \leq T(x) \in \text{End}(E_x)$ to get $T(x) = \text{diag}(c_1(x), c_2(x), \dots, c_m(x))$, where $0 < c_j \in L_{\text{loc}}^1(M)$, $j = 1, 2, \dots, m$ and $m = \dim E_x$.

Let $u_j(x)$ ($j = 1, 2, \dots, m$) be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of E_x . Then for all $x \in M$

$$\langle Tu, u \rangle = \sum_{j=1}^m c_j(x) |u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u \rangle d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j |u_j|^2 \in L^1(M)$, for all $j = 1, 2, \dots, m$.

Now, for all $x \in M$ and $j = 1, 2, \dots, m$

$$2|c_j u_j| = 2|c_j| |u_j| \leq |c_j| + |c_j| |u_j|^2, \quad (2.9)$$

The right hand side of (2.9) is clearly in $L_{\text{loc}}^1(M)$. Therefore $c_j u_j \in L_{\text{loc}}^1(M)$.

But $(Tu)(x)$ has components $c_j(x) u_j(x)$ ($j = 1, 2, \dots, m$) with respect to chosen bases of E_x . Therefore $Tu \in L_{\text{loc}}^1(E)$, and the lemma is proven. \square

Corollary 2.9. *If $u \in D(h)$, then $Vu \in L_{\text{loc}}^1(E)$.*

Proof. By adding a constant to V , we may assume that $V \geq 1$ (in the operator sense). By Lemma 2.8 it follows that $Vu \in L_{\text{loc}}^1(E)$, and the corollary is proven. \square

2.10. Self-adjoint operator H associated to h . Since h is a densely defined, closed and semi-bounded below form in $L^2(E)$, by Theorem VI.2.1 from [6] there exists a semi-bounded below self-adjoint operator H in $L^2(E)$ such that $\text{Dom}(H) \subset D(h)$ and

$$h(u, v) = (Hu, v), \quad \text{for all } u \in \text{Dom}(H) \text{ and } v \in D(h).$$

We will prove that $H = H_{V, \max}$.

Lemma 2.11. *Let V be as in Assumption A, where q is a positive constant. Then $H \subset H_{V,\max}$.*

Proof. Let $u \in \text{Dom}(H)$. By Corollary 2.9, it follows that $Vu \in L^1_{\text{loc}}(E)$. Then for any $v \in C_c^\infty(E)$,

$$(Hu, v) = h(u, v) = (\nabla u, \nabla v) + \int \langle Vu, v \rangle d\mu. \quad (2.10)$$

The first equality holds by Sect. 2.10, and the second equality holds by the definition of h .

Hence, using integration by parts in the first term on the right hand side of the second equality in (2.10) (see, for example, Lemma 8.8 from [2]), we get

$$(u, \nabla^* \nabla v) = \int \langle Hu - Vu, v \rangle d\mu, \quad \text{for all } v \in C_c^\infty(E). \quad (2.11)$$

Since $Vu \in L^1_{\text{loc}}(E)$ and $Hu \in L^2(E)$, it follows that $(Hu - Vu) \in L^1_{\text{loc}}(E)$, and (2.11) implies $\nabla^* \nabla u = Hu - Vu$ (as distributional sections of E). Therefore,

$$\nabla^* \nabla u + Vu = Hu.$$

Since $Hu \in L^2(E)$, it follows that $u \in \text{Dom}(H_{V,\max})$ and $H_{V,\max}u = Hu$. This proves the relation $H \subset H_{V,\max}$. \square

2.12. Proof of Theorem 1.3 in the case $q = \text{const}$. To conclude the proof of Theorem 1.3 when q is a positive constant, it is enough to show that $\text{Dom}(H_{V,\max}) \subset \text{Dom}(H)$. Suppose $u \in \text{Dom}(H_{V,\max})$. Since H is self-adjoint and semibounded below, it follows that for sufficiently large $b > 0$, $(H + b)^{-1}$ is a bounded linear operator on $L^2(E)$. Set

$$v = (H + b)^{-1}(H_{V,\max} + b)u.$$

Then $(H + b)v = (H_{V,\max} + b)u$. Since $H \subset H_{V,\max}$, it follows that $(H_{V,\max} + b)w = 0$, where $w = u - v$. Therefore, $\nabla^* \nabla w + (V + b)w = 0$. Since $Vw \in L^1_{\text{loc}}(E)$, it follows that $\nabla^* \nabla w \in L^1_{\text{loc}}(E)$. By Lemma 2.2, we obtain

$$\Delta_M |w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign } w \rangle = \langle -(V + b)w, \text{sign } w \rangle \leq (q - b)|w|. \quad (2.12)$$

Choosing b large enough so that $\tilde{b} := b - q > 0$, we obtain

$$(\Delta_M + \tilde{b})|w| \leq 0. \quad (2.13)$$

By Lemma 2.5 it follows that $|w| \leq 0$. So $w = 0$, and, hence, $u = v$. Thus $u \in \text{Dom}(H)$. This proves $H_{V,\max} \subset H$. \square

3. PROOF OF THEOREM 1.3

We will adopt the scheme of proof of Main Theorem in [5] to our setting with the help of a family of cut-off functions on manifolds of bounded geometry given in Sect. 3.3 below.

We begin with an important estimate.

Lemma 3.1. *Assume that q is a positive constant. If $u \in L^2(E)$, $Vu \in L^1_{\text{loc}}(E)$ and $H_V u \in W^{-1,2}(E)$, then $u \in D(h)$ and the following estimate holds:*

$$\|u\|_1 \leq \sqrt{2} \left((q+1)^{1/2} \|u\| + 2^{-1/2} \|H_V u\|_{-1} \right), \quad (3.1)$$

where $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$, $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$ and h as in (2.5).

Proof. Let \mathcal{K} denote the Hilbert space $D(h)$ in (2.6) with the norm $(h(u) + z\|u\|^2)^{1/2}$, where z is a positive constant such that $z \geq (q+2)$. Let H be the self-adjoint operator associated with h . Then we have the following continuous inclusions.

$$\text{Dom}(H) \subset \mathcal{K} \subset W^{1,2}(E) \subset L^2(E) \subset W^{-1,2}(E) \subset \mathcal{K}^*, \quad (3.2)$$

where \mathcal{K}^* denotes the dual of \mathcal{K} .

By a well-known abstract fact, $H: \text{Dom}(H) \rightarrow L^2(E)$ can be extended to a continuous linear operator $H': \mathcal{K} \rightarrow \mathcal{K}^*$. In fact, H' is the restriction of the differential expression H_V to \mathcal{K} . This is clear since $w \in \mathcal{K}$ implies $Vw \in L^1_{\text{loc}}(E)$ by Corollary 2.9. Moreover, the same calculation as in (2.11) shows that $\nabla^* \nabla w + Vw = H'w$.

By an abstract fact (see the Remark after Theorem 2.1 in [3]), $H' + z: \mathcal{K} \rightarrow \mathcal{K}^*$ is an isomorphism of Hilbert spaces.

Let u be as in hypothesis of this lemma. Then $(z + H_V)u \in W^{-1,2}(E) \subset \mathcal{K}^*$. Thus we can find $w \in \mathcal{K}$ such that

$$(H' + z)w = (H_V + z)u.$$

Since $w \in \mathcal{K}$, we get $Vw \in L^1_{\text{loc}}(E)$. Also, H' is the restriction of H_V to \mathcal{K} , hence $(H_V + z)(w - u) = 0$. Denoting $s = w - u$, we get $H_V s = -zs$. Note that $Vs \in L^1_{\text{loc}}(E)$. Since $s \in L^2(E)$, we immediately get $s \in \text{Dom}(H_{V,\text{max}})$. Therefore

$$(H_{V,\text{max}} + z)s = 0. \quad (3.3)$$

By the proof in Sect. 2, $H_{V,\text{max}} + z$ is a positive self-adjoint operator, so (3.3) implies $s = 0$, i.e. $u = w$. This shows that $u \in D(h)$.

It remains to prove (3.1). Since $(H' + z)^{-1}$ maps \mathcal{K}^* onto \mathcal{K} continuously, it also maps $W^{-1,2}(E)$ into $W^{1,2}(E)$ continuously.

For $w \in \text{Dom}(H)$, we have

$$h(w) = (Hw, w) \leq \|Hw\|_{-1} \|w\|_1 \leq \frac{1}{4\epsilon} \|Hw\|_{-1}^2 + \epsilon \|w\|_1^2,$$

where $\epsilon > 0$ is arbitrary.

Taking $\epsilon = 1/2$ and using (2.8), we obtain

$$\|w\|_1 \leq \sqrt{2} \left((q+1)^{1/2} \|w\| + 2^{-1/2} \|Hw\|_{-1} \right). \quad (3.4)$$

We know that $(H+z)$ is a positive self-adjoint operator, and hence $(H+z)\text{Dom}(H) = L^2(E)$. Let $f = (H+z)w$, i.e. $w = (H+z)^{-1}f$. From (3.4) we obtain

$$\|(H+z)^{-1}f\|_1 \leq \sqrt{2}(q+1)^{1/2} \|(H+z)^{-1}f\| + \|f - z(H+z)^{-1}f\|_{-1}. \quad (3.5)$$

Operator $(H+z)^{-1}$ can be extended to a continuous linear operator $(H'+z)^{-1}$ which maps $W^{-1,2}(E)$ into $W^{1,2}(E)$ continuously. Hence (3.5) can be extended to all $f \in W^{-1,2}(E)$, whereby H is replaced by H' . Now write $u = (H'+z)^{-1}f$, i.e. $f = (H'+z)u = (H_V+z)u$ (the last equality is true since $u \in \mathcal{K}$ and H' is the restriction of differential expression H_V to \mathcal{K}). This immediately leads to (3.1) (whereby u is any element of \mathcal{K} such that $(H_V+z)u \in W^{-1,2}(E)$, i.e. $H_V u \in W^{-1,2}(E)$). Since u in the hypotheses of this lemma satisfies these conditions, the lemma is proven. \square

3.2. Regularized distance. Since M has bounded geometry, there exists a regularized distance on M ; more precisely, there exists a smooth function $\tilde{d}: M \times M \rightarrow [0, +\infty)$ satisfying the following conditions

(i) There exists $\rho > 0$ such that

$$|\tilde{d}(x, y) - d(x, y)| < \rho \quad \text{for every } x, y \in M, \quad (3.6)$$

(ii) for every multiindex α with $|\alpha| > 0$, there exists a constant $C_\alpha > 0$ such that

$$|\partial_y^\alpha \tilde{d}(x, y)| \leq C_\alpha \quad x, y \in M, \quad (3.7)$$

where the derivative ∂_y^α is taken with respect to canonical coordinates.

For the construction of \tilde{d} with these properties, see, for example, Lemma 2.1 of [7].

Let us fix $x_0 \in M$, and denote $\tilde{d}(x) = \tilde{d}(x, x_0)$.

In what follows we will also use the notation

$$\tilde{B}_r = \{x \in M : \tilde{d}(x) < r\}, \quad \text{for } r > 0. \quad (3.8)$$

3.3. Cut-off functions. Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 1$, for $t \leq \frac{1}{4}$ and $\chi = 0$, for $t > \frac{3}{4}$ with $|\chi'| \leq C_1$ and $|\chi''| \leq C_2$, where $C_1 > 0$ and $C_2 > 0$ are constants.

For $0 < r < R$, define $\phi_{r,R}: M \rightarrow [0, 1]$ by

$$\phi_{r,R}(x) = \chi \left(\frac{\tilde{d}(x) - r}{R - r} \right). \quad (3.9)$$

Clearly, $\phi_{r,R} \in C_c^\infty(M)$, $\phi_{r,R} = 0$ for $\tilde{d}(x) > R - \delta$, and $\phi_{r,R} = 1$ for $\tilde{d}(x) \leq r + \delta$, where $\delta = (R - r)/4$. Moreover, by (3.7) we obtain

$$\|\phi_{r,R}\|_\infty \leq 1, \quad \|d\phi_{r,R}\|_\infty \leq \frac{K_1}{R-r}, \quad \|\Delta_M \phi_{r,R}\|_\infty \leq \frac{K_2}{R-r} + \frac{K_3}{(R-r)^2}. \quad (3.10)$$

Here $\|\cdot\|_\infty$ denotes the L^∞ norm, and K_1, K_2, K_3 are positive constants.

We also need a statement about the regularity of sections in $\text{Dom}(H_{V,\max})$. From now on, q is as in Assumption A.

Lemma 3.4. *Let $u \in \text{Dom}(H_{V,\max})$. Then $u \in W_{\text{loc}}^{1,2}(E)$ and the following estimate holds for any $0 < r < R$:*

$$\|u\|_{W^{1,2}(\tilde{B}_r, E)} \leq \sqrt{2} \left(2^{-1/2} \|H_{V,\max} u\| + (q(R+\rho))^{1/2} \|u\| \right) + C_{R-r} \|u\|, \quad (3.11)$$

where \tilde{B}_r is as in (3.8), ρ is as in (3.6) and $C_{R-r} > 0$ is a constant depending on $R - r$, n , and $\dim E$.

Proof. Let $u \in \text{Dom}(H_{V,\max})$. Take $\phi = \phi_{r,R}$, with $\phi_{r,R}$ as in (3.9). Clearly $\phi V u \in L_{\text{loc}}^1(E)$, and

$$\begin{aligned} H_V(\phi u) &= \nabla^* \nabla(\phi u) + \phi(Vu) = -(g \otimes 1) \circ \nabla^1 \circ \nabla(\phi u) + \phi(Vu) \\ &= -(g \otimes 1) \circ \nabla^1(d\phi \otimes u + \phi \nabla u) + \phi(Vu) \\ &= -(g \otimes 1)((\nabla^{LC} d\phi) \otimes u) - (g \otimes 1)(d\phi \otimes \nabla u) - (g \otimes 1)(d\phi \otimes \nabla u) - (g \otimes 1)(\phi \nabla^1 \nabla u) + \phi(Vu) \\ &= \phi H_V u - 2(g \otimes 1)(d\phi \otimes \nabla u) + (\Delta_M \phi)u, \end{aligned} \quad (3.12)$$

where ∇^{LC} and ∇^1 are as in Sect. 1.

Clearly $(g \otimes 1)(d\phi \otimes \nabla u) \in W^{-1,2}(E)$. We will now estimate its $W^{-1,2}$ norm. Let $v \in C_c^\infty(E)$. Then

$$\begin{aligned} ((g \otimes 1)(d\phi \otimes \nabla u), v) &= (\nabla u, d\phi \otimes v) = (u, \nabla^*(d\phi \otimes v))_{L^2(E)} \\ &= (u, -\nabla_X v)_{L^2(E)} + (u, -\text{div}(X)v)_{L^2(E)}, \end{aligned} \quad (3.13)$$

where X is the vector field associated to $d\phi$ via metric g , i.e. $X = \text{grad } \phi$, and (\cdot, \cdot) on both sides of the first equality denotes the duality between $W^{-1,2}$ and $W^{1,2}$. In the second equality in (3.13) we used integration by parts; see, for example, Lemma 8.8 from [2]. The last equality in (3.13) follows from Proposition 1.4 of Appendix C in [8]. Since $-\text{div}(\text{grad } \phi) = \Delta_M \phi$, we obtain

$$((g \otimes 1)(d\phi \otimes \nabla u), v) = (u, -\nabla_X v) + (u, (\Delta_M \phi)v). \quad (3.14)$$

By (3.10) we get

$$|(u, (\Delta_M \phi)v)| \leq \left(\frac{L'}{R-r} + \frac{L''}{(R-r)^2} \right) \|u\| \|v\|, \quad (3.15)$$

where $L' > 0$, $L'' > 0$ are constants and $\|\cdot\|$ is the norm in $L^2(E)$.

Since $X = \text{grad } \phi$ and $\|\text{grad } \phi\|_\infty \leq L'''/(R-r)$, where $L''' > 0$ is a constant, we obtain

$$\|\nabla_X v\|_{L^2(E)} = \|(i_X \otimes 1)\nabla v\|_{L^2(E)} \leq \|X\|_\infty \|\nabla v\|_{L^2(T^*M \otimes E)} \leq \frac{L'''}{R-r} \|\nabla v\|_{L^2(T^*M \otimes E)}. \quad (3.16)$$

Here $i_X : \Omega^j(M) \rightarrow \Omega^{j-1}(M)$ denotes the usual interior product with vector field X .

Combining (3.15) and (3.16) we get

$$|(u, -\nabla_X v) + (u, (\Delta_M \phi)v)| \leq \left(\frac{L}{R-r} + \frac{L''}{(R-r)^2} \right) \|u\| \|v\|_1 \quad \text{for all } v \in C_c^\infty(E), \quad (3.17)$$

where $\|\cdot\|_1$ is the norm $W^{1,2}(E)$ and $L = L' + L'' > 0$, $L'' > 0$ are constants (independent of R). Since $C_c^\infty(E)$ is dense in $W^{1,2}(E)$, we conclude that

$$\|(g \otimes 1)(d\phi \otimes \nabla u)\|_{-1} \leq C' \|u\|, \quad (3.18)$$

where $C' = L/(R-r) + L''/(R-r)^2$.

From (3.12) we get $H_V(\phi u) \in W^{-1,2}(E)$. Now we can apply Lemma 3.1 with q replaced by $q(R+\rho)$ to get $\phi u \in W^{1,2}(E)$. This proves $u \in W_{\text{loc}}^{1,2}(E)$. To prove the estimate (3.11), we use (3.4) with q replaced by $q(R+\rho)$ and $w = \phi u$. From (3.12) and (3.18), we obtain

$$\|H_V(\phi u)\|_{-1} \leq \|H_{V,\max} u\| + 2C' \|u\| + C'' \|u\|,$$

where $C' > 0$ and $C'' > 0$ are constants (depending on $R-r$, n and $\dim E$). This and (3.4) immediately give (3.11). \square

Lemma 3.4 and (3.12) immediately imply the following

Corollary 3.5. *Let $\phi \in C_c^\infty(M)$ and $u \in \text{Dom}(H_{V,\max})$. Then $\phi u \in \text{Dom}(H_{V,\max})$.*

In what follows, we denote $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$.

3.6. A sequence of cut-off functions. Let $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $t \leq 1$; $\chi(t) = 0$ for $t \geq 2$; $|\chi'| \leq C_1$, and $|\chi''| \leq C_2$, for some constants $C_1 > 0$ and $C_2 > 0$. For every $k \in \mathbb{Z}_+$ and $x \in M$ define

$$\phi_k(x) = \chi\left(\frac{\tilde{d}(x)}{k}\right). \quad (3.19)$$

Clearly, $\phi_k \in C_c^\infty(M)$, $\phi_k = 1$ on \tilde{B}_k , and $\text{supp } \phi_k \subset \tilde{B}_{2k}$. Also, for every $x \in M$, $\lim_{k \rightarrow \infty} \phi_k(x) = 1$. Moreover, properties (3.10) hold with $\phi = \phi_k$ and $R-r = k$.

Lemma 3.7. *Let ϕ_k be as in (3.19) and $u \in \text{Dom}(H_{V,\max})$. Then $H_{V,\max}(\phi_k u) \rightharpoonup H_{V,\max} u$ weakly in $L^2(E)$, as $k \rightarrow +\infty$.*

Proof. Let $u \in \text{Dom}(H_{V,\max})$. By Corollary 3.5, $\phi_k u \in \text{Dom}(H_{V,\max})$. By (3.12) we have

$$H_{V,\max}(\phi_k u) = \phi_k(H_V u) - 2(g \otimes 1)(d\phi_k \otimes \nabla u) + (\Delta_M \phi_k)u.$$

As $k \rightarrow +\infty$, we clearly have $\phi_k H_V u \rightarrow H_V u$ in $L^2(E)$, and $(\Delta_M \phi_k)u \rightarrow 0$ in $L^2(E)$ since $\|\Delta_M \phi_k\|_\infty \leq \frac{K_4}{k}$, where K_4 is a constant independent of k (take, for example, $K_4 = K_2 + K_3$, where $K_2 > 0$, $K_3 > 0$ are as in (3.10) with $\phi = \phi_k$ and $R - r = k$.)

It remains to show that

$$(g \otimes 1)(d\phi_k \otimes \nabla u) \rightarrow 0 \quad \text{weakly in } L^2(E). \quad (3.20)$$

We claim that $(g \otimes 1)(d\phi_k \otimes \nabla u)$ is bounded in $L^2(E)$. Indeed, by (3.11) with $r = 2k$ and $R = 2k + 1$ we have

$$\|\nabla u\|_{L^2(\tilde{B}_{2k}, T^*M \otimes E)} \leq \sqrt{2} \left(2^{-1/2} \|H_{V,\max} u\| + (q(2k + 1 + \rho))^{1/2} \|u\| \right) + C_1 \|u\|, \quad (3.21)$$

where ρ is as in (3.6).

From the proof of Lemma 3.4 it follows that C_1 does not depend on k because $R - r = 1$. Therefore $C_1 \|u\|$ is bounded.

By assumption on q , we have $(q(2k + 1 + \rho))^{1/2} = O(k)$ as $k \rightarrow +\infty$, and by (3.10), it follows that $\|d\phi_k\|_\infty \leq \frac{K_1}{k}$, ($K_1 > 0$ is a constant independent of k). This and (3.21) immediately imply that $(g \otimes 1)(d\phi_k \otimes \nabla u)$ is bounded in $L^2(E)$.

To prove (3.20), it is enough to show that for all $v \in C_c^\infty(E)$, we have

$$((g \otimes 1)(d\phi_k \otimes \nabla u), v) \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

By (3.14) and (3.17) with $\phi = \phi_k$ and $R - r = k$, we obtain

$$|((g \otimes 1)(d\phi_k \otimes \nabla u), v)| = |(u, (\Delta_M \phi_k)v) + (u, -\nabla_{X_k} v)| \leq \left(\frac{L}{k} + \frac{L''}{k^2} \right) \|u\| \|v\|_1, \quad (3.22)$$

where X_k is the vector field associated to $d\phi_k$ via metric g , $L > 0$ and $L'' > 0$ are constants independent of k , and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

Letting $k \rightarrow +\infty$ in (3.22) gives (3.20) and proves the lemma. \square

3.8. Truncation operators. For $N \in \mathbb{Z}_+$ define

$$H_V^{(N)} = \nabla^* \nabla + V^{(N)}, \quad (3.23)$$

where for every $x \in M$,

$$V^{(N)}(x) = \begin{cases} V(x) & \text{for } x \in \tilde{B}_{2N}, \\ V(x) + (q(d(x)) - q(2N)) \text{Id} & \text{for } x \notin \tilde{B}_{2N}. \end{cases} \quad (3.24)$$

Here V is as in Assumption A, and Id is the identity endomorphism of E_x .

As in Sect. 1, we define the maximal operator $H_{V,\max}^{(N)}$ associated to $H_V^{(N)}$.

From (1.3) and (3.6) it follows that $V^{(N)} \in L^1_{\text{loc}}(\text{End } E)$ is bounded below by a constant, and by Sect. 2, $H_{V,\text{max}}^{(N)}$ is a self-adjoint operator.

Lemma 3.9. $\text{Dom}(H_{V,\text{max}}^{(N)})$ is independent of N . We will denote this domain by G .

Proof. Let $N, k \in \mathbb{Z}_+$ and $N > k$. Then for every $x \in M$,

$$(V^{(N)} - V^{(k)})(x) = \begin{cases} 0 & \text{for } x \in \tilde{B}_{2k}, \\ (-q(d(x)) + q(2k)) \text{Id} & \text{for } x \in \tilde{B}_{2N} \setminus \tilde{B}_{2k}, \\ (q(2k) - q(2N)) \text{Id} & \text{for } x \in M \setminus \tilde{B}_{2N}, \end{cases}$$

where Id is the identity endomorphism of E_x .

Since $V^{(N)} - V^{(k)} \in L^\infty(\text{End } E)$, it follows that $\text{Dom}(H_{V,\text{max}}^{(N)}) = \text{Dom}(H_{V,\text{max}}^{(k)})$, and the lemma is proven. \square

Lemma 3.10. If $u \in \text{Dom}(H_{V,\text{max}})$ or $u \in G$, then $\phi u \in \text{Dom}(H_{V,\text{max}}) \cap G$, for all $\phi \in C_c^\infty(M)$. Moreover, if $\text{supp } \phi \subset \tilde{B}_{2N}$, then $H_{V,\text{max}}^{(N)}(\phi u) = H_{V,\text{max}}(\phi u)$.

Proof. The first statement follows immediately from Corollary 3.5. If $\text{supp } \phi \subset \tilde{B}_{2N}$ then $\text{supp}(\phi u) \subset \tilde{B}_{2N}$. By (3.24) it is clear that if v is supported on \tilde{B}_{2N} , then $v \in \text{Dom}(H_{V,\text{max}}^{(N)})$ if and only if $v \in \text{Dom}(H_{V,\text{max}})$. In this case $H_{V,\text{max}}^{(N)}(v) = H_{V,\text{max}}(v)$. Applying this to $v = \phi u$ the second statement is also proven. \square

Lemma 3.11. $H_{V,\text{max}}$ is a symmetric operator, i.e.

$$(H_{V,\text{max}}u, v) = (u, H_{V,\text{max}}v) \quad \text{for all } u, v \in \text{Dom}(H_{V,\text{max}}). \quad (3.25)$$

Proof. Let $u, v \in \text{Dom}(H_{V,\text{max}})$, and let ϕ_k be as in (3.19). By Lemma 3.10 and self-adjointness of $H_{V,\text{max}}^{(k)}$ we obtain

$$(H_{V,\text{max}}(\phi_k u), \phi_k v) = (H_{V,\text{max}}^{(k)}(\phi_k u), \phi_k v) = (\phi_k u, H_{V,\text{max}}^{(k)}(\phi_k v)) = (\phi_k u, H_{V,\text{max}}(\phi_k v)). \quad (3.26)$$

Letting $k \rightarrow +\infty$ in (3.26) and applying Lemma 3.7, we immediately get (3.25). \square

Lemma 3.12. $H_{V,\text{max}}$ is a closed operator.

Proof. Let $\{u_N\}$ be a sequence in $\text{Dom}(H_{V,\text{max}})$ such that $u_N \rightarrow u$ in $L^2(E)$ and $H_{V,\text{max}}u_N \rightarrow f$ in $L^2(E)$. We need to show that $u \in \text{Dom}(H_{V,\text{max}})$ and $H_{V,\text{max}}u = f$.

By (3.11) it follows that $\{u_N\}$ is a Cauchy sequence in $W^{1,2}(\tilde{B}_r, E)$ for any $r > 0$. Thus u_N converges in $W_{\text{loc}}^{1,2}(E)$, and the limit must be u . Hence $u \in W_{\text{loc}}^{1,2}(E)$. Let ϕ_k be as in (3.19). Then Lemma 3.10 and (3.12) with $\phi = \phi_k$ and $u = u_N$ give

$$H_{V,\text{max}}^{(k)}(\phi_k u_N) = \phi_k H_V u_N - 2(g \otimes 1)(d\phi_k \otimes \nabla u_N) + (\Delta_M \phi_k)u_N. \quad (3.27)$$

Fix k and let $N \rightarrow +\infty$ in (3.27). By assumption it is clear that $\phi_k H_V u_N \rightarrow \phi_k f$ in $L^2(E)$ and $(\Delta_M \phi_k) u_N \rightarrow (\Delta_M \phi_k) u$ in $L^2(E)$. Since $u_N \rightarrow u$ in $W_{\text{loc}}^{1,2}(E)$, the middle term on the right hand side of (3.27) converges to $2(g \otimes 1)(d\phi_k \otimes \nabla u)$ in $L^2(E)$.

Since $H_{V,\text{max}}^{(k)}$ is closed, $H_{V,\text{max}}^{(k)}(\phi_k u_N) \rightarrow H_{V,\text{max}}^{(k)}(\phi_k u)$ in $L^2(E)$. Thus we obtain

$$H_{V,\text{max}}^{(k)}(\phi_k u) = \phi_k f - 2(g \otimes 1)(d\phi_k \otimes \nabla u) + (\Delta_M \phi_k) u. \quad (3.28)$$

Evaluating (3.28) at $x \in \tilde{B}_k$ (on this set $\phi_k(x) = 1$) and using (3.24), we have

$$(H_V u)(x) = f(x), \quad x \in \tilde{B}_k.$$

Since $\phi_k u \in \text{Dom}(H_{V,\text{max}}^{(k)})$, we get $\phi_k V u \in L^1(E)$. Since k is arbitrary, $V u \in L_{\text{loc}}^1(E)$, and $H_V u = f$ for all $x \in M$. This shows that $u \in \text{Dom}(H_{V,\text{max}})$ and $H_{V,\text{max}} u = f$. \square

In what follows we will denote by $\text{Ran } A$ the range of operator A .

Lemma 3.13. *The following holds: $\text{Ran}(H_{V,\text{max}} \pm i) = L^2(E)$.*

Proof. Suppose $f \in L^2(E)$. Since $H_{V,\text{max}}^{(N)}$ is self-adjoint, we can find $u_N \in \text{Dom}(H_{V,\text{max}}^{(N)})$ such that

$$(H_{V,\text{max}}^{(N)} - i)u_N = f, \quad \|u_N\| \leq \|f\|, \quad \|H_{V,\text{max}}^{(N)} u_N\| \leq \|f\|. \quad (3.29)$$

Let ϕ_N be as in (3.19). Then equation (3.27) with $k = N$ gives

$$H_{V,\text{max}}^{(N)}(\phi_N u_N) = \phi_N H_V u_N - 2(g \otimes 1)(d\phi_N \otimes \nabla u_N) + (\Delta_M \phi_N) u_N. \quad (3.30)$$

Multiplying both sides of the equation in (3.29) by ϕ_N , we get $\phi_N H_V u_N = \phi_N(iu_N + f)$. So (3.30) can be written as

$$H_{V,\text{max}}^{(N)}(\phi_N u_N) = \phi_N(iu_N + f) - 2(g \otimes 1)(d\phi_N \otimes \nabla u_N) + (\Delta_M \phi_N) u_N. \quad (3.31)$$

From (3.31), it follows that

$$v_N := \phi_N f - 2(g \otimes 1)(d\phi_N \otimes \nabla u_N) + (\Delta_M \phi_N) u_N \in \text{Ran}(H_{V,\text{max}} - i).$$

We claim that as $N \rightarrow +\infty$, $v_N \rightarrow f$ weakly in $L^2(E)$. Indeed, it is clear that $\phi_N f \rightarrow f$ in $L^2(E)$, and due to (3.10) and (3.29) $(\Delta_M \phi_N) u_N \rightarrow 0$ in $L^2(E)$. The proof that $(g \otimes 1)(d\phi_N \otimes \nabla u_N) \rightarrow 0$ weakly in $L^2(E)$, is the same as in Lemma 3.7 and is based on the estimates (3.21) and (3.22) with u replaced by u_N , $H_{V,\text{max}}$ replaced by $H_{V,\text{max}}^{(N)}$, and k replaced by N .

Thus f is the weak limit of a sequence of elements in $\text{Ran}(H_{V,\text{max}} - i)$. But $\text{Ran}(H_{V,\text{max}} - i)$ is closed by Lemmas 3.11 and 3.12, so, in particular, weakly closed. Therefore $f \in \text{Ran}(H_{V,\text{max}} - i)$. Since $f \in L^2(E)$ is arbitrary, we get $\text{Ran}(H_{V,\text{max}} - i) = L^2(E)$. The proof of the equality $\text{Ran}(H_{V,\text{max}} + i) = L^2(E)$ is similar. \square

Proof of Theorem 1.3 By Lemma 3.11, Lemma 3.12 and Lemma 3.13, $H_{V,\text{max}}$ is a closed symmetric operator with $\text{Ran}(H_{V,\text{max}} \pm i) = L^2(E)$. By Theorem V.3.16 in [6], it follows that $H_{V,\text{max}}$ is self-adjoint. \square

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