POSITIVE PERTURBATIONS OF SELF-ADJOINT SCHRÖDINGER OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract. We consider a Schrödinger differential expression $L_0 = \Delta_M + V_0$ on a (not necessarily complete) Riemannian manifold (M, g) with metric g, where Δ_M is the scalar Laplacian on M and V_0 is a real-valued locally square integrable function on M. We consider a perturbation $L_0 + V$, where V is a non-negative locally square-integrable function on M, and give sufficient conditions for $L_0 + V$ to be essentially self-adjoint on $C_c^\infty(M)$. This is an extension of a result of T. Kappeler. The proof adopts Kappeler's technique, but requires the use of positivity preserving property of resolvents of certain self-adjoint operators in $L^2(M)$.

1. Introduction and the main result

1.1. **The setting.** Let (M, g) be a C^∞-Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected and oriented. We do not assume that M is complete. By $d\nu$ we will denote the Riemannian volume element of M. In any local coordinates $x^1, \dots, x^n$, we have $d\nu = \sqrt{\det(g_{jk})}\, dx^1 dx^2 \dots dx^n$.

By $L^2(M)$ we denote the space of complex-valued square integrable functions on M with the inner product

$$(u, v) = \int_M (u\bar{v})\, d\nu. \tag{1.1}$$

In what follows, by $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$—the space of smooth compactly supported functions on M, by $(C_c^\infty(M))^+$—non-negative elements of $C_c^\infty(M)$, by $\Omega^1(M)$—the space of smooth 1-forms on M, and by $\mathbb{Z}_+$—the set of positive integers.

By $d\colon C^\infty(M) \to \Omega^1(M)$ we denote the standard differential, and by $d^*\colon \Omega^1(M) \to C^\infty(M)$ we denote the formal adjoint of d with respect to the inner product (1.1).

By $\Delta_M := d^*d$ we will denote the scalar Laplacian on M.

We consider a Schrödinger-type differential expression

$$L_0 = \Delta_M + V_0,$$

where $V_0 \in L^2_{\mathrm{loc}}(M)$ is a real-valued function.

1.2. **Operators H_0 and T_0.** Define an operator H_0 by the formula $H_0 u = L_0 u$ with the domain $\mathrm{Dom}(H_0) = C_c^\infty(M)$. Since H_0 is a symmetric operator in $L^2(M)$, it follows that H_0 is closable. Let T_0 be the closure $\widetilde{H_0}$ of H_0.

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Assumption A. Assume that $V \in L^2_{\mathrm{loc}}(M)$ and $V \geq 0$.

1.3. **Operators H and T.** Let V be as in Assumption (A). Define an operator H by the formula $Hu = L_0 u + Vu$ with the domain $\mathrm{Dom}(H) = C_c^\infty(M)$. Since H is a symmetric operator in $L^2(M)$, it follows that H is closable. Let T be the closure $\widetilde{H}$ of H.

We make an additional Assumption on V.

Assumption B. Assume that there exist constants $a \geq 0$ and $b \geq 0$ such that

$$\|Vu\| \leq a\|T_0 u\| + b\|u\|, \qquad \text{for all } u \in C_c^\infty(M), \tag{1.2}$$

where $\|\cdot\|$ denotes the norm in $L^2(M)$ corresponding to (1.1).

We now state the main results.

Theorem 1.4. *Assume that (M, g) is a (not necessarily complete) C^∞-Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that the operator T_0 is semi-bounded below and self-adjoint. Assume that V satisfies Assumptions (A) and (B). Then the operator T is self-adjoint.*

Corollary 1.5. *Assume that the hypotheses of Theorem 1.4 are satisfied. Additionally, assume that T_0 satisfies the following property:*

(P) for every $0 \leq u \in \mathrm{Dom}(T_0)$, there exists a sequence $\{u_k\}$ in $(C_c^\infty(M))^+$ such that $u_k \to u$ and $T_0 u_k \to T_0 u$ in $L^2(M)$, as $k \to \infty$.

Then T also satisfies the property (P) (with T_0 replaced by T).

Remark 1.6. Theorem 1.4 and Corollary 1.5 extend the results of T. Kappeler (see Theorem 1 and Corollary 3 in [4]) concerning the Schrödinger differential expressions $L_0 = -\Delta + V_0$ and $L_0 + V$, where Δ is the standard Laplacian on an open set $\Omega \subset \mathbb{R}^n$ and V_0 and V are as in the hypotheses of Theorem 1.4.

2. Proof of Theorem 1.4

Throughout this section, we assume that all hypotheses of Theorem 1.4 are satisfied. By using positivity preserving property of resolvents of certain self-adjoint operators in $L^2(M)$, we will adopt Kappeler's technique from [4] to our setting.

We know by hypothesis that T_0 is semi-bounded below. Without loss of generality we may and we will assume that T_0 is non-negative, i.e.

$$(T_0 u, u) \geq 0, \qquad \text{for all } u \in \mathrm{Dom}(T_0).$$

Since $0 \leq V \in L^2_{\mathrm{loc}}(M)$, by definition of T it follows that T is a symmetric and non-negative operator. Let T_F denote the Friedrichs extension of T; see, for example, Section VI.2.3 in Kato [5]. Since T_F is a self-adjoint extension of T, to prove the Theorem, it suffices to show that $\mathrm{Dom}(T_F) \subset \mathrm{Dom}(T)$.

The following Lemma provides a key step in the proof of the Theorem.

Lemma 2.1. *Assume that* $u \in \mathrm{Dom}(T_F)$. *Then there exists a sequence* $\{u_k\}$ *in* $\mathrm{Dom}(T_0)$ *such that*

$$u_k \to u \qquad \textit{and} \qquad T_F u_k \to T_F u \quad \textit{in } L^2(M), \quad \textit{as } k \to \infty. \tag{2.1}$$

Remark 2.2. In Lemma 2.14 below, we will show that $\mathrm{Dom}(T_0) \subset \mathrm{Dom}(T)$, so that $T_F u_k$ in (2.1) is a sequence in $L^2(M)$ (because $\mathrm{Dom}(T) \subset \mathrm{Dom}(T_F)$).

In the proof of Lemma 2.1, we will use the following notations and Lemmas.

2.3. **Potentials** V_m**.** For every $m \in \mathbb{Z}_+$ and $x \in M$, define

$$V_m(x) \;=\; \begin{cases} V(x) & \text{if } \; V(x) \le m \;, \\ m & \text{if } \; V(x) > m. \end{cases}$$

2.4. **Operator** T_m**.** Since $V_m \in L^\infty(M)$, the multiplication operator $(V_m u)(x) = V_m(x)u(x)$ in $L^2(M)$ is defined for all $u \in L^2(M)$. We define an operator T_m in $L^2(M)$ by the formula $T_m = T_0 + V_m$ with the domain $\mathrm{Dom}(T_m) = \mathrm{Dom}(T_0) \cap \mathrm{Dom}(V_m) = \mathrm{Dom}(T_0)$.

Lemma 2.5. *For all* $m \in \mathbb{Z}_+$, *the operator* T_m *is self-adjoint. Moreover,* T_m *is the closure of* $(\Delta_M + V_0 + V_m)|_{C_c^\infty(M)}$, *and*

$$(T_m u, u) \ge (T_0 u, u) \ge 0, \qquad \textit{for all } u \in \mathrm{Dom}(T_m) = \mathrm{Dom}(T_0). \tag{2.2}$$

Proof. Since $V_m \in L^\infty(M)$, the multiplication operator V_m is T_0-bounded with the relative bound 0. Thus, by Theorem V.4.4 in [5], it follows that T_m is self-adjoint.

Since the multiplication operator V_m is H_0-bounded with relative bound 0 and since H_0 (with $\mathrm{Dom}(H_0) = C_c^\infty(M)$) is essentially self-adjoint (because $T_0 = \widetilde{H_0}$ is self-adjoint by hypothesis), by Theorem V.4.4 in [5] it follows that $H_0 + V_m$ is essentially self-adjoint on $C_c^\infty(M)$ and $(H_0 + V_m)^\sim = \widetilde{H_0} + \widetilde{V_m}$, where V_m is the multiplication operator as in Subsection 2.4. Since $\mathrm{Dom}(V_m) = L^2(M)$ and since V_m is a bounded linear operator in $L^2(M)$, it follows that the operator V_m is closed. Thus, we have

$$T_m = T_0 + V_m = \widetilde{H_0} + \widetilde{V_m} = (H_0 + V_m)^\sim = ((\Delta_M + V_0 + V_m)|_{C_c^\infty(M)})^\sim.$$

Since $V_m \ge 0$ and since T_0 is a non-negative operator, the inequality (2.2) immediately follows, and the Lemma is proven. □

Since T_m and T_0 are non-negative self-adjoint operators, it follows that for all k, $m \in \mathbb{Z}_+$, the operators

$$\left(\frac{T_m}{k} + 1\right)^{-1} : L^2(M) \to L^2(M) \qquad \text{and} \qquad \left(\frac{T_0}{k} + 1\right)^{-1} : L^2(M) \to L^2(M) \tag{2.3}$$

are bounded linear operators; see, for example, Section V.3.10 in [5].

2.6. **Positivity Preserving Property.** For the following definition, see, for example, the Definition below the formulation of Theorem X.30 in [7].

Definition 2.7. Let (X, μ) be a measure space. A bounded linear operator $A\colon L^2(X,\mu) \to L^2(X,\mu)$ is said to be *positivity preserving* if for every $u \in L^2(X,\mu)$ such that $u \geq 0$ a.e. on X, we have $Au \geq 0$ a.e. on X.

Remark 2.8. Let $A\colon L^2(X,\mu) \to L^2(X,\mu)$ be a positivity preserving bounded linear operator. Then the following inequality holds for all $u \in L^2(X,\mu)$:

$$|(Au)(x)| \leq A|u(x)|, \quad \text{a.e. on } X, \tag{2.4}$$

where $|\cdot|$ denotes the absolute value of a complex number. For the proof of (2.4), see the proof of the inequality (X.103) in [7].

An important ingredient in the proof of Lemma 2.1 is the following Proposition.

Proposition 2.9. *Assume that (M, g) is a (not necessarily complete) C^∞-Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $Q_0 \in L^2_{\text{loc}}(M)$ is real-valued. Additionally, assume that*

$$((\Delta_M + Q_0)u, u) \geq 0, \qquad \text{for all } u \in C_c^\infty(M).$$

Let S_0 be the Friedrichs extension of $(\Delta_M + Q_0)|_{C_c^\infty(M)}$. Assume that λ is a positive real number. Then the operator $(S_0 + \lambda)^{-1}$ is positivity preserving.

Remark 2.10. For the proof of Proposition 2.9, which is based on Kato's inequality technique on Riemannian manifolds, see the proof of Proposition 2.13 in [6]. Proposition 2.9 is an extension to Riemannian manifolds of Lemma 2 from Goelden [3]. For more on Kato's inequality technique on Riemannian manifolds and its application to essential self-adjointness of Schrödinger-type operators, see [1] and references there.

Corollary 2.11. *The operators in* (2.3) *are positivity preserving.*

Proof. Let k, $m \in \mathbb{Z}_+$. It suffices to show that the operators $(T_0 + k)^{-1}$ and $(T_m + k)^{-1}$ are positivity preserving. By hypothesis and the definition of T_0, the operator T_0 is the self-adjoint closure of H_0. Thus, T_0 is the Friedrichs extension of H_0; hence, by Proposition 2.9, the operator $(T_0 + k)^{-1}$ is positivity preserving. Since $0 \leq V_m \in L^\infty(M)$ and since T_0 is non-negative, it follows that $Q_m := V_0 + V_m$ satisfies the hypotheses of Proposition 2.9. By Lemma 2.5 it follows that T_m is the self-adjoint closure of $(\Delta_M + Q_m)|_{C_c^\infty(M)}$. Thus, T_m is the Friedrichs extension of $(\Delta_M + Q_m)|_{C_c^\infty(M)}$; hence, by Proposition 2.9, the operator $(T_m + k)^{-1}$ is positivity preserving. This concludes the proof. □

2.12. **Sequence v_{km}.** Let $u \in \text{Dom}(T_F)$ and let k, $m \in \mathbb{Z}_+$. Define the following sequence:

$$v_{km} := \left(\frac{T_m}{k} + 1\right)^{-1} u. \tag{2.5}$$

Since $u \in \text{Dom}(T_F) \subset L^2(M)$, by (2.3) it follows that the sequence v_{km} is well-defined. Moreover, we have $v_{km} \in \text{Dom}(T_m) = \text{Dom}(T_0)$.

Lemma 2.13. *Let u and v_{km} be as in* (2.5)*. Then*

$$|v_{km}| \leq \left(\frac{T_0}{k}+1\right)^{-1}|u|. \tag{2.6}$$

Proof. By Corollary 2.11, the operator $((T_m/k)+1)^{-1}$ is positivity preserving. Hence, by (2.4) we get

$$|v_{km}| \leq \left(\frac{T_m}{k}+1\right)^{-1}|u|,$$

where v_{km} and u are as in (2.5).

To prove (2.6), we will show that

$$\left(\frac{T_m}{k}+1\right)^{-1}|u| \leq \left(\frac{T_0}{k}+1\right)^{-1}|u|. \tag{2.7}$$

We have

$$\begin{aligned}\left(\frac{T_0}{k}+1\right)^{-1}|u| - \left(\frac{T_m}{k}+1\right)^{-1}|u| &= \\ = \left(\frac{T_0}{k}+1\right)^{-1}\left(\left(\frac{T_m}{k}+1\right)-\left(\frac{T_0}{k}+1\right)\right)\left(\frac{T_m}{k}+1\right)^{-1}|u| \\ = \left(\frac{T_0}{k}+1\right)^{-1}\left(\frac{V_m}{k}\right)\left(\frac{T_m}{k}+1\right)^{-1}|u|.\end{aligned} \tag{2.8}$$

Since $0 \leq V_m \in L^\infty(M)$, by Corollary 2.11 it follows that the right hand side of the last inequality in (2.8) is non-negative. This shows (2.7), and the Lemma is proven. □

Lemma 2.14. *Under the hypotheses of Theorem 1.4, the following hold:*

(i) $\operatorname{Dom}(T_0) \subset \operatorname{Dom}(T)$.

(ii) *For all $b \in \operatorname{Dom}(T_0)$, we have*

$$T_F b = T_0 b + V b, \tag{2.9}$$

where V is understood as the maximal multiplication operator with

$$\operatorname{Dom}(V) = \{u \in L^2(M) \colon Vu \in L^2(M)\}.$$

Proof. We first prove part (i). Let $b \in \operatorname{Dom}(T_0)$. Since $T_0 = \widetilde{H_0}$ and since $\operatorname{Dom}(H_0) = C_c^\infty(M)$, it follows that there exists a sequence $b_k \in C_c^\infty(M)$ such that $b_k \to b$ in $L^2(M)$ and $H_0 b_k \to T_0 b$ in $L^2(M)$, as $k \to \infty$; see, for example, Section III.5.3 in [5]. Hence, by (1.2) it follows that the sequence $\{Vb_k\}$ is a Cauchy sequence in $L^2(M)$. Therefore, the sequence $\{H_0 b_k + Vb_k\}$ is a Cauchy sequence in $L^2(M)$. By the definition of T in Subsection 1.3, we have $T = \widetilde{H}$. Thus, since $b_k \to b$ in $L^2(M)$ and since $\{Hb_k\} = \{H_0 b_k + V b_k\}$ is a Cauchy sequence in $L^2(M)$, it follows that $b \in \operatorname{Dom}(T)$ (and $Tb = \lim_{k\to\infty} Hb_k$ in $L^2(M)$). This concludes the proof of part (i).

We now prove part (ii). Let $b \in \operatorname{Dom}(T_0)$ and let $\{b_k\}$ be the sequence as in the proof of part (i) of this Lemma. It is well-known the maximal multiplication operator V is self-adjoint (hence, closed); see, for example, Problem V.3.22 in [5]. By the proof of part (i), we know that

$b_k \in C_c^\infty(M) \subset \mathrm{Dom}(V)$ and $b_k \to b$ in $L^2(M)$ and $\{Vb_k\}$ is a Cauchy sequence in $L^2(M)$. Since V is closed, it follows that $Vb_k \to Vb$ in $L^2(M)$, as $k \to \infty$. Since $b \in \mathrm{Dom}(T_0) \subset \mathrm{Dom}(T)$ and since T_F is an extension of T, we have $T_F b = Tb$. By the proof of part (i) we have

$$T_F b = Tb = \lim_{k\to\infty} (Hb_k) = \lim_{k\to\infty} (H_0 b_k) + \lim_{k\to\infty} Vb_k = T_0 b + Vb, \tag{2.10}$$

where the limits in (2.10) denote the convergence in $L^2(M)$.

This shows (2.9), and the Lemma is proven. □

Lemma 2.15. *Let u and v_{km} be as in (2.5). Let $k \in \mathbb{Z}_+$ be fixed. Then, the following holds:*

$$v_{km} \to \left(\frac{T_F}{k} + 1\right)^{-1} u \quad \text{in } L^2(M), \quad \text{as } m \to \infty. \tag{2.11}$$

Proof. Let $k \in \mathbb{Z}_+$ be fixed. Since $v_{km} \in \mathrm{Dom}(T_m) = \mathrm{Dom}(T_0)$, by Lemma 2.14 we have

$$T_F v_{km} = T_0 v_{km} + V v_{km}.$$

Therefore,

$$\begin{aligned} v_{km} - \left(\frac{T_F}{k} + 1\right)^{-1} u \ &= \ \left(\frac{T_F}{k} + 1\right)^{-1} \left(\left(\frac{T_F}{k} + 1\right) - \left(\frac{T_m}{k} + 1\right)\right) v_{km} \ = \\ &= \ \left(\frac{T_F}{k} + 1\right)^{-1} \left(\frac{V - V_m}{k}\right) v_{km}. \end{aligned} \tag{2.12}$$

Since T_F is non-negative and self-adjoint, it follows that $((T_F/k) + 1)^{-1} \colon L^2(M) \to L^2(M)$ is a bounded linear operator. Thus, to finish the proof of the Lemma, it is enough to show that

$$(V - V_m) v_{km} \to 0 \quad \text{in } L^2(M), \quad \text{as } m \to \infty. \tag{2.13}$$

By (2.6) and by the definition of V_m it follows that

$$(V - V_m) v_{km} \to 0 \quad \text{a.e. on } M, \quad \text{as } m \to \infty. \tag{2.14}$$

Moreover, by Lemma 2.13 we have

$$|(V - V_m) v_{km}| = (V - V_m)|v_{km}| \le (V - V_m)\left(\frac{T_0}{k} + 1\right)^{-1} |u| \le V \left(\frac{T_0}{k} + 1\right)^{-1} |u|. \tag{2.15}$$

The last inequality in (2.15) follows by the definition of V_m.

Since, by hypothesis, $H_0 = (\Delta_M + V_0)|_{C_c^\infty(M)}$ is an essentially self-adjoint operator with closure $T_0 = \widetilde{H_0}$ and since $(((T_0/k) + 1)^{-1}|u|) \in \mathrm{Dom}(T_0)$, by using (1.2) and by repeating the same arguments as in the beginning of the proof of part (ii) of Lemma 2.14, it follows that $(V((T_0/k) + 1)^{-1}|u|) \in L^2(M)$.

Using (2.14), (2.15) and Dominated Convergence Theorem, we obtain (2.13). This concludes the proof of the Lemma.

□

Lemma 2.16. *Let u and v_{km} be as in* (2.5). *Let $k \in \mathbb{Z}_+$ be fixed. Then, the following holds:*

$$T_F v_{km} \to T_F \left(\frac{T_F}{k} + 1\right)^{-1} u \quad \text{in } L^2(M), \quad \text{as } m \to \infty. \tag{2.16}$$

Proof. Since $v_{km} \in \mathrm{Dom}(T_m) = \mathrm{Dom}(T_0)$, by Lemma 2.14 we have

$$T_F v_{km} = T_0 v_{km} + V v_{km} = T_0 v_{km} + V_m v_{km} + (V - V_m) v_{km} = T_m v_{km} + (V - V_m) v_{km}.$$

By the proof of (2.13), we have $(V - V_m) v_{km} \to 0$ in $L^2(M)$ as $m \to \infty$.

Writing

$$T_m v_{km} = k \left(\frac{T_m}{k} + 1\right) v_{km} - k v_{km} = ku - k v_{km},$$

and using (2.11), we have, as $m \to \infty$:

$$T_m v_{km} \to ku - k \left(\frac{T_F}{k} + 1\right)^{-1} u \quad \text{in } L^2(M).$$

Since

$$ku - k \left(\frac{T_F}{k} + 1\right)^{-1} u = T_F \left(\frac{T_F}{k} + 1\right)^{-1} u,$$

the Lemma is proven. □

Lemma 2.17. *Let u be as in* (2.5). *Then, as $k \to \infty$, we have*

$$\left(\frac{T_F}{k} + 1\right)^{-1} u \to u \quad \text{and} \quad T_F \left(\frac{T_F}{k} + 1\right)^{-1} u \to T_F u \quad \text{in } L^2(M). \tag{2.17}$$

Proof. Let $k \in \mathbb{Z}_+$. By hypothesis, the (non-negative self-adjoint) operator T_F/k is the Friedrichs extension of T/k. Let $t_{F,k}$ be the (densely defined, closed and non-negative) quadratic form associated to T_F/k by Theorem VI.2.7 in [5]. Since $t_{F,k}$ is non-negative, we have the following inequality of quadratic forms:

$$t_{F,k} \geq 0, \tag{2.18}$$

where 0 on the right hand side denotes the zero-form $0(\cdot,\cdot)$ with the domain $L^2(M)$.

Since $C_c^\infty(M) \subset \mathrm{Dom}(T/k) \subset \mathrm{Dom}(T_F/k)$, we have

$$\left(\frac{T_F}{k} s, s\right) = \frac{1}{k}(T_F s, s) \to 0 = (0s, s), \qquad \text{for all } s \in C_c^\infty(M), \quad \text{as } k \to \infty, \tag{2.19}$$

where 0 in $(0s, s)$ is the zero operator $0\colon L^2(M) \to L^2(M)$.

Since by Theorem VI.2.1 in [5], the domain $\mathrm{Dom}(T_F/k) \subset \mathrm{Dom}(t_{F,k})$, it follows that $C_c^\infty(M) \subset \mathrm{Dom}(t_{F,k})$. We also know that $C_c^\infty(M)$ is dense in $\mathrm{Dom}(0(\cdot,\cdot)) = L^2(M)$. This, together with (2.18) and (2.19), shows that the hypotheses of abstract Theorem 7.9 from [2] are satisfied.

By Theorem 7.9 from [2] it follows that $T_F/k \to 0$ in the strong resolvent sense, as $k \to \infty$. Since -1 belongs to the resolvent sets of operators T_F/k and 0, we have

$$\left(\frac{T_F}{k} + 1\right)^{-1} \to 1, \quad \text{as } k \to \infty \tag{2.20}$$

strongly (as a sequence of bounded linear operators $L^2(M) \to L^2(M)$). Here, $1\colon L^2(M) \to L^2(M)$ denotes the identity operator defined on the whole $L^2(M)$.

Thus, by the definition of strong convergence (for sequences of bounded linear operators), the leftmost convergence relation in (2.17) follows from (2.20).

We know that T_F is a non-negative self-adjoint operator. Since -1 belongs to the resolvent set of T_F/k and since $u \in \mathrm{Dom}(T_F)$, by Problem III.6.2 in [5] it follows that

$$T_F\left(\frac{T_F}{k}+1\right)^{-1}u = \left(\frac{T_F}{k}+1\right)^{-1}T_F u. \tag{2.21}$$

Since $T_F u \in L^2(M)$, the rightmost convergence relation in (2.17) follows from (2.21) and (2.20). This concludes the proof of the Lemma. □

Proof of Lemma 2.1. By (2.11) and (2.16), for every $k \in \mathbb{Z}_+$, we can choose $m(k)$ such that

$$\left\| v_{k,m(k)} - \left(\frac{T_F}{k}+1\right)^{-1}u \right\| \leq \frac{1}{k}. \tag{2.22}$$

and

$$\left\| T_F v_{k,m(k)} - T_F\left(\frac{T_F}{k}+1\right)^{-1}u \right\| \leq \frac{1}{k}. \tag{2.23}$$

Let $u_k := v_{k,m(k)}$. Then the sequence $\{u_k\}$ is in $\mathrm{Dom}(T_0)$, and (2.22) and (2.23) hold with $v_{k,m(k)}$ and replaced by u_k.

Using (2.22) and (2.23) (with $v_{k,m(k)}$ and replaced by u_k), and using (2.17), it follows that $\{u_k\}$ satisfies the properties (2.1). This concludes the proof of the Lemma. □

Proof of Theorem 1.4. Let $u \in \mathrm{Dom}(T_F)$. By Lemma 2.1 there exists a sequence $\{u_k\}$ in $\mathrm{Dom}(T_0)$ satisfying (2.1). By Lemma 2.14 it follows that $\mathrm{Dom}(T_0) \subset \mathrm{Dom}(T)$. Since $T \subset T_F$, from (2.1) it follows that $\{u_k\}$ is a sequence in $\mathrm{Dom}(T_0) \subset \mathrm{Dom}(T)$ such that $u_k \to u$ in $L^2(M)$ and $\{Tu_k\}$ converges in $L^2(M)$. Since, by hypothesis, T is a closed operator, it follows that $u \in \mathrm{Dom}(T)$. This shows that $\mathrm{Dom}(T_F) \subset \mathrm{Dom}(T)$, and the Theorem is proven. □

3. Proof of Corollary 1.5

Throughout this section, we assume that all hypotheses of Corollary 1.5 are satisfied.

We begin with the following Lemma.

Lemma 3.1. *Assume that $0 \leq u \in \mathrm{Dom}(T)$. Then, there exists a sequence $\{u_k\} \in \mathrm{Dom}(T_0)$ such that $u_k \geq 0$ and*

$$u_k \to u, \quad \text{and} \quad Tu_k \to Tu \quad \text{in } L^2(M), \quad \text{as } k \to \infty. \tag{3.1}$$

Proof. By Theorem 1.4 the operator T is self-adjoint (and $T = T_F$). We define $v_{km} \in \mathrm{Dom}(T_0)$ as in (2.5). Since $u \geq 0$, by Corollary 2.11 it follows that $v_{km} \geq 0$. Let $u_k := v_{k,m(k)}$ be as in the proof of Lemma 2.1. By the proof of Lemma 2.1, it follows that the sequence $\{u_k\}$ satisfies the properties (3.1). □

Proof of Corollary 1.5. Let $0 \le u \in \mathrm{Dom}(T)$. By Lemma 3.1, for any $\epsilon > 0$, there exists a function $0 \le w \in \mathrm{Dom}(T_0)$ such that

$$\|w - u\| < \epsilon \qquad \text{and} \qquad \|Tw - Tu\| < \epsilon. \tag{3.2}$$

By hypotheses of the Corollary, the operator T_0 satisfies the property (P). Thus, for any $\epsilon > 0$, there exists a function $s \in (C_c^\infty(M))^+$ such that

$$\|s - w\| < \epsilon \qquad \text{and} \qquad \|T_0 s - T_0 w\| < \epsilon. \tag{3.3}$$

By hypotheses of the Corollary, the function V satisfies (1.2). By the proof of the inequality (4.6) from Section V.4.1 from [5], it follows that (1.2) holds for all $u \in \mathrm{Dom}(T_0)$ (where V is understood as in part (ii) of Lemma 2.14). Therefore,

$$\|V(s - w)\| \le a\|T_0(s - w)\| + b\|s - w\|, \tag{3.4}$$

where a and b are as in (1.2).

Using, (3.2), (3.3) and (3.4), we obtain

$$\|s - u\| < 2\epsilon \tag{3.5}$$

and

$$\begin{aligned}\|Ts - Tu\| \;&=\; \|(Ts - Tw) + (Tw - Tu)\| \;=\; \|((T_0 s + Vs) - (T_0 w + Vw)) + (Tw - Tu)\| \le \\ &\le\; \|T_0 s - T_0 w\| + \|Vs - Vw\| + \|Tw - Tu\| < (2 + a + b)\epsilon.\end{aligned} \tag{3.6}$$

On the right hand side of the second equality in (3.6) we used (2.9) with $T_F = T$.

By (3.5) and (3.6), it follows that the operator T satisfies the property (P). This concludes the proof of the Corollary. □

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