

Separation property for Schrödinger operators on Riemannian manifolds

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Abstract

We consider a Schrödinger differential expression $L = \Delta_M + q$ on a complete Riemannian manifold (M, g) with metric g , where Δ_M is the scalar Laplacian on M and $q \geq 0$ is a locally square integrable function on M . In the terminology of W. N. Everitt and M. Giertz, the differential expression L is said to be separated in $L^2(M)$ if for all $u \in L^2(M)$ such that $Lu \in L^2(M)$, we have $qu \in L^2(M)$. We give sufficient conditions for L to be separated in $L^2(M)$.

Key words: Riemannian manifold, Schrödinger operator, separation
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1 Introduction and the main result

1.1 The setting

Let (M, g) be a Riemannian manifold without boundary (i.e. M is a C^∞ -manifold without boundary, (g_{jk}) is a Riemannian metric on M) and $\dim M = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates x^1, x^2, \dots, x^n there exists a strictly positive C^∞ -density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$.

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In the sequel, $L^2(M)$ is the space of complex-valued square integrable functions on M with the inner product

$$(u, v) = \int_M (u\bar{v}) d\mu, \quad (1)$$

and $\|\cdot\|$ is the norm in $L^2(M)$ corresponding to the inner product (1).

We use the notation $L^2(\Lambda^1 T^* M)$ for the space of complex-valued square integrable 1-forms on M with the inner product

$$(\omega, \psi)_{L^2(\Lambda^1 T^* M)} = \int_M \langle \omega, \bar{\psi} \rangle d\mu, \quad (2)$$

where for 1-forms $\omega = \omega_j dx^j$ and $\psi = \psi_k dx^k$, we define

$$\langle \omega, \psi \rangle := g^{jk} \omega_j \psi_k,$$

where (g^{jk}) is the inverse matrix to (g_{jk}) , and

$$\bar{\psi} = \bar{\psi}_k dx^k.$$

(Above we used the standard Einstein summation convention.)

The notation $\|\cdot\|_{L^2(\Lambda^1 T^* M)}$ stands for the norm in $L^2(\Lambda^1 T^* M)$ corresponding to the inner product (2).

In what follows, by $C^\infty(M)$ we denote the space of smooth functions on M , by $C_c^\infty(M)$ —the space of smooth compactly supported functions on M , by $\Omega^1(M)$ —the space of smooth 1-forms on M and by $\Omega_c^1(M)$ —the space of smooth compactly supported 1-forms on M .

In the sequel, the operator $d: C^\infty(M) \rightarrow \Omega^1(M)$ is the standard differential, and

$$d^*: \Omega^1(M) \rightarrow C^\infty(M)$$

is the formal adjoint of d defined by the identity

$$(du, \omega)_{L^2(\Lambda^1 T^* M)} = (u, d^* \omega), \quad u \in C_c^\infty(M), \omega \in \Omega^1(M).$$

By $\Delta_M := d^* d$ we will denote the scalar Laplacian on M .

We consider a Schrödinger-type differential expression

$$L = \Delta_M + q, \quad (3)$$

where $q \in L^2_{\text{loc}}(M)$ is a real-valued function.

1.2 The set D_1

Let L be as in (3). In the sequel, we will use the notation

$$D_1 := \{u \in L^2(M) : Lu \in L^2(M)\}, \quad (4)$$

where Lu is understood in the sense of distributions.

Remark 1 In general, it is not true that for all $u \in D_1$ we have $\Delta_M u \in L^2(M)$ and $qu \in L^2(M)$ separately.

Using the terminology of W. N. Everitt and M. Giertz [4], we will say that the differential expression $L = \Delta_M + q$ is *separated* in $L^2(M)$ when the following statement holds true:

$$\text{for all } u \in D_1, \text{ we have } qu \in L^2(M).$$

We will give sufficient conditions for L to be separated in $L^2(M)$.

First, we make assumptions on q .

Assumption (A1) Assume that there exists a function $0 \leq V \in C^1(M)$ such that

$$V(x) \leq q(x) \leq cV(x) \quad (5)$$

and

$$|dV(x)| \leq \sigma V^{3/2}(x), \quad \text{for all } x \in M, \quad (6)$$

where $c > 0$ and $0 \leq \sigma < 2$ are constants.

In (6), the notation $|dV(x)|$ denotes the norm of $dV(x) \in T_x^*M$ with respect to the inner product in T_x^*M induced by the metric g .

Remark 2 From (5) it follows that $0 \leq q \in L_{\text{loc}}^\infty(M)$.

We now state the main result.

Theorem 3 *Assume that (M, g) is a connected C^∞ -Riemannian manifold without boundary, with metric g and a positive smooth measure $d\mu$. Additionally, assume that (M, g) is complete. Assume that q satisfies the Assumption*

(A1). Then

$$\|\Delta_M u\| + \|qu\| \leq C(\|Lu\| + \|u\|), \quad \text{for all } u \in D_1, \quad (7)$$

where $C \geq 0$ is a constant (independent of u).

The following Corollary is an immediate consequence of Theorem 3.

Corollary 4 *Under the hypotheses of Theorem 3, the differential expression L is separated in $L^2(M)$.*

Remark 5 Theorem 3 extends a result of K. Kh. Boimatov [1, Theorem 4] concerning the separation property for the Schrödinger operator $-\Delta + q$ in $L^2(\mathbb{R}^n)$, where Δ is the standard Laplacian on \mathbb{R}^n with standard metric and measure and $0 \leq q \in C^1(\mathbb{R}^n)$. The problem of separation of differential expressions in $L^2(\mathbb{R}^n)$ has been studied by many authors; see, for instance, [1] and [4] and references there.

2 Proof of Theorem 3

2.1 Differential expression L_V

Let $0 \leq V \in C^1(M)$. In the sequel, by L_V we will denote the differential expression $L_V = \Delta_M + V$.

In the two preliminary Lemmas, we will adopt the scheme of Boimatov [1] and Everitt–Giertz [4] to our context. In the proof of Theorem 3, we use the positivity preserving property of resolvents of self-adjoint closures of $L_V|_{C_c^\infty(M)}$ and $L|_{C_c^\infty(M)}$.

Lemma 6 *Assume that (M, g) is a connected C^∞ -Riemannian manifold without boundary, with metric g and a positive smooth measure $d\mu$. Assume that $0 \leq V \in C^1(M)$ satisfies (6) with $\sigma \in [0, 2)$. Then the following inequalities hold:*

$$\|\Delta_M u\| + \|Vu\| \leq \tilde{C}\|L_V u\|, \quad \text{for all } u \in C_c^\infty(M), \quad (8)$$

and

$$\|V^{1/2} du\|_{L^2(\Lambda^1 T^*M)} \leq \tilde{C}\|L_V u\|, \quad \text{for all } u \in C_c^\infty(M), \quad (9)$$

where L_V is as in Subsection 2.1 and \tilde{C} is a constant depending on n and σ .

Proof

We will first prove that the following equality holds for any $\nu > 0$:

$$\begin{aligned} \|L_V u\|^2 &= \|Vu\|^2 + \nu \|\Delta_M u\|^2 + (1 + \nu) \operatorname{Re}(Vu, \Delta_M u) + \\ &+ (1 - \nu) (\Delta_M u, L_V u), \quad \text{for all } u \in C_c^\infty(M). \end{aligned} \quad (10)$$

Let $\nu > 0$ be arbitrary. By the definition of L_V , for all $u \in C_c^\infty(M)$ we have

$$\begin{aligned} \|L_V u\|^2 &= \|Vu\|^2 + \|\Delta_M u\|^2 + 2 \operatorname{Re}(\Delta_M u, Vu) = \\ &= \|Vu\|^2 + \nu \|\Delta_M u\|^2 + (1 - \nu) \|\Delta_M u\|^2 + 2 \operatorname{Re}(\Delta_M u, Vu) = \\ &= \|Vu\|^2 + \nu \|\Delta_M u\|^2 + (1 - \nu) \operatorname{Re}(\Delta_M u, L_V u - Vu) + 2 \operatorname{Re}(\Delta_M u, Vu) = \\ &= \|Vu\|^2 + \nu \|\Delta_M u\|^2 + (1 - \nu) \operatorname{Re}(\Delta_M u, L_V u) + (1 + \nu) \operatorname{Re}(\Delta_M u, Vu), \end{aligned}$$

where (\cdot, \cdot) is as in (1) and $\|\cdot\|$ is the corresponding norm in $L^2(M)$.

Since $u \in C_c^\infty(M)$, using integration by parts and the product rule, we have

$$\begin{aligned} \operatorname{Re}(\Delta_M u, Vu) &= \operatorname{Re}(d^* du, Vu) = \operatorname{Re}(du, d(Vu))_{L^2(\Lambda^1 T^* M)} = \\ &= \operatorname{Re}(du, (dV)u + V du)_{L^2(\Lambda^1 T^* M)} = \operatorname{Re}(du, (dV)u)_{L^2(\Lambda^1 T^* M)} + \\ &+ (du, V du)_{L^2(\Lambda^1 T^* M)} = (\operatorname{Re} Z) + W, \end{aligned} \quad (11)$$

where

$$Z = \int_M \langle du, \bar{u}(dV) \rangle d\mu \quad (12)$$

and

$$W = (V^{1/2} du, V^{1/2} du)_{L^2(\Lambda^1 T^* M)}. \quad (13)$$

From (11) we get

$$\begin{aligned} (1 + \nu) \operatorname{Re}(\Delta_M u, Vu) &= (1 + \nu) \operatorname{Re} Z + (1 + \nu) W \geq \\ &\geq -(1 + \nu) |Z| + (1 + \nu) W. \end{aligned} \quad (14)$$

We will now estimate $|Z|$, where Z is as in (12). Using the Cauchy-Schwarz inequality and the inequality

$$2ab \leq ka^2 + k^{-1}b^2, \quad (15)$$

where a , b and k are positive real numbers, we get for any $\delta > 0$:

$$\begin{aligned} |Z| &\leq \int_M |du| |dV| |u| d\mu \leq \sigma \int_M V^{3/2} |u| |du| d\mu = \\ &= \sigma \int_M |V^{1/2} du| |Vu| d\mu \leq \frac{\nu\delta}{2} \|V^{1/2} du\|_{L^2(\Lambda^1 T^* M)}^2 + \frac{\sigma^2}{2\nu\delta} \|Vu\|^2. \end{aligned} \quad (16)$$

Here, for $\xi \in T_x^* M$, the notation $|\xi|$ denotes the norm of ξ with respect to the inner product in $T_x^* M$ induced by the metric g . In the second inequality in (16), we used (6), and in the third inequality in (16) we used (15).

Using (15), for any $\alpha > 0$ we get

$$|\operatorname{Re}(\Delta_M u, L_V u)| \leq |(\Delta_M u, L_V u)| \leq \frac{\alpha}{2} \|\Delta_M u\|^2 + \frac{1}{2\alpha} \|L_V u\|^2. \quad (17)$$

Combining (10), (14), (16) and (17) we obtain

$$\begin{aligned} \|L_V u\|^2 &\geq \|Vu\|^2 + \nu \|\Delta_M u\|^2 - \frac{(1+\nu)\nu\delta}{2} \|V^{1/2} du\|_{L^2(\Lambda^1 T^* M)}^2 - \\ &\quad - \frac{(1+\nu)\sigma^2}{2\nu\delta} \|Vu\|^2 + (1+\nu) \|V^{1/2} du\|_{L^2(\Lambda^1 T^* M)}^2 - \\ &\quad - \frac{|1-\nu|\alpha}{2} \|\Delta_M u\|^2 - \frac{|1-\nu|}{2\alpha} \|L_V u\|^2. \end{aligned} \quad (18)$$

From (18) we get

$$\begin{aligned} \left(1 + \frac{|1-\nu|}{2\alpha}\right) \|L_V u\|^2 &\geq \left(1 - \frac{(1+\nu)\sigma^2}{2\nu\delta}\right) \|Vu\|^2 + \\ &\quad + \left(\nu - \frac{|1-\nu|\alpha}{2}\right) \|\Delta_M u\|^2 + \\ &\quad + \left((1+\nu) - \frac{(1+\nu)\nu\delta}{2}\right) \|V^{1/2} du\|_{L^2(\Lambda^1 T^* M)}^2. \end{aligned} \quad (19)$$

The inequalities (8) and (9) will immediately follow from (19) if

$$|1-\nu| < \frac{2\nu}{\alpha}, \quad \nu\delta < 2, \quad \text{and} \quad (1+\nu)\sigma^2 < 2\nu\delta. \quad (20)$$

Since, by hypothesis, $0 \leq \sigma < 2$, there exist numbers $\nu > 0$, $\alpha > 0$ and $\delta > 0$ such that the inequalities (20) hold.

This concludes the proof of the Lemma. \square

In the sequel, we will also use the following terms and notations.

2.2 Minimal operators S and T

Let L be as in (3) and let L_V be as in Subsection 2.1. We define the minimal operators S and T in $L^2(M)$ associated to L and L_V by the formulas $Su = Lu$ and $Tu = L_V u$ with the domains $\text{Dom}(S) = \text{Dom}(T) = C_c^\infty(M)$. Since S and T are symmetric operators, it follows that S and T is closable; see, for example, Section V.3.3 in [6]. In what follows, we will denote by \tilde{S} and \tilde{T} the closures in $L^2(M)$ of the operators S and T respectively.

2.3 Maximal operators H and K

Let L be as in (3). We define the maximal operator H in $L^2(M)$ associated to L by the formula $Hu = S^*u$, where S^* is the adjoint of the operator S in $L^2(M)$. In the case when $q \in L_{\text{loc}}^2(M)$ is real-valued, it is well-known that $\text{Dom}(H) = D_1$, where D_1 is as in (4). Let L_V be as in Subsection 2.1. We define the maximal operator K in $L^2(M)$ associated to L_V by the formula $Ku = T^*u$, and we have $\text{Dom}(K) = \{u \in L^2(M) : L_V u \in L^2(M)\}$.

Remark 7 By Lemma 5.1 from [9] it follows that $\text{Dom}(K) \subset W_{\text{loc}}^{2,2}(M)$.

2.4 Essential self-adjointness of S and T

If (M, g) is a complete Riemannian manifold with metric g and positive smooth measure $d\mu$ and if $0 \leq q \in L_{\text{loc}}^2(M)$, then by Theorem 1.1 in [9] (or by Corollary 2.9 in [2]) the operator S is essentially self-adjoint in $L^2(M)$. In this case, we have $\tilde{S} = S^*$; see, for example, Section V.3.3 in [6]. In particular, since $0 \leq V \in C^1(M) \subset L_{\text{loc}}^2(M)$, it follows that the operator T is essentially self-adjoint in $L^2(M)$.

Lemma 8 *Assume that (M, g) is a connected C^∞ -Riemannian manifold without boundary, with metric g and a positive smooth measure $d\mu$. Additionally, assume that (M, g) is complete. Assume that $0 \leq V \in C^1(M)$ satisfies (6). Then the inequalities (8) and (9) hold for all $u \in \text{Dom}(K)$, where K is as in Subsection 2.3.*

Proof Under the hypotheses of this Lemma, by Subsection 2.4 it follows that T is essentially self-adjoint and $K = T^* = \tilde{T}$. In particular, $\text{Dom}(\tilde{T}) = \text{Dom}(K)$.

Let $u \in \text{Dom}(K)$. Then there exists a sequence $\{u_k\}$ in $C_c^\infty(M)$ such that

$$u_k \rightarrow u \quad \text{and} \quad L_V u_k \rightarrow \tilde{T}u \quad \text{in } L^2(M), \quad \text{as } k \rightarrow \infty.$$

Since by Lemma 6 the sequence $\{u_k\}$ satisfies (8) and (9), it follows that the sequences $\{Vu_k\}$ and $\{\Delta_M u_k\}$ are Cauchy sequences in $L^2(M)$, and $\{V^{1/2} du_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^*M)$.

We will first show that

$$Vu_k \rightarrow Vu \quad \text{in } L^2(M), \quad \text{as } k \rightarrow \infty. \quad (21)$$

Since $\{Vu_k\}$ is a Cauchy sequence in $L^2(M)$, it follows that Vu_k converges to $s \in L^2(M)$. Let ϕ be an arbitrary element of $C_c^\infty(M)$. Then

$$0 = (Vu_k, \phi) - (u_k, V\phi) \rightarrow (s, \phi) - (u, V\phi) = (s - Vu, \phi), \quad (22)$$

where (\cdot, \cdot) is as in (1).

Since $C_c^\infty(M)$ is dense in $L^2(M)$, we get $s = Vu$, and (21) is proven.

If (M, g) is complete, it is well-known that Δ_M is essentially self-adjoint on $C_c^\infty(M)$, and we have the following equality:

$$(\Delta_M|_{C_c^\infty(M)})^\sim = \Delta_{M, \max},$$

where $\Delta_{M, \max} u := \Delta_M u$ with the domain

$$\text{Dom}(\Delta_{M, \max}) = \{u \in L^2(M) : \Delta_M u \in L^2(M)\};$$

see, for example, Theorem 3.5 in [3].

Since $u_k \rightarrow u$ in $L^2(M)$ and since $\{\Delta_M u_k\}$ is a Cauchy sequence in $L^2(M)$, by the definition of $(\Delta_M|_{C_c^\infty(M)})^\sim$ it follows that $u \in \text{Dom}((\Delta_M|_{C_c^\infty(M)})^\sim)$. Since $(\Delta_M|_{C_c^\infty(M)})^\sim = \Delta_{M, \max} u$, we have

$$\Delta_M u_k \rightarrow \Delta_M u \quad \text{in } L^2(M), \quad \text{as } k \rightarrow \infty. \quad (23)$$

Since $\{\Delta_M u_k\}$ and $\{u_k\}$ are Cauchy sequences in $L^2(M)$ and since

$$\|du_k\|_{L^2(\Lambda^1 T^* M)}^2 = (du_k, du_k)_{L^2(\Lambda^1 T^* M)} = (\Delta_M u_k, u_k) \leq \|\Delta_M u_k\| \|u_k\|,$$

it follows that $\{du_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^* M)$, and, hence, du_k converges to some element $\omega \in L^2(\Lambda^1 T^* M)$. Let $\psi \in \Omega_c^1(M)$ be arbitrary. Then, using integration by parts (see, for example, Lemma 8.8 in [2]) and Remark 7, we get

$$\begin{aligned} 0 &= (du_k, \psi)_{L^2(\Lambda^1 T^* M)} - (u_k, d^* \psi) \rightarrow (\omega, \psi)_{L^2(\Lambda^1 T^* M)} - (u, d^* \psi) = \\ &= (\omega, \psi)_{L^2(\Lambda^1 T^* M)} - (du, \psi)_{L^2(\Lambda^1 T^* M)}, \end{aligned} \quad (24)$$

where (\cdot, \cdot) is the inner product in $L^2(M)$.

From (24) we get $du = \omega \in L^2(\Lambda^1 T^* M)$, and, hence,

$$du_k \rightarrow du, \quad \text{in } L^2(\Lambda^1 T^* M), \quad \text{as } k \rightarrow \infty. \quad (25)$$

Since $\{V^{1/2} du_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^* M)$, using (25) and the same argument as in (22), we obtain

$$V^{1/2} du_k \rightarrow V^{1/2} du, \quad \text{in } L^2(\Lambda^1 T^* M), \quad \text{as } k \rightarrow \infty. \quad (26)$$

Using (21), (23), (26) and taking limits as $k \rightarrow \infty$ in all terms in (8) and (9) (with u replaced by u_k), shows that (8) and (9) hold for all $u \in \text{Dom}(\tilde{T}) = \text{Dom}(K)$. This concludes the proof of the Lemma. \square

2.5 Operators R_1 and R_2

Let S and T be as in Subsection 2.2 and let H and K be as in Subsection 2.3. By Subsection 2.4 the operators $H = \tilde{S}$ and $K = \tilde{T}$ are non-negative self-adjoint in $L^2(M)$. Thus,

$$\begin{aligned} R_1 &:= (H + 1)^{-1}: L^2(M) \rightarrow L^2(M) \\ R_2 &:= (K + 1)^{-1}: L^2(M) \rightarrow L^2(M) \end{aligned} \quad (27)$$

are bounded linear operators.

2.6 Positivity Preserving Property

For the following definition, see, for example, the Definition below the formulation of Theorem X.30 in [8].

Let (X, μ) be a measure space. A bounded linear operator $A: L^2(X, \mu) \rightarrow L^2(X, \mu)$ is said to be *positivity preserving* if for every $u \in L^2(X, \mu)$ such that $u \geq 0$ a.e. on X , we have $Au \geq 0$ a.e. on X .

Remark 9 Let $A: L^2(X, \mu) \rightarrow L^2(X, \mu)$ be a positivity preserving bounded linear operator. Then the following inequality holds for all $u \in L^2(X, \mu)$:

$$|(Au)(x)| \leq A|u(x)|, \quad \text{a.e. on } X, \quad (28)$$

where $|\cdot|$ denotes the absolute value of a complex number. For the proof of (28), see the proof of the inequality (X.103) in [8].

In the sequel, we will use the following proposition.

Proposition 10 *Assume that (M, g) is a (not necessarily complete) C^∞ -Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $Q_0 \in L^2_{\text{loc}}(M)$ is real-valued. Additionally, assume that*

$$((\Delta_M + Q_0)u, u) \geq 0, \quad \text{for all } u \in C_c^\infty(M).$$

Let S_0 be the Friedrichs extension of $(\Delta_M + Q_0)|_{C_c^\infty(M)}$. Assume that λ is a positive real number. Then the operator $(S_0 + \lambda)^{-1}$ is positivity preserving.

Remark 11 For the proof of Proposition 10, which is based on Kato's inequality technique on Riemannian manifolds, see the proof of Proposition 2.13 in [7]. Proposition 10 is an extension to Riemannian manifolds of Lemma 2 from Goelden [5]. For more on Kato's inequality technique on Riemannian manifolds and its application to essential self-adjointness of Schrödinger-type operators, see [2] and references there.

From Proposition 10 we obtain the following Corollary.

Corollary 12 *Assume that (M, g) is a complete C^∞ -Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $0 \leq q \in L^2_{\text{loc}}(M)$ and $0 \leq V \in C^1(M)$. Let R_1 and R_2 be as in (27). Then the operators R_1 and R_2 are positivity preserving.*

Proof Since (M, g) is complete, by Subsection 2.4 it follows that S and T are non-negative essentially self-adjoint operators in $L^2(M)$. Thus, the Friedrichs

extensions of $(\Delta_M + q)|_{C_c^\infty(M)}$ and $(\Delta_M + V)|_{C_c^\infty(M)}$ are H and K respectively. By Proposition 10 it follows that R_1 and R_2 are positivity preserving operators in $L^2(M)$. \square

Proof of Theorem 3 By Lemma 8 it follows that

$$\|Vu\| \leq \tilde{C}\|Ku\|, \quad \text{for all } u \in \text{Dom}(K). \quad (29)$$

Let R_2 be as in (27) and let $V: L^2(M) \rightarrow L^2(M)$ denote the maximal multiplication operator corresponding to the function V .

From (29) it follows that $VR_2: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator.

Let $q: L^2(M) \rightarrow L^2(M)$ denote the maximal multiplication operator corresponding to the function q . Since $VR_2: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator, by (5) it follows that $qR_2: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator.

We will now show that qR_1 is a bounded linear operator: $L^2(M) \rightarrow L^2(M)$.

First, we will show that

$$qR_1f \leq qR_2f, \quad \text{for all } 0 \leq f \in L^2(M). \quad (30)$$

Here, the inequality is understood in a pointwise sense. (Since $(R_1f) \in D_1 \subset L^2(M)$ and since $q \in L^2_{\text{loc}}(M)$, it follows that $(qR_1f) \in L^1_{\text{loc}}(M)$ is a function.)

Let $0 \leq f \in L^2(M)$ be arbitrary. Since

$$R_2f \in \text{Dom}(K) = \{z \in L^2(M): L_V z \in L^2(M)\},$$

by Lemma 8 we have $(\Delta_M(R_2f)) \in L^2(M)$ and $(VR_2f) \in L^2(M)$. Since $qR_2: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator, we get $(qR_2f) \in L^2(M)$. Thus $R_2f \in D_1$, and, hence,

$$qR_2f = qR_1(H + 1)R_2f. \quad (31)$$

By the definition of R_2 have

$$qR_1f = qR_1(K + 1)R_2f. \quad (32)$$

From (31) and (32) we have

$$\begin{aligned}
qR_2f - qR_1f &= qR_1((H + 1)R_2f - (K + 1)R_2f) = \\
&= qR_1((q - V)R_2f).
\end{aligned} \tag{33}$$

Since R_2 is positivity preserving in $L^2(M)$, we have $R_2f \geq 0$. Since VR_2 and qR_2 are bounded linear operators $L^2(M) \rightarrow L^2(M)$ and since $q - V \geq 0$, we have

$$0 \leq (q - V)R_2f \in L^2(M). \tag{34}$$

Since R_1 is positivity preserving in $L^2(M)$, from (34) we get

$$R_1((q - V)R_2f) \geq 0. \tag{35}$$

Since $q \geq 0$, from (33) and (35) we get

$$qR_2f - qR_1f = qR_1((q - V)R_2f) \geq 0,$$

and (30) is proven.

Let $w \in L^2(M)$ be arbitrary. Since $R_1: L^2(M) \rightarrow L^2(M)$ is a positivity preserving bounded linear operator, by (28) we have the following pointwise inequality:

$$|R_1w| \leq R_1|w|. \tag{36}$$

Since $q \geq 0$, from (36) and (30) we obtain the following pointwise inequality:

$$|qR_1w| = q|R_1w| \leq qR_1|w| \leq qR_2|w|. \tag{37}$$

Since $qR_2: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator, from (37) it follows that

$$\|qR_1w\| \leq \|qR_2|w|\| \leq C_3\|w\|, \quad \text{for all } w \in L^2(M),$$

where $C_3 \geq 0$ is a constant.

Hence, qR_1 is a bounded linear operator: $L^2(M) \rightarrow L^2(M)$.

Let $u \in D_1$ be arbitrary. Then $qu = qR_1(u + Lu)$, where $Lu = \Delta_M u + qu$.

Since $qR_1: L^2(M) \rightarrow L^2(M)$ is a bounded linear operator, we have

$$\|qu\| \leq C_1(\|u\| + \|Lu\|), \quad \text{for all } u \in D_1, \tag{38}$$

where $C_1 \geq 0$ is a constant.

Using (38) we get

$$\begin{aligned} \|\Delta_M u\| &= \|Lu - qu\| \leq \|Lu\| + \|qu\| \leq \\ &\leq C_2(\|u\| + \|Lu\|), \quad \text{for all } u \in D_1, \end{aligned} \tag{39}$$

where $C_2 \geq 0$ is a constant.

From (38) and (39) we obtain (7). This concludes the proof of the Theorem. \square

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