ON m-ACCRETIVE SCHRÖDINGER-TYPE OPERATORS WITH SINGULAR POTENTIALS ON MANIFOLDS OF BOUNDED GEOMETRY

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Abstract. We consider a Schrödinger type differential expression $H_V = \nabla^*\nabla + V$, where ∇ is a C^∞-bounded Hermitian connection on a Hermitian vector bundle E of bounded geometry over a manifold of bounded geometry (M, g) with positive C^∞-bounded measure $d\mu$, and $V \in L^1_{\text{loc}}(\text{End}\, E)$ is a linear bundle endomorphism. We give a sufficient condition for $m-$accretiveness of the operator S defined by $Su = H_V u$ for $u \in \text{Dom}(S) = \{u \in W^{1,2}(E)\colon H_V u \in L^2(E)\}$. The proof essentially follows the scheme of T. Kato, but it requires the use of more general version of Kato's inequality for Bochner Laplacian operator as well as a result on the positivity of $u \in L^2(M)$ satisfying the equation $(\Delta_M + b)u = \nu$, where Δ_M is the scalar Laplacian on M, $b > 0$ is a constant and $\nu \geq 0$ is a positive distribution on M.

1. Introduction and the main result

1.1. **The setting.** Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric g, $\dim M = n$. We will assume that M is connected. We will also assume that M has bounded geometry. Moreover, we will assume that we are given a positive C^∞-bounded measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-bounded density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$.

Let E be a Hermitian vector bundle over M. We will assume that E is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of E on every canonical coordinate neighborhood U such that the corresponding matrix transition functions $h_{U,U'}$ on all intersections $U \bigcap U'$ of such neighborhoods are C^∞-bounded, i.e. all derivatives $\partial_y^\alpha h_{U,U'}(y)$, where α is a multiindex, with respect to canonical coordinates are bounded with bounds C_α which do not depend on the chosen pair U, U').

We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u, v) \ = \int_M \langle u(x), v(x)\rangle_{E_x}\, d\mu(x). \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle_{E_x}$ denotes the fiberwise inner product.

Let

$$\nabla\colon C^\infty(E) \to C^\infty(T^*M \otimes E)$$

be a Hermitian connection on E which is C^∞-bounded as a linear differential operator, i.e. in any canonical coordinate system U (with the chosen trivializations of $E|_U$ and $(T^*M \otimes E)|_U$),

∇ is written in the form

$$\nabla = \sum_{|\alpha|\leq 1} a_\alpha(y)\partial_y^\alpha,$$

where α is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial_y^\beta a_\alpha(y)$ for any multiindex β are bounded by a constant C_β which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

$$H_V = \nabla^*\nabla + V,$$

Here

$$\nabla^*\colon C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1), and V is a linear bundle endomorphism of E, i.e. for every $x \in M$,

$$V(x)\colon E_x \to E_x \tag{1.2}$$

is a linear operator.

We make the following assumption on V.

Assumption A. $V \in L^p_{\mathrm{loc}}(\operatorname{End} E)$, where

(i) $p = \frac{2n}{n+2}$ for $n \geq 3$
(ii) $p > 1$ for $n = 2$
(iii) $p = 1$ for $n = 1$.

We will use the following notations

$$V_1(x) := \frac{V(x) + (V(x))^*}{2}, \qquad V_2(x) := \frac{V(x) - (V(x))^*}{2i}, \qquad x \in M, \tag{1.3}$$

where $i = \sqrt{-1}$ and $(V(x))^*$ denotes the adjoint of the linear operator (1.2) (in the sense of linear algebra).

By (1.3), for all $x \in M$, we have the following decomposition

$$V(x) = V_1(x) + iV_2(x).$$

1.2. **Sobolev space $W^{1,2}(E)$.** By $W^{1,2}(E)$ we will denote the set of all $u \in L^2(E)$, such that $\nabla u \in L^2(T^*M \otimes E)$. It is well-known (see, for example, Sect. A1.1 in [10]) that $W^{1,2}(E)$ is the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u, v)_1 \ := \ (u, v) \ + \ (\nabla u, \nabla v) \qquad u, v \in C_c^\infty(E).$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

Since (M, g) and E have bounded geometry, by Sect A1.1 in [10], it follows that the usual Sobolev embedding theorem (see, for example, Theorem 2.21 in [1]) holds for $W^{1,2}(E)$.

1.3. **A Realization of H_V in $L^2(E)$.** Let V be as in Assumption A. We define an operator S associated to H_V as an operator in $L^2(E)$ given by $Su = H_Vu$ with domain

$$\mathrm{Dom}(S) = \{u \in W^{1,2}(E) : H_Vu \in L^2(E)\}. \tag{1.4}$$

Remark 1.4. We will show that for all $u \in W^{1,2}(E)$, we have $Vu \in L^1_{\mathrm{loc}}(E)$, so that H_Vu in (1.4) can be understood in distributional sense.

Let $u \in W^{1,2}(E)$. For $n \geq 3$, by Sect. 1.2 and the first part of Theorem 2.21 from Aubin [1], we have the following continuous embedding

$$W^{1,2}(E) \subset L^{p'}(E), \tag{1.5}$$

where $1/p' = 1/2 - 1/n$.

Let $p = \frac{2n}{n+2}$ be as in Assumption A. Since $1/p + 1/p' = 1$, by Hölder's inequality it follows that $Vu \in L^1_{\mathrm{loc}}(E)$.

For $n = 2$, by the first part of Theorem 2.21 from Aubin [1], we get continuous embedding (1.5) for all $2 < p' < \infty$. By Assumption A, for $n = 2$, we have $p > 1$. We may assume that $1 < p < 2$ (if $V \in L^t_{\mathrm{loc}}(\mathrm{End}\, E)$ with $t \geq 2$, then $V \in L^p_{\mathrm{loc}}(\mathrm{End}\, E)$ for all $1 < p < 2$). Given $1 < p < 2$, we can take $p' > 2$ such that $1/p + 1/p' = 1$. By Hölder's inequality we have $Vu \in L^1_{\mathrm{loc}}(E)$.

For $n = 1$, it is well-known (see e.g. the second part of Theorem 2.21 in [1]) that (1.5) holds with $p' = \infty$. By Assumption A for $n = 1$, we have $p = 1$. Thus by Hölder's inequality we have $Vu \in L^1_{\mathrm{loc}}(E)$.

We now state the main result.

Theorem 1.5. *Assume that (M, g) is a manifold of bounded geometry with a positive C^∞-bounded measure $d\mu$. Assume that E is a Hermitian vector bundle of bounded geometry over M. Assume ∇ be a C^∞-bounded Hermitian connection on E. Let V be as in assumption A. Moreover, assume that for all $x \in M$,*

$$V_1(x) \geq 0, \quad \textit{as an operator } E_x \to E_x,$$

where $V_1(x)$ is as in (1.3).

Then S is m-accretive.

Remark 1.6. The main source of inspiration for Theorem 1.5 was a result of T. Kato [8, Theorem I] which was proven for the operator $-\Delta + V$ on an open set $\Omega \subset \mathbb{R}^n$, where $-\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V \in L^p_{\mathrm{loc}}(\Omega)$, with p as in Assumption A, is a complex-valued function such that $\mathrm{Re}\, V \geq 0$.

Let $d\colon C^\infty(M) \to \Omega^1(M)$ be the standard differential. Then $d^*d\colon C^\infty(M) \to C^\infty(M)$ is called the scalar Laplacian and will be denoted by Δ_M.

2. Proof of Theorem 1.5

We will adopt the proof of Theorem I in Kato [8] to our context. Throughout this section, we assume that all hypotheses of Theorem 1.5 are satisfied. We begin by introducing another realization of H_V.

2.1. **Maximal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$.** We define an operator T associated to H_V as an operator $W^{1,2}(E) \to W^{-1,2}(E)$ given by $Tu = H_V u$ with domain

$$\mathrm{Dom}(T) = \{u \in W^{1,2}(E) : H_V u \in W^{-1,2}(E)\}. \tag{2.1}$$

Remark 2.2. Condition $H_V u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$ makes sense since $H_V u$ is a distributional section of E by Remark 1.4. Since $\nabla^*\nabla u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$, it follows that the condition $H_V u \in W^{-1,2}(E)$ in (2.1) is equivalent to $Vu \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$.

Lemma 2.3. *The following inclusion holds: $C_c^\infty(E) \subset \mathrm{Dom}(T)$.*

Proof. Let $u \in C_c^\infty(E)$. Then $Vu \in L^p(E)$, where p is as in assumption A. By Remark 1.4, it follows that $W^{1,2}(E) \subset L^{p'}(E)$, where $1/p + 1/p' = 1$. By duality, we have $L^p(E) \subset W^{-1,2}(E)$. Thus $Vu \in W^{-1,2}(E)$, and hence $u \in \mathrm{Dom}(T)$. □

2.4. **Minimal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$.** By T_0 we will denote the restriction of T with $\mathrm{Dom}(T_0) = C_c^\infty(E)$. Clearly, T_0 is a densely defined operator.

Remark 2.5. Since $\mathrm{Dom}(S)$, where S is as in (1.4), does not necessarily contain $C_c^\infty(E)$, there is no minimal realization of H_V in $L^2(E)$ (in the sense of Sect. 2.4).

2.6. **Maximal and minimal realization of H_{V^*}.** In what follows, we will denote by T' and T_0' the maximal and minimal realization of H_{V^*} in the sense of Sect. 2.1 and Sect. 2.4 respectively, where V^* is the adjoint of V as in (1.3).

Lemma 2.7. *The following holds: $T = (T_0')^*$, where $*$ denotes the adjoint of an operator.*

Proof. We need to show that for any $u \in W^{1,2}(E)$ and $f \in W^{-1,2}(E)$, the equation $Tu = f$ is true if and only if

$$(u, T's) = (f, s), \qquad \text{for all } s \in C_c^\infty(E), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{1,2}_{\mathrm{loc}}(E)$ and $W^{-1,2}_{\mathrm{comp}}(E)$ extending the inner product in $L^2(E)$ by continuity from $C_c^\infty(E)$.

1. Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and $Tu = f$. Then $Vu \in W^{-1,2}(E)$. By Lemma 2.3, for all $s \in C_c^\infty(E)$, we have $V^*s \in W^{-1,2}_{\mathrm{comp}}(E)$. Since $s \in C_c^\infty(E)$, we have $V^*s \in L^p_{\mathrm{comp}}(E)$ with p as in Assumption A. By the proof in Remark 1.4, we have $u \in W^{1,2}(E) \subset L^{p'}(E)$ (continuous embedding), where $1/p + 1/p' = 1$. By Hölder's inequality, $L^{p'}_{\mathrm{loc}}(E)$ is in a continuous duality with $L^p_{\mathrm{comp}}(E)$ by the usual integration. Thus, for all $s \in C_c^\infty(E)$, we have (after approximating u by sections $u_j \in C_c^\infty(E)$ in $W^{1,2}$-norm in a neighborhood of $\mathrm{supp}\, s$)

$$\begin{aligned}(u, V^*s) &= \lim_{j\to\infty}(u_j, V^*s) = \lim_{j\to\infty}\int \langle u_j(x), (V^*s)(x)\rangle\, d\mu(x)\\ &= \int \langle u(x), (V^*s)(x)\rangle\, d\mu(x), \end{aligned}\tag{2.3}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{1,2}_{\mathrm{loc}}(E)$ and $W^{-1,2}_{\mathrm{comp}}(E)$ extending the inner product in $L^2(E)$ by continuity from $C_c^\infty(E)$. The second equality in (2.3) holds since $V^*s \in L^1_{\mathrm{loc}}(E)$ by Remark 1.4 and $u_j \in C_c^\infty(E)$.

Therefore, we obtain

$$(u, V^*s) \;=\; \int \langle u(x), (V^*s)(x)\rangle\, d\mu(x) \;=\; \int \langle (Vu)(x), s(x)\rangle\, d\mu(x) \;=\; (Vu, s), \tag{2.4}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{1,2}_{\mathrm{loc}}(E)$ and $W^{-1,2}_{\mathrm{comp}}(E)$ extending the inner product in $L^2(E)$ by continuity from $C_c^\infty(E)$. The first equality in (2.4) follows from (2.3). The second equality in (2.4) holds by the definition of $(V(x))^*\colon E_x \to E_x$. The third equality in (2.4) holds for all $s \in C_c^\infty(E)$ since $Vu \in W^{-1,2}(E)$ and $Vu \in L^1_{\mathrm{loc}}(E)$ by Remark 1.4.

Using (2.4), we obtain

$$(u, T's) \;=\; (u, \nabla^*\nabla s + V^*s) \;=\; (u, \nabla^*\nabla s) + (u, V^*s) \;=\; (\nabla^*\nabla u, s) + (Vu, s) \;=\; (Tu, s), \tag{2.5}$$

where V^* is the adjoint of V as in (1.3) and $(\cdot,\cdot)$ denotes the duality between $W^{1,2}_{\mathrm{loc}}(E)$ and $W^{-1,2}_{\mathrm{comp}}(E)$. In the third equality we also used the integration by parts (see, for example, Lemma 8.8 in [2]).

2. Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and (2.2) holds. Then the first two equalities in (2.4) hold (we do not know a priori that $Vu \in W^{-1,2}(E)$ so the third equality in (2.4) is not yet justified). Thus for all $s \in C_c^\infty(E)$,

$$(f, s) \;=\; (u, T's) \;=\; (u, \nabla^*\nabla s) + (u, V^*s) \;=\; (\nabla^*\nabla u, s) + \int \langle (Vu)(x), s(x)\rangle\, d\mu(x),$$

where the second equality follows as in (2.5), and the third equality follows from integration by parts and the second equality in (2.4).

Since $\nabla^*\nabla u \in W^{-1,2}(E)$ and $f \in W^{-1,2}(E)$, we obtain

$$(f - \nabla^*\nabla u, s) \;=\; \int \langle (Vu)(x), s(x)\rangle\, d\mu(x), \quad \text{for all } s \in C_c^\infty(E), \tag{2.6}$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}_{\mathrm{loc}}(E)$ and $W^{1,2}_{\mathrm{comp}}(E)$.

Since $u \in W^{1,2}(E)$, from Remark 1.4 we know that $Vu \in L^1_{\mathrm{loc}}(E)$. By (2.6) we get $Vu \in W^{-1,2}(E)$ since $C_c^\infty(E)$ is dense in $W^{1,2}(E)$. Thus, as in (2.4),

$$\int \langle (Vu)(x), s(x)\rangle\, d\mu(x) \;=\; (Vu, s), \quad \text{for all } s \in C_c^\infty(E), \tag{2.7}$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}_{\mathrm{loc}}(E)$ and $W^{1,2}_{\mathrm{comp}}(E)$.

From (2.6) and (2.7), we obtain

$$(f - \nabla^*\nabla u, s) \;=\; (Vu, s), \quad \text{for all } s \in C_c^\infty(E), \tag{2.8}$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}_{\mathrm{loc}}(E)$ and $W^{1,2}_{\mathrm{comp}}(E)$.

Therefore

$$(f, s) \;=\; (\nabla^*\nabla u, s) + (Vu, s) \;=\; (Tu, s), \quad \text{for all } s \in C_c^\infty(E),$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}_{\mathrm{loc}}(E)$ and $W^{1,2}_{\mathrm{comp}}(E)$.

This shows that $Tu = f$, and the Lemma is proven. □

In what follows, we will adopt the terminology of Kato [8] and distinguish between monotone and accretive operators. Accretive operators act within the same Hilbert space, while monotone operators act from a Hilbert space into its adjoint space (anti-dual).

Lemma 2.8. *The operator T_0 is monotone, i.e.*

$$\mathrm{Re}(T_0 s, s) \geq 0, \quad \textit{for all } s \in C_c^\infty(E),$$

where $(\cdot,\cdot)$ denotes the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$.

Proof. We have for all $s \in C_c^\infty(E)$,

$$\begin{aligned}\mathrm{Re}(T_0 s, s) &= \mathrm{Re}\left[(\nabla^*\nabla s, s) + \int \langle Vs, s\rangle\, d\mu\right] \\ &= \|\nabla s\|^2 + \mathrm{Re}\left[\int \langle V_1 s, s\rangle\, d\mu + i\int \langle V_2 s, s\rangle\, d\mu\right] \geq \|\nabla s\|^2, \quad (2.9)\end{aligned}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$, $\|\cdot\|$ denotes the L^2-norm, and $V_1 \geq 0$ and V_2 are linear self-adjoint bundle endomorphisms as in (1.3).

The Lemma is proven. □

Lemma 2.9. *The operator $1 + T_0$ is coercive in the sense that*

$$\|(1+T_0)s\|_{-1} \geq \|s\|_1, \qquad \textit{for all } s \in \mathrm{Dom}(T_0) = C_c^\infty(E), \tag{2.10}$$

where $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$, and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

Proof. As in (2.9), we have for all $s \in C_c^\infty(E)$,

$$\mathrm{Re}((T_0+1)s, s) \;\geq\; \|s\|^2 + \|\nabla s\|^2 \;=\; \|s\|_1^2, \tag{2.11}$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$.

Since the left hand side of (2.11) does not exceed $\|(1+T_0)s\|_{-1}\|s\|_1$, the inequality (2.10) immediately follows from (2.11). □

In what follows, $\mathrm{Ker}\, A$ and $\mathrm{Ran}\, A$ denote the kernel and the range of operator A respectively, and $\bar{A}$ denotes the closure of A.

Lemma 2.10. *The following hold:*

(i) *The operator T_0 is closable with closure T_0^{**}.*

(ii) $\mathrm{Ran}(1+T_0^{**})$ *is closed.*

Proof. By Lemma 2.7 it follows that $T' = T_0^*$, where T' is as in Sect. 2.6. Since $T_0' \subset T'$ (as operators), it follows that T' is densely defined. Thus T_0^{**} exists and equals $\overline{T_0}$. This proves property (i).

We will now prove property (ii). Since $1+T_0$ is coercive by Lemma 2.9, it follows by definition of $\overline{T_0}$ that $1+T_0^{**} = 1+\overline{T_0}$ is also coercive, i.e.

$$\|(1+T_0^{**})u\|_{-1} \geq \|u\|_1, \qquad \text{for all } u \in \mathrm{Dom}(T_0^{**}), \tag{2.12}$$

where $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$, and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

We will now show that $\operatorname{Ran}(1+T_0^{**})$ is closed.

Let $f_j \in \operatorname{Ran}(1+T_0^{**})$ and $\|f_j - f\|_{-1} \to 0$ as $j \to \infty$. Let $u_j \in \operatorname{Dom}(1+T_0^{**})$ be a sequence such that $(1+T_0^{**})u_j = f_j$. Since f_j is a Cauchy sequence in $\|\cdot\|_{-1}$, by (2.12) it follows that u_j is a Cauchy sequence in $\|\cdot\|_1$. Thus u_j converges in $\|\cdot\|_1$, and we will denote its limit by u. Since $1+T_0^{**}$ is a closed operator, it follows that $u \in \operatorname{Dom}(1+T_0^{**})$ and $f = (1+T_0^{**})u$. Thus $f \in \operatorname{Ran}(1+T_0^{**})$, and property (ii) is proven. □

In what follows, we will use the general version of Kato's inequality whose proof is given in Theorem 5.7 from [2].

Lemma 2.11. *Assume that (M,g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E. Assume that $w \in L^1_{\text{loc}}(E)$ and $\nabla^*\nabla w \in L^1_{\text{loc}}(E)$. Then*

$$\Delta_M |w| \;\leq\; \operatorname{Re}\langle \nabla^*\nabla w, \operatorname{sign} w\rangle, \tag{2.13}$$

where Δ_M is the scalar Laplacian on M and

$$\operatorname{sign} w(x) \;=\; \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } \; w(x) \neq 0\ , \\ 0 & \text{otherwise.} \end{cases}$$

We now state and prove the key proposition.

Proposition 2.12. $\operatorname{Ran}(1+T_0^{**}) = W^{-1,2}(E)$.

Proof. By Lemma 2.10 it suffices to show that if $u \in W^{1,2}(E)$ and

$$((1+T_0)s, u) = 0, \qquad \text{for all } s \in C_c^\infty(E), \tag{2.14}$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$, then $u = 0$.

Using condition (2.14) and the same arguments as in the proof of the first two equalities in (2.4) and the equation (2.7), we have

$$0 \;=\; (s,u) + (\nabla^*\nabla s, u) + (Vs, u) \;=\; (s,u) + (s, \nabla^*\nabla u) + (s, V^*u), \qquad \text{for all } s \in C_c^\infty(E),$$

where $(\cdot,\cdot)$ is the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$ and V^* is as in (1.3).

Therefore, the following distributional equality holds (recall that by Remark 1.4, we have $V^*u \in L^1_{\text{loc}}(E)$)

$$\nabla^*\nabla u + V^*u + u = 0. \tag{2.15}$$

From (2.15), we have $\nabla^*\nabla u = -V^*u - u \in L^1_{\text{loc}}(E)$. Therefore, by Lemma 2.11, we get

$$\begin{aligned}\Delta_M |u| \;\leq\; \operatorname{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle \;&=\; \operatorname{Re}\langle -u - V_1 u + iV_2 u, \operatorname{sign} u\rangle \\ &=\; -|u| - \langle V_1 u, \operatorname{sign} u\rangle \leq -|u|, \end{aligned} \tag{2.16}$$

where Δ_M, $\langle\cdot,\cdot\rangle$ and $\operatorname{sign} u$ are as in (2.13), and $V_1 \geq 0$, V_2 are linear self-adjoint bundle endomorphisms as in (1.3).

By (2.16) we get the following distributional inequality

$$(\Delta_M + 1)|u| \leq 0. \tag{2.17}$$

Since (M, g) is a manifold of bounded geometry, by Proposition B.3 from [2], the inequality (2.17) implies that $|u| = 0$, i.e. $u = 0$. This concludes the proof of the Proposition. $\square$

Corollary 2.13. *T_0^{**} is a maximal monotone operator (in the sense that it is monotone and has no proper monotone extension).*

Proof. The corollary follows immediately from Proposition 2.12, the inequality (2.12), and the remark after the equation (3.38) in Sect. 5.3.10 of [6]. $\square$

Proposition 2.14. *The following hold:*

(i) $T = T_0^{**}$.
(ii) *The operator T is maximal monotone.*

Proof. We first prove property (i). Since $T_0 \subset T$ (as operators), it follows that $T_0^{**} \subset T$ because T is closed by Lemma 2.7. By Proposition 2.12, $\mathrm{Ran}(1 + T_0^{**}) = W^{-1,2}(E)$. By the same proposition (with V replaced by V^*) it follows that $\mathrm{Ran}(1 + (T_0')^{**}) = W^{-1,2}(E)$, where T_0' is as in Sect. 2.6. Since $1 + T = 1 + (T_0')^*$ (cf. Lemma 2.7), it follows that $\mathrm{Ker}(1 + T) = \{0\}$. Hence T cannot be a proper extension of T_0^{**}. This shows that $T_0^{**} = T$.

Property (ii) follows immediately from property (i) and Corollary 2.13. $\square$

3. Proof of Theorem 1.5

First note that the following holds: $u \in \mathrm{Dom}(S)$ if and only if $u \in \mathrm{Dom}(T)$ and $Tu \in L^2(E)$ (in which case $Su = Tu$).

By Proposition 2.12 and Proposition 2.14, it follows that $\mathrm{Ran}(1 + T) = W^{-1,2}(E)$. Therefore $\mathrm{Ran}(1 + S) = L^2(E)$. Furthermore, since T is maximal monotone by Proposition 2.14, it follows that

$$\mathrm{Re}(Su, u)_{L^2(E)} = \mathrm{Re}(Tu, u) \geq 0, \quad \text{for all } u \in \mathrm{Dom}(S),$$

where $(\cdot, \cdot)_{L^2(E)}$ denotes the inner product in $L^2(E)$, and $(\cdot, \cdot)$ is the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$.

Thus we proved that S is accretive and $\mathrm{Ran}(1+S) = L^2(E)$. By the remark after the equation (3.37) in Sect. 5.3.10 of [6], it follows that S is m-accretive. $\square$

References

[1] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.

[2] M. Braverman, O. Milatovic, M. Shubin, *Essential self-adjointness of Schrödinger type operators on manifolds*, Russian Math. Surveys, **57**(4) (2002), 641–692.

[3] H. Brézis, T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl., **58**(9) (1979), 137–151.

[4] W. G. Faris, *Self-adjoint Operators*, Lecture Notes in Mathematics No. 433, Springer-Verlag, Berlin e.a., 1975.

[5] H. Hess, R. Schrader, A. Uhlenbrock, *Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifold*, J. Differential Geom., **15** (1980), 27-37.

[6] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980.

[7] T. Kato, *A second look at the essential selfadjointness of the Schrödinger operators*, Physical reality and mathematical description, Reidel, Dordrecht, 1974, 193–201.

[8] T. Kato, *On some Schrödinger operators with a singular complex potential*, Ann. Sc. Norm. Sup. Pisa, Ser. IV, Vol. 5 (1978), 105–114.

[9] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I, II: Functional analysis. Fourier analysis, self-adjointness*, Academic Press, New York e.a., 1975.

[10] M. A. Shubin, *Spectral theory of elliptic operators on noncompact manifolds,* Astérisque No. 207 (1992), 35–108.

[11] M. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations*, Springer-Verlag, New York e.a., 1996.

2000 *Mathematics Subject Classification.* Primary 35P05, 58G25; Secondary 47B25, 81Q10.

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