Title: “Localized” self-adjointness of Schrödinger-type operators on Riemannian manifolds.

Proposed running head: Schrödinger-type operators on manifolds.

Author: Ognjen Milatovic

Department Address: Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

Address (for correspondence): 78 Apsley Street Apt. 1, Hudson, MA 01749, USA.

Phone: (978) 562-6891

E-mail: bmilatovic@msn.com

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ABSTRACT

We prove self-adjointness of the Schrödinger type operator $H_V = \nabla^*\nabla + V$, where ∇ is a Hermitian connection on a Hermitian vector bundle E over a complete Riemannian manifold M with positive smooth measure $d\mu$ which is fixed independently of the metric, and $V \in L^1_{\text{loc}}(\operatorname{End} E)$ is a Hermitian bundle endomorphism. Self-adjointness of H_V is deduced from the self-adjointness of the corresponding "localized" operator. This is an extension of a result by Cycon. The proof uses the scheme of Cycon, but requires a refined integration by parts technique as well as the use of a family of cut-off functions which are constructed by a non-trivial smoothing procedure due to Karcher.

Key words: complete, manifold, operator, Riemannian, Schrödinger, self-adjoint, self-adjointness

1. INTRODUCTION AND THE MAIN RESULT

Let (M, g) be a Riemannian manifold (i.e. M is a C^∞-manifold, (g_{jk}) is a Riemannian metric on M), $\dim M = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$. We will assume that (M, g) is complete.

Let E be a Hermitian vector bundle over M. We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u, v) \;=\; \int_M \langle u(x), v(x)\rangle_{E_x}\, d\mu(x). \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle_{E_x}$ denotes the fiberwise inner product.

Let

$$\nabla\colon C^\infty(E) \to C^\infty(T^*M \otimes E)$$

be a Hermitian connection on E.

We will consider a Schrödinger type operator of the form

$$H_V = \nabla^*\nabla + V,$$

where V is a linear self-adjoint bundle map $V \in L^1_{\text{loc}}(\operatorname{End} E)$. Here

$$\nabla^*\colon C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1)

We define the maximal operator $H_{V,\max}$ associated to H_V as an operator in $L^2(E)$ given by $H_{V,\max}u = H_V u$ with domain

$$\operatorname{Dom}(H_{V,\max}) = \{u \in L^2(E) : Vu \in L^1_{\text{loc}}(E),\ H_V u \in L^2(E)\}. \tag{1.2}$$

Here $\nabla^*\nabla u$ in $H_V u = \nabla^*\nabla u + Vu$ is understood in distributional sense.

We want to give a sufficient condition for self-adjointness of $H_{V,\max}$.

For every $x \in M$ we have the following canonical decomposition:

$$V(x) = V^+(x) - V^-(x). \tag{1.3}$$

Here $V^+(x) = P_+(x)V(x)$, where $P_+(x) := \chi_{[0,+\infty)}(V(x))$, and $V^-(x) = -P_-(x)V(x)$, where $P_-(x) := \chi_{(-\infty,0)}(V(x))$. Here χ_A denotes the characteristic function of the set A.

1.1. **Sesquilinear form.**

Consider the sesquilinear form $h\colon C_c^\infty(E) \times C_c^\infty(E) \to \mathbb{C}$ defined by

$$h(u, v) = \int \langle H_V u, v\rangle\, d\mu. \tag{1.4}$$

We will make the following assumption on h.

Assumption A. The quadratic form (which we also denote by h) corresponding to h in (1.4) is closable and $h \geq 0$.

Remark 1.2. If $h \geq 0$ on $C_c^\infty(E)$ and $V \in L^2_{\text{loc}}(\operatorname{End} E)$, then h is closable, so Assumption A is satisfied, cf., for example, Theorem 14.1 in [5]. Assumption A is also satisfied when, for example, $0 \leq V \in L^1_{\text{loc}}(\operatorname{End} E)$ as shown in Lemma 2.2 below.

In what follows, we will denote by $H_{\bar{h}}$ the self-adjoint operator associated to the closure $\bar{h}$ of the form h in Assumption A so that

$$H_{\bar{h}} \geq 0. \tag{1.5}$$

1.3. **Quadratic forms of semibounded operators.** For any semibounded below self-adjoint operator $-\alpha \leq T\colon \operatorname{Dom}(T) \subset L^2(E) \to L^2(E)$, we will denote by $Q(T)$ the domain of the quadratic form t associated to T. Then $Q(T)$ is a Hilbert space with the inner product

$$(\cdot,\cdot)_t := t(\cdot,\cdot) + (1+\alpha)(\cdot,\cdot),$$

where $(\cdot,\cdot)$ is the inner product in $L^2(E)$.

A set which is dense in the Hilbert space $Q(T)$ with the inner product $(\cdot,\cdot)_t$ is called a form core of T.

In what follows, we will denote $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$.

For a fixed $x_0 \in M$ and for all $k \in \mathbb{Z}_+$, denote

$$B_k := \{x \in M : d(x_0, x) \leq k\}, \tag{1.6}$$

where d is the distance function on M induced by the metric g.

In what follows we will use the following result on the existence of cut-off functions due to Karcher [6] (see also [13]).

1.4. **Cut-off functions.** Let (M, g) be a complete Riemannian manifold. Then there exists a sequence of functions $\phi_j\colon M \to \mathbb{R}$, $j = 1, 2, \dots$ such that
(a) $\phi_j \in C_c^\infty(M)$
(b) $0 \leq \phi_j(x) \leq 1$, $x \in M$, $j = 1, 2, \dots$
(c) for every compact set $K \subset M$, there exists j_0 such that $\phi_j = 1$ on K, for $j \geq j_0$.
(d) $\epsilon_j := \sup_{x\in M} |\nabla\phi_j| \to 0$ as $j \to \infty$.

Clearly, $\operatorname{supp}\phi_j \subset B_{k_j}$ for k_j large enough. Denote $\widetilde{B}_j := B_{k_j}$.

Assumption B.

(i) For every $k \in \mathbb{Z}_+$ and $x \in M$, denote $V_k^-(x) := \chi_k V^-(x)$, where V^- is as in (1.3) and χ_k is the characteristic function of $\widetilde{B}_k$. Let

$$H_k := \nabla^*\nabla + V_k, \tag{1.7}$$

where $V_k := V^+ - V_k^-$ and V^+ is as in (1.3).

Denote by $H_{k,\max}$ the maximal operator associated to H_k as in (1.2).

Assume that $H_{k,\max}$ is self-adjoint.

(ii) Assume that $C_c^\infty(E)$ is a core of the quadratic form associated to $|H_{k,\max}|$.

(iii) Assume that for every $k \in \mathbb{Z}_+$ there exists a constant $c_k > 0$ such that

$$(\nabla^*\nabla w, w) + (V^+ w, w) \leq c_k[(H_k w, w) + \|w\|^2], \qquad \text{for all } w \in C_c^\infty(E). \tag{1.8}$$

Remark 1.5. If $V^- \in L^p_{\text{loc}}(\operatorname{End} E)$ with $p \geq \frac{n}{2}$ for $n \geq 5$, $p > 2$ for $n = 4$, and $p = 2$ for $n \leq 3$, then the following holds: there exist constants $0 < \alpha_k < 1$ and $\beta_k > 0$ such that

$$\left(\int_M |V_k^-|^2 |u|^2 \, d\mu\right)^{1/2} \leq \alpha_k \|\Delta_M u\| + \beta_k \|u\|, \qquad \text{for all} \quad u \in C_c^\infty(M). \tag{1.9}$$

Here $|V_k^-|$ denotes the norm of the endomorphism $V_k^-(x)\colon E_x \to E_x$, and $\Delta_M := d^*d$ is the scalar Laplacian on M.

(In fact, for every $\alpha_k > 0$, there exists $\beta_k > 0$ such that (1.9) holds).

If $\operatorname{supp} V_k^-$ is contained in a coordinate neighborhood, (1.9) follows from Theorem IX.28, arguments from the proof of Theorem X.15, and Theorems X.20, X.21 of [11]. The general case may be proven using a localization technique as it is explained in [13, §5.2]. Another option is to require that $V^- \in S_{n,\text{loc}}$ where $S_{n,\text{loc}}$ is a local Stummel class, cf. Appendix C of [1].

By Lemma 6.2 from [1], there exist constants $0 < a_k < 1$ and $b_k \geq 0$ such that

$$|(V_k^- w, w)| \leq a_k(\nabla^*\nabla w, w) + b_k\|w\|^2, \qquad \text{for all } w \in C_c^\infty(E). \tag{1.10}$$

Since $\nabla^*\nabla$ and V^+ are positive self-adjoint operators in $L^2(E)$, it follows by Theorem 4.1 in [5] that the form sum $\nabla^*\nabla \tilde{+} V^+$ is a positive self-adjoint operator. Since V_k^- satisfies (1.10), Theorem 7.11 in [5] immediately supplies a semibounded self-adjoint operator F_k which is the form sum of $\nabla^*\nabla \tilde{+} V^+$ and $-V_k^-$. By the same theorem, $C_c^\infty(E)$ is a form core of F_k. However, a great deal of work is needed to check that F_k coincides with the maximal operator $H_{k,\max}$. In the case of the operator $-\Delta + V^+ - V_k^-$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with standard metric and $V^+ \in L^1_{\text{loc}}(\mathbb{R}^n)$, $V_k^- \in L^p(\mathbb{R}^n)$, where p is as above, this was done in [3], but the arguments given there extend to the case of operator $\Delta_M + V^+ - V_k^-$ on a manifold (M, g) of bounded geometry.

Thus if (M, g) has bounded geometry, $V^+ \in L^1_{\text{loc}}(M)$ and $V^- \in L^p_{\text{loc}}(M)$, where p is as in the beginning of this remark, then the maximal operator $H_{k,\max}$ associated to $H_k = \Delta_M + V^+ - V_k^-$ as in (1.2) coincides with semibounded self-adjoint operator F_k above (in case $\nabla = d$). So $H_{k,\max}$ is self-adjoint, hence (i) of Assumption B is satisfied. Since $C_c^\infty(M)$ is a form core of F_k, it follows that $C_c^\infty(M)$ is a form core of $|H_{k,\max}|$, so (ii) of Assumption B is also satisfied. From (1.10) it follows that

$$(\nabla^*\nabla w, w) + (V^+ w, w) \leq \frac{1}{1 - a_k}[(H_k w, w) + (b_k + 1)\|w\|^2] \qquad \text{for all } w \in C_c^\infty(E). \tag{1.11}$$

Taking $\nabla = d$ and $E = M \times \mathbb{C}$ in (1.11), the condition (iii) of Assumption B is also satisfied in case $H_k = \Delta_M + V^+ - V_k^-$.

We now state the main result.

Theorem 1.6. *Assume that (M, g) is complete and Assumptions A and B hold.*

Then $H_{V,\max}$ is a self-adjoint operator.

Remark 1.7. This result extends the result of Cycon [4] which was proven for the case of operator $-\Delta + V$, on $\mathbb{R}^n$ with standard metric. For results on the essential self-adjointness of Schrödinger-type operators $H_V = D^*D + V$, where $D\colon C_c^\infty(E) \to C_c^\infty(F)$ is a first order linear differential operator (with injective principal symbol) acting on sections of Hermitian vector bundles E and F over a non-compact manifold M, and $V \in L^2_{\text{loc}}(\operatorname{End} E)$, see [1]. Appendix D of [1] also contains useful historical remarks and references on the essential self-adjointness of Schrödinger operators.

In what follows, $W^{1,2}(E)$ denotes the set of all $u \in L^2(E)$, such that $\nabla u \in L^2(T^*M \otimes E)$. For a complete Riemannian manifold (M, g), it is well-known that $W^{1,2}(E)$ is the closure of $C_c^\infty(E)$ with respect to norm $\|\cdot\|_1$ defined by the inner product

$$(u, v)_1 \;:=\; (u, v) \;+\; (\nabla u, \nabla v) \qquad u, v \in C_c^\infty(E),$$

where $(\cdot,\cdot)$ is the inner product in L^2.

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

In what follows we will use the following facts and notations from differential geometry.

Let ∇^1 be the connection on $T^*M \otimes E$ induced by ∇ and Levi-Civita connection ∇^{LC} on T^*M. Then

$$\nabla^1\colon C^\infty(T^*M \otimes E) \to C^\infty(T^*M \otimes T^*M \otimes E). \tag{1.12}$$

Define an operator $A\colon C^\infty(E) \to C^\infty(E)$ as

$$A := -(g \otimes 1) \circ \nabla^1 \circ \nabla.$$

By Proposition 2.1 in Appendix C of [14], $A = \nabla^*\nabla$.

If we take $\nabla = d$, the following holds: $\Delta_M = -g \circ \nabla^{LC} \circ d$.

2. PROOF OF THEOREM 1.6

We begin with the following

Lemma 2.1. *Assume that $0 \le T \in L^1_{\text{loc}}(\operatorname{End} E)$ is a linear self-adjoint bundle map. Assume also that $u \in Q(T)$, where $Q(T) = \{u \in L^2(E)\colon \langle Tu, u\rangle \in L^1(M)\}$.*

Then $Tu \in L^1_{\text{loc}}(E)$.

Proof. By adding a constant we can assume that $T \geq 1$ (in operator sense).

Let $u \in Q(T)$. By hypothesis, we have $\langle Tu, u\rangle \in L^1(M)$.

We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $1 \leq T(x) \in \operatorname{End}(E_x)$ to get $T(x) = \operatorname{diag}(c_1(x), c_2(x), \ldots, c_m(x))$, where $0 < c_j \in L^1_{\text{loc}}(M)$, $j = 1, 2, \ldots, m$ and $m = \dim E_x$.

Let $u_j(x)$ ($j = 1, 2, \ldots, m$) be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of E_x. Then for all $x \in M$

$$\langle Tu, u\rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u\rangle \, d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j|u_j|^2 \in L^1(M)$, for all $j = 1, 2, \ldots, m$.

Now, for all $x \in M$ and $j = 1, 2, \cdots, m$

$$2|c_j u_j| = 2|c_j||u_j| \leq |c_j| + |c_j||u_j|^2, \tag{2.1}$$

The right hand side of (2.1) is clearly in $L^1_{\text{loc}}(M)$. Therefore $c_j u_j \in L^1_{\text{loc}}(M)$.

But $(Tu)(x)$ has components $c_j(x)u_j(x)$ ($j = 1, 2, \ldots, m$) with respect to chosen bases of E_x. Therefore $Tu \in L^1_{\text{loc}}(E)$ and the lemma is proven. □

We will also need the following well-known lemma whose proof parallels Theorem 1 from [10] which dealt with magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, cf. also Lemma 2.1 in [9] for the case of magnetic Schrödinger operators on Riemannian manifolds.

Lemma 2.2. *$C_c^\infty(E)$ is dense in $\mathcal{H}_1 := W^{1,2}(E) \bigcap Q(V^+)$ with respect to norm*

$$\|u\|_+^2 = \int |\nabla u|^2 \, d\mu + \int \langle V^+ u, u\rangle \, d\mu + \|u\|^2, \tag{2.2}$$

where $\|\cdot\|$ is $L^2(E)$ norm.

Proof. 1. We will first show that compactly supported elements of $\mathcal{H}_1$ are dense in $\mathcal{H}_1$ with respect to the norm $\|\cdot\|_+$. Take any $u \in \mathcal{H}_1$, and let $u_k = \phi_k u$, where ϕ_k is as in Sect. 1.4. Since

$$\|\nabla(\phi_k u)\| = \|d\phi_k \otimes u + \phi_k \nabla u\| \leq \epsilon_k \|u\| + \|\nabla u\|,$$

it follows immediately that $\|\nabla(\phi_k u)\| \to \|\nabla u\|$ as $k \to \infty$, where $\|\cdot\|$ denotes the $L^2(T^*M \otimes E)$ norm.

Clearly $\phi_k u \to u$ in $L^2(E)$.

It remains to show that

$$\int \langle V^+ \phi_k u, \phi_k u\rangle \, d\mu \to \int \langle V^+ u, u\rangle \, d\mu \qquad \text{as } k \to \infty. \tag{2.3}$$

We have

$$\langle V^+ \phi_k u, \phi_k u\rangle = \phi_k^2 \langle V^+ u, u\rangle \leq \langle V^+ u, u\rangle \in L^1(M),$$

and, as $k \to \infty$,

$$\langle V^+\phi_k u, \phi_k u\rangle \to \langle V^+ u, u\rangle \quad \text{pointwise.}$$

Hence by the dominated convergence theorem we obtain (2.3).

2. We now show that compactly supported elements of $\mathcal{H}_1 \bigcap L^\infty(E)$ are dense in $\mathcal{H}_1$ in norm $\|\cdot\|_+$.

Take any compactly supported $u \in \mathcal{H}_1$. For every $R > 0$ define the truncation u_R of u by the formula

$$u_R(x) \;=\; \begin{cases} u(x), & \text{if} \quad |u(x)| \leq R; \\ R\frac{u(x)}{|u(x)|}, & \text{if} \quad |u(x)| > R. \end{cases}$$

The section u_R is a compactly supported element of $\mathcal{H}_1 \bigcap L^\infty(E)$.

It follows from Theorem A of the Appendix in [10] that $u_R \in W^{1,2}_{\text{comp}}(E)$ for all $R > 0$ and that $u_R \to u$ as $R \to \infty$ in $W^{1,2}_{\text{comp}}(E)$.

It remains to show that

$$\int \langle V^+ u_R, u_R\rangle\, d\mu \to \int \langle V^+ u, u\rangle\, d\mu \qquad \text{as } R \to \infty. \tag{2.4}$$

We have

$$\langle V^+ u_R, u_R\rangle \leq \langle V^+ u, u\rangle \in L^1(M),$$

and, as $R \to \infty$,

$$\langle V^+ u_R, u_R\rangle \to \langle V^+ u, u\rangle \quad \text{pointwise.}$$

By the dominated convergence theorem we obtain (2.4).

Therefore, $\|u_R - u\|_+ \to 0$ as $R \to \infty$. So our second claim is proven.

3. It remains to show that $C^\infty_c(E)$ is dense in the set of compactly supported elements of $\mathcal{H}_1 \bigcap L^\infty(E)$.

Let $u \in \mathcal{H}_1 \bigcap L^\infty(E)$ be compactly supported. Using a partition of unity we may assume that u is supported on a coordinate neighborhood. Let $u^\rho = \mathcal{J}^\rho u$, where $\mathcal{J}^\rho$ the Friedrichs mollifying operator as in Section 5.12 of [1]. It is well-known (cf. Lemma 5.12 in [1]) that $u^\rho \in C^\infty_c(E)$ and $u^\rho \to u$ as $\rho \to 0+$ both in the space $W^{1,2}(E)$ and in the space $L^2(E)$. In particular, $\|u^\rho\|_{W^{1,2}}$ is bounded for $0 < \rho < 1$. It remains to show that

$$\int \langle V^+ u^\rho, u^\rho\rangle\, d\mu \to \int \langle V^+ u, u\rangle\, d\mu \qquad \text{as } \rho \to 0+\,. \tag{2.5}$$

We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $0 \leq V^+(x) \in \text{End}(E_x)$ to get $V^+(x) = \text{diag}(c_1(x), c_2(x), \ldots, c_m(x))$, where $0 \leq c_j \in L^1_{\text{loc}}(M)$, $j = 1, 2, \ldots, m$ and $m = \dim E_x$.

Let $u_j^\rho(x)$ $(j = 1, 2, \ldots, m)$ be the components of $u^\rho(x) \in E_x$ with respect to the chosen orthogonal basis of E_x. Then for all $x \in M$

$$\langle V^+ u^\rho, u^\rho\rangle = \sum_{j=1}^{m} c_j |u_j^\rho|^2. \tag{2.6}$$

It remains to prove that for all $j = 1, 2, \ldots, m$

$$\int c_j |u_j^\rho|^2 \, d\mu \to \int c_j |u_j|^2 \, d\mu \qquad \text{as } \rho \to 0+\,. \tag{2.7}$$

Since $u \in L^\infty(E)$ is compactly supported and $c_j \in L^1_{\mathrm{loc}}(M)$, the dominated convergence theorem immediately implies (2.7).

Therefore $\|u^\rho - u\|_+ \to 0$ as $\rho \to 0+$. This proves the third claim and the lemma. □

In what follows, we adopt the scheme of proof from Cycon [4] to our setting with the help of refined integration by parts technique and the family of cut-off functions from Sect. 1.4.

Lemma 2.3. (i) *$C_c^\infty(E)$ is a form core of $H_{k,\max}$.*
(ii) *The operator $H_{k,\max}$ is semibounded below by -1.*

Proof. Using (1.8) and the spectral representation of $H_{k,\max}$ and $|H_{k,\max}|$, the following holds: for every $k \in \mathbb{Z}_+$ there exists a constant $c_k > 0$ such that

$$(\nabla^*\nabla w, w) + (V^+ w, w) \le c_k[(H_k w, w) + \|w\|^2] \le c_k[(|H_k| w, w) + \|w\|^2], \tag{2.8}$$

for all $w \in C_c^\infty(E)$.

By (ii) of Assumption B, Lemma 2.2, and by (2.8), it follows that $C_c^\infty(E)$ is a form core of $H_{k,\max}$. Thus (2.8) holds for all $w \in \mathrm{Dom}(H_{k,\max})$. Therefore $H_{k,\max}$ is semibounded below by -1. □

The following corollary follows immediately from Lemma 2.3.

Corollary 2.4. *$Q(H_{k,\max})$ is well-defined in the sense of Sect. 1.3.*

Lemma 2.5. *The following holds: $Q(H_{k,\max}) \subset W^{1,2}(E) \bigcap Q(V^+) \subset Q(V_k^-)$.*

Proof. By (1.8), Lemma 2.2 and Lemma 2.3, it follows that $Q(H_{k,\max}) \subset W^{1,2}(E) \bigcap Q(V^+)$.

By (1.8), it follows that

$$(V_k^- w, w) \le (\nabla^*\nabla w, w) + (V^+ w, w) + \|w\|^2, \qquad \text{for all } w \in C_c^\infty(E). \tag{2.9}$$

By Lemma 2.2, the inclusion $W^{1,2}(E) \bigcap Q(V^+) \subset Q(V_k^-)$ follows from (2.9). □

In what follows, we let $\{\phi_k\}_{k \in \mathbb{Z}_+}$ be as in Sect. 1.4.

Then

$$V_k(\phi_k u) = V^+(\phi_k u) - \chi_k V^-(\phi_k u) = V(\phi_k u), \tag{2.10}$$

where χ_k is as in Assumption B.

Lemma 2.6. *Assume that* $u \in \mathrm{Dom}(H_{V,\max})$. *Then* $\phi_k u \in Q(H_{k,\max}) \subset W^{1,2}(E) \bigcap Q(V^+)$.

Proof. In the proof we will use the arguments due to Kato, cf. Lemma 1 in [7].

$Q(H_{k,\max})$ is a Hilbert space with the inner product

$$(u,v)_k := h_k(u,v) + 2(u,v)_{L^2(E)},$$

where $h_k(\cdot,\cdot)$ is the sesquilinear form obtained by polarization from the quadratic form $h_k(\cdot)$ associated to $H_{k,\max}$. In view of Lemma 2.5, we have the following continuous inclusions

$$\mathrm{Dom}(H_{k,\max}) \subset Q(H_{k,\max}) \subset W^{1,2}(E) \subset L^2(E) \subset W^{-1,2}(E) \subset Q(H_{k,\max})^*, \tag{2.11}$$

where $Q(H_{k,\max})^*$ denotes the dual of $Q(H_{k,\max})$.

By a well-known abstract fact $H_{k,\max}\colon \mathrm{Dom}(H_{k,\max}) \to L^2(E)$ can be extended to a continuous linear operator $H^{'}_{k,\max}\colon Q(H_{k,\max}) \to Q(H_{k,\max})^*$. In fact, $H^{'}_{k,\max}$ is the restriction of the differential expression H_k to $Q(H_{k,\max})$ because for all $w \in Q(H_{k,\max})$, by Lemmas 2.1 and 2.5 it follows that $V^+w \in L^1_{\mathrm{loc}}(E)$ and $V_k^- w \in L^1_{\mathrm{loc}}(E)$, and hence $V_k w \in L^1_{\mathrm{loc}}(E)$.

By an abstract fact (cf. Remark after the proof of Theorem 2.1 in [5]), $H^{'}_{k,\max}+2\colon Q(H_{k,\max}) \to Q(H_{k,\max})^*$ is a topological isomorphism.

Let u be as in hypothesis of this lemma. Then

$$\begin{aligned} H_k(\phi_k u) &= \nabla^*\nabla(\phi_k u) + \phi_k(Vu) = -(g\otimes 1)\circ\nabla^1\circ\nabla(\phi_k u) + \phi_k(Vu) \\ &= -(g\otimes 1)\circ\nabla^1(d\phi_k\otimes u + \phi_k\nabla u) + \phi_k(Vu) \\ &= -(g\otimes 1)((\nabla^{LC}d\phi_k)\otimes u) - (g\otimes 1)(d\phi_k\otimes\nabla u) - (g\otimes 1)(d\phi_k\otimes\nabla u) - (g\otimes 1)(\phi_k\nabla^1\nabla u) + \phi_k(Vu) \\ &= \phi_k H_{V,\max}u - 2(g\otimes 1)(d\phi_k\otimes\nabla u) + (\Delta_M\phi_k)u, \end{aligned} \tag{2.12}$$

where ∇^{LC} and ∇^1 are as in Sect. 1.

Clearly $(H_k+2)(\phi_k u) \in W^{-1,2}(E) \subset Q(H_{k,\max})^*$. Thus we can find $s_k \in Q(H_{k,\max})$ such that

$$(H^{'}_{k,\max}+2)s_k = (H_k+2)(\phi_k u).$$

Since $s_k \in Q(H_{k,\max})$, Lemma 2.5 and Lemma 2.1 give $V_k s_k \in L^1_{\mathrm{loc}}(E)$. As above, $H^{'}_{k,\max}$ is the restriction of differential expression H_k to $Q(H_{k,\max})$, hence $(H_k+2)(s_k-\phi_k u)=0$. Denoting $w_k = s_k - \phi_k u$, we get $H_k w_k = -2w_k$. Since $V_k u \in L^1_{\mathrm{loc}}(E)$ and $V_k s_k \in L^1_{\mathrm{loc}}(E)$, it follows that $V_k w_k \in L^1_{\mathrm{loc}}(E)$. Since $w_k \in L^2(E)$, we immediately get $w_k \in \mathrm{Dom}(H_{k,\max})$. Therefore

$$(H_{k,\max}+2)w_k = 0. \tag{2.13}$$

But $H_{k,\max}+2$ is a positive self-adjoint operator, so (2.13) implies $w_k=0$, i.e. $\phi_k u = s_k$. This shows that $\phi_k u \in Q(H_{k,\max})$.

By Lemma 2.5, it follows immediately that $\phi_k u \in W^{1,2}(E)\bigcap Q(V^+)$. □

Lemma 2.7. *Assume that* $u \in \mathrm{Dom}(H_{V,\max})$. *Then* $\phi_k u \in Q(H_{\bar{h}})$.

Proof. By an abstract fact, $Q(H_{\bar{h}})$ is the closure of $C_c^\infty(E)$ in the norm

$$\|u\|_h = h(u) + \|u\|^2 = \int |\nabla u|^2\, d\mu + \int \langle Vu, u\rangle\, d\mu + \|u\|^2, \tag{2.14}$$

where $\|\cdot\|$ is the $L^2(E)$ norm.

To prove the lemma, we need to find a sequence $v_j \in C_c^\infty(E)$ such that $\|v_j - \phi_k u\|_h \to 0$ as $j \to \infty$.

By Lemma 2.6, $\phi_k u \in W^{1,2}(E) \bigcap Q(V^+)$.

By Lemma 2.2 there exists a sequence $v_j \in C_c^\infty(E)$ such that

$$\|v_j - \phi_k u\|_+ \to 0 \qquad \text{as } j \to \infty, \tag{2.15}$$

where norm $\|\cdot\|_+$ is as in (2.2).

By Assumption A we have

$$(V^- v_j, v_j) + \|v_j\|^2 \leq (\nabla v_j, \nabla v_j) + (V^+ v_j, v_j) + \|v_j\|^2.$$

From (2.15) it follows that $\{v_j\}$ is a Cauchy sequence with respect to $\|\cdot\|_+$. Thus $\{v_j\}$ is a Cauchy sequence in $Q(V^-)$ with respect to the norm $\|\cdot\|_{V^-}$, where $Q(V^-) = \{u \in L^2(E)\colon \langle V^- u, u\rangle \in L^1(M)\}$ is the domain of the quadratic form

$$b(u) := \int \langle V^- u, u\rangle\, d\mu$$

and $\|\cdot\|_{V^-}^2 := b(\cdot) + \|\cdot\|^2$.

Since b is a closed quadratic form (see Example 1.5 in Sec. VI.1.2 and Example 1.15 in Sec. VI.1.3 of [8]), it follows that the sequence $\{v_j\}$ converges in $Q(V^-)$ with respect to the norm $\|\cdot\|_{V^-}$ to some element $z \in Q(V^-)$. In particular, $v_j \to z$ in $L^2(E)$. By (2.15) we know, in particular, that $v_j \to \phi_k u$ in $L^2(E)$. Thus $\phi_k u = z \in Q(V^-)$ and

$$(V^-(v_j - \phi_k u), (v_j - \phi_k u)) \to 0 \qquad \text{as } j \to \infty. \tag{2.16}$$

Now (2.16) and (2.15) imply $\|v_j - \phi_k u\|_h \to 0$ as $j \to \infty$, and the lemma is proven. □

Lemma 2.8. *If $v \in Q(H_{k,\max})$, then $\phi_k v \in Q(H_{k,\max}) \bigcap Q(H_{\bar{h}})$.*

Proof. $Q(H_{k,\max})$ is a Hilbert space with the norm

$$\|v\|_{h_k}^2 = h_k(v) + 2\|v\|^2, \tag{2.17}$$

where h_k is the quadratic form associated to self-adjoint operator $H_{k,\max} \geq -1$, cf. (i) of Assumption B and Lemma 2.3.

Since $C_c^\infty(E)$ is a form core of h_k (cf. Lemma 2.3), there exists a sequence $v_j \in C_c^\infty(E)$ such that

$$\|v_j - v\|_{h_k} \to 0 \qquad \text{as } j \to \infty. \tag{2.18}$$

Since $h_k \geq -1$, we have

$$(V_k^- \phi_k(v_j - v), \phi_k(v_j - v)) \leq \|\nabla(\phi_k(v_j - v))\|^2 + (V^+ \phi_k(v_j - v), \phi_k(v_j - v)) + \|\phi_k(v_j - v)\|^2. \quad (2.19)$$

We will now estimate the terms on the right hand side of (2.19).

$$\|\nabla(\phi_k(v_j - v))\| = \|d\phi_k \otimes (v_j - v) + \phi_k \nabla(v_j - v)\| \leq \epsilon_k \|v_j - v\| + \|\nabla(v_j - v)\|, \quad (2.20)$$

where ϵ_k is as in (d) of Sect. 1.4, and $\|\cdot\|$ is L^2 norm.

Since $0 \leq \phi_k \leq 1$, we obtain

$$(V^+ \phi_k(v_j - v), \phi_k(v_j - v)) \leq (V^+(v_j - v), (v_j - v)). \quad (2.21)$$

By (2.18), (2.19), (2.20) and (2.21) it follows that $\phi_k v \in Q(H_{k,\max})$ and

$$\|\phi_k v_j - \phi_k v\|_{h_k} \to 0 \qquad \text{as } j \to \infty.$$

From (2.14) and (2.17) we obtain

$$\|\phi_k v_j - \phi_k v\|_{\bar{h}}^2 = \|\phi_k v_j - \phi_k v\|_{h_k}^2 - \|\phi_k v_j - \phi_k v\|^2, \quad (2.22)$$

where $\|\cdot\|$ is the norm in $L^2(E)$. Letting $j \to \infty$ in (2.22) we get $\phi_k v \in Q(H_{\bar{h}})$. □

Lemma 2.9. *If* $u \in \mathrm{Dom}(H_{\bar{h}})$*, then* $\phi_k u \in Q(H_{k,\max})$.

Proof. Since $u \in \mathrm{Dom}(H_{\bar{h}}) \subset Q(H_{\bar{h}})$ and $Q(H_{\bar{h}})$ is the closure of $C_c^\infty(E)$ in the norm (2.14), there exists a sequence $u_j \in C_c^\infty(E)$ such that

$$\|u_j - u\|_h \to 0 \qquad \text{as } j \to \infty, \quad (2.23)$$

where $\|\cdot\|_h$ is as in (2.14).

Let $v \in Q(H_{k,\max})$. By Lemma 2.8 it follows that $\phi_k v \in Q(H_{k,\max}) \bigcap Q(H_{\bar{h}})$. Since $C_c^\infty(E)$ is a form core of h_k (cf. Lemma 2.3), there exists a sequence $v_l \in C_c^\infty(E)$ such that

$$\|v_l - v\|_{h_k} \to 0 \qquad \text{as } l \to \infty, \quad (2.24)$$

Thus

$$(H_{\bar{h}} u, \phi_k v) = \lim_{j\to\infty, l\to\infty} \bar{h}(u_j, \phi_k v_l) = \lim_{j\to\infty, l\to\infty} (H_{\bar{h}} u_j, \phi_k v_l) = \lim_{j\to\infty, l\to\infty} (H_V u_j, \phi_k v_l). \quad (2.25)$$

By (2.12) we obtain

$$(H_V(\phi_k u_j), v_l) = (\phi_k H_V u_j, v_l) - (2(g \otimes 1)(d\phi_k \otimes \nabla u_j), v_l) + ((\Delta_M \phi_k) u_j, v_l). \quad (2.26)$$

We will now rewrite the second term on the right hand side of (2.26).

$$\begin{aligned} 2((g \otimes 1)(d\phi_k \otimes \nabla u_j), v_l)_{L^2(E)} &= 2(\nabla u_j, d\phi_k \otimes v_l)_{L^2(T^*M \otimes E)} = 2(u_j, \nabla^*(d\phi_k \otimes v_l))_{L^2(E)} \\ &= 2(u_j, -\nabla_{X_k} v_l)_{L^2(E)} + 2(u_j, -\operatorname{div}(X_k) v_l)_{L^2(E)}, \quad (2.27) \end{aligned}$$

where X_k is the vector field associated to $d\phi_k$ via metric g, i.e. $X_k = \operatorname{grad}\phi_k$. The last equality in (2.27) follows from Proposition 1.4 of Appendix C in [14]. Since $-\operatorname{div}(\operatorname{grad}\phi_k) = \Delta_M\phi_k$, we obtain

$$2((g\otimes 1)(d\phi_k\otimes\nabla u_j), v) = 2(u_j, -\nabla_{X_k}v_l) + 2(u_j, (\Delta_M\phi_k)v_l). \tag{2.28}$$

By (2.26) and (2.28) we get

$$(H_V(\phi_k u_j), v_l) = (\phi_k H_V u_j, v_l) + 2(u_j, \nabla_{X_k}v_l) - (u_j, (\Delta_M\phi_k)v_l). \tag{2.29}$$

We will now show that $\phi_k u_j$ converges weakly in the Hilbert space $Q(H_{k,\max})$ (with norm $\|\cdot\|_{h_k}$ as in (2.17)).

Fix $v \in Q(H_{k,\max})$, and let u_j and v_l be as in (2.23) and (2.24). Using (2.25) and (2.29) we obtain

$$\begin{aligned}\lim_{j\to\infty}(\phi_k u_j, v)_{h_k} &= \lim_{j\to\infty}[(H_k(\phi_k u_j), v) + 2(\phi_k u_j, v)] = \lim_{j\to\infty, l\to\infty}(H_k(\phi_k u_j), v_l) + 2(\phi_k u, v)\\ &= \lim_{j\to\infty, l\to\infty}[(H_{\bar h}u_j, \phi_k v_l) + 2(u_j, \nabla_{X_k}v_l) - (u_j, (\Delta_M\phi_k)v_l)] + 2(\phi_k u, v)\\ &= (H_{\bar h}u, \phi_k v) + 2(u, \nabla_{X_k}v) - (u, (\Delta_M\phi_k)v) + 2(\phi_k u, v).\end{aligned}$$

Here $(\cdot,\cdot)_{h_k}$ denotes the inner product in $Q(H_{k,\max})$ whose norm is given in (2.17), and $(\cdot,\cdot)$ is the inner product in $L^2(E)$.

This shows that $\phi_k u_j$ converges weakly in $Q(H_{k,\max})$ as $j\to\infty$. We will denote the weak limit (as $j\to\infty$) of $\phi_k u_j$ in $Q(H_{k,\max})$ by z_k. We will show that $z_k = \phi_k u$.

We know that for every $f \in Q(H_{k,\max})^*$, the following holds:

$$f(\phi_k u_j) \to f(z_k) \qquad \text{as } j\to\infty. \tag{2.30}$$

Since $Q(H_{k,\max}) \subset L^2(E) \subset Q(H_{k,\max})^*$, cf. (2.11), it follows that (2.30) holds for all $f \in L^2(E)$. This means that $\phi_k u_j \to z_k$ weakly in $L^2(E)$, as $j\to\infty$.

Since $\|\phi_k u_j - \phi_k u\|_{L^2(E)} \to 0$ as $j\to\infty$, it follows that $\phi_k u = z_k$. But $z_k \in Q(H_{k,\max})$, so $\phi_k u \in Q(H_{k,\max})$ and the lemma is proven. □

Lemma 2.10. *The following operator relation holds:* $H_{\bar h} \subset H_{V,\max}$.

Proof. Let $u \in \operatorname{Dom}(H_{\bar h})$. By Lemma 2.9 we get $\phi_k u \in Q(H_{k,\max})$. By Lemma 2.5, we obtain $\phi_k u \in W^{1,2}(E)\bigcap Q(V^+) \subset Q(V_k^-)$. From Lemma 2.1 with u replaced by $\phi_k u$, we obtain $V_k\phi_k u \in L^1(E)$. By (2.10), $V_k\phi_k u = V\phi_k u$, so $V\phi_k u \in L^1(E)$. Since $k \in \mathbb{Z}_+$ is arbitrary, it follows that $Vu \in L^1_{\text{loc}}(E)$.

Since $H_V u = H_{\bar h}u \in L^2(E)$ we get $u \in \operatorname{Dom}(H_{V,\max})$ and $H_{\bar h}u = H_{V,\max}u$. This concludes the proof of the lemma. □

Lemma 2.11. *Suppose that* $u \in \operatorname{Dom}(H_{V,\max})$*. Then*

$$(\nabla(\phi_k u), \nabla(\phi_k u)) + \int \langle V^+ \phi_k u, \phi_k u\rangle \, d\mu - \int \langle V_k^- \phi_k u, \phi_k u\rangle \, d\mu$$
$$= \operatorname{Re}(\phi_k H_{V,\max} u, \phi_k u) + \|d\phi_k \otimes u\|^2. \quad (2.31)$$

Proof. By Lemma 2.6 it follows that $\phi_k u \in Q(H_{k,\max})$. Using (2.10) and Lemma 2.5, we immediately obtain $\phi_k u \in W^{1,2}(E) \bigcap Q(V^+) \bigcap Q(V^-)$.

Using integration by parts (cf. Lemma 8.2 in [1]), we get

$$\begin{aligned}(\nabla(\phi_k u), \nabla(\phi_k u)) &= (d\phi_k \otimes u, \nabla(\phi_k u)) + (\phi_k \nabla u, \nabla(\phi_k u)) = (d\phi_k \otimes u, \nabla(\phi_k u)) + (\nabla u, \phi_k \nabla(\phi_k u)) \\ &= (d\phi_k \otimes u, d\phi_k \otimes u) + (d\phi_k \otimes u, \phi_k \nabla u) - (\nabla u, \phi_k (d\phi_k \otimes u)) + (\nabla u, \nabla(\phi_k^2 u)) \\ &= \|d\phi_k \otimes u\|^2 + 2i \operatorname{Im}(d\phi_k \otimes u, \phi_k \nabla u) + (\phi_k \nabla^* \nabla u, \phi_k u).\end{aligned}$$

Adding this formula with its complex conjugate and dividing by 2, we obtain

$$(\nabla(\phi_k u), \nabla(\phi_k u)) = \|d\phi_k \otimes u\|^2 + \operatorname{Re}(\phi_k \nabla^* \nabla u, \phi_k u). \quad (2.32)$$

Since $0 \leq \int \langle V^+ \phi_k u, \phi_k u\rangle \, d\mu < +\infty$ and $0 \leq \int \langle V_k^- \phi_k u, \phi_k u\rangle \, d\mu < +\infty$, we can add $\int \langle V^+ \phi_k u, \phi_k u\rangle \, d\mu - \int \langle V_k^- \phi_k u, \phi_k u\rangle \, d\mu$ to both sides of (2.32) to get

$$(\nabla(\phi_k u), \nabla(\phi_k u)) + \int \langle V^+ \phi_k u, \phi_k u\rangle \, d\mu - \int \langle V_k^- \phi_k u, \phi_k u\rangle \, d\mu$$
$$= \|d\phi_k \otimes u\|^2 + \operatorname{Re}(\phi_k \nabla^* \nabla u, \phi_k u) + \int \langle (V^+ - V_k^-) \phi_k u, \phi_k u\rangle \, d\mu. \quad (2.33)$$

Since $u \in \operatorname{Dom}(H_{V,\max})$, by Lemma 2.6 it follows that $u \in W^{1,2}_{\text{loc}}(E)$ and hence $\nabla^* \nabla u \in W^{-1,2}_{\text{loc}}(E)$. By definition of $\operatorname{Dom}(H_{V,\max})$, it follows that $H_V u \in L^2(E)$ and $Vu \in L^1_{\text{loc}}(E)$. Therefore, $Vu = (H_V - \nabla^* \nabla) u \in W^{-1,2}_{\text{loc}}(E) \bigcap L^1_{\text{loc}}(E)$. Also,

$$\langle (V^+ - V_k^-)(\phi_k u), \phi_k u\rangle \geq -\langle V_k^-(\phi_k u), \phi_k u\rangle \in L^1(M).$$

By the main theorem from [2], we obtain

$$\int \big\langle (V^+ - V_k^-)(\phi_k u), \phi_k u \big\rangle \, d\mu = \big((V^+ - V_k^-)(\phi_k u), \phi_k u \big), \quad (2.34)$$

where $(\cdot, \cdot)$ on the right hand side denotes the duality between $W^{-1,2}_{\text{loc}}(E)$ and $W^{1,2}_{\text{comp}}(E)$.

By (2.34), we can rewrite the right hand side of (2.33) as

$$\|d\phi_k \otimes u\|^2 + \operatorname{Re}(\phi_k \nabla^* \nabla u, \phi_k u) + ((V^+ - V_k^-) \phi_k u, \phi_k u), \quad (2.35)$$

where $(\cdot, \cdot)$ in the last two terms denotes the duality between $W^{-1,2}_{\text{loc}}(E)$ and $W^{1,2}_{\text{comp}}(E)$.

Adding the last two terms in (2.35), we obtain the following form of the right hand side of (2.33)

$$\|d\phi_k \otimes u\|^2 + \operatorname{Re}(\phi_k (\nabla^* \nabla + V^+ - V_k^-) u, \phi_k u), \quad (2.36)$$

where $(\cdot,\cdot)$ denotes the duality between $W^{-1,2}_{\mathrm{loc}}(E)$ and $W^{1,2}_{\mathrm{comp}}(E)$.

Since $H_V u \in L^2(E)$, we can apply Lemma 8.3 from [1] to conclude that the duality $(\cdot,\cdot)$ in the last term in (2.36) coincides with the inner product in $L^2(E)$.

From (2.36) and (2.33), the equation (2.31) follows immediately. □

2.12. **Proof of Theorem 1.6.** By Lemma 2.10 it follows that $H_{\bar{h}} \subset H_{V,\max}$, so it remains to prove $H_{V,\max} \subset H_{\bar{h}}$. Clearly, we only need to show $\mathrm{Dom}(H_{V,\max}) \subset \mathrm{Dom}(H_{\bar{h}})$.

Assume that $v \in \mathrm{Dom}(H_{V,\max})$. By (1.5), $H_{\bar{h}}$ is a non-negative self-adjoint operator, hence $(H_{\bar{h}}+1)^{-1}$ is a bounded linear operator on $L^2(E)$. Let

$$s = (H_{\bar{h}}+1)^{-1}(H_{V,\max}+1)v.$$

Then $(H_{\bar{h}}+1)s = (H_{V,\max}+1)v$. Since $H_{\bar{h}} \subset H_{V,\max}$, it follows that $s \in \mathrm{Dom}(H_{V,\max})$ and $(H_{\bar{h}}+1)s = (H_{V,\max}+1)s = (H_{V,\max}+1)v$. Denoting $u = v - s$, we get

$$(H_{V,\max}+1)u = 0. \tag{2.37}$$

Since $u \in \mathrm{Dom}(H_{V,\max})$, Lemma 2.7 shows that $\phi_k u \in Q(H_{\bar{h}})$. By (1.5) $H_{\bar{h}} \geq 0$, so the quadratic form $\bar{h}$ associated to $H_{\bar{h}}$ is also positive and hence $\bar{h}(\phi_k u) \geq 0$. This and Lemma 2.11 give

$$0 \leq \mathrm{Re}(\phi_k H_{V,\max} u, \phi_k u) + \|d\phi_k \otimes u\|^2. \tag{2.38}$$

From (2.38) and (2.37) we obtain

$$\|\phi_k u\|^2 \leq \|d\phi_k \otimes u\|^2. \tag{2.39}$$

By property (d) of Sect. 1.4 we have

$$\|d\phi_k \otimes u\|^2 \leq \epsilon_k^2 \|u\|^2.$$

Using (c) of Sect. 1.4 and (2.39), for any compact $K \subset M$, we obtain for $k \geq k_0(K)$:

$$\int_K |u|^2 \, d\mu = \int_K |\phi_k u|^2 \, d\mu \leq \epsilon_k^2 \int_M |u|^2 \, d\mu. \tag{2.40}$$

Letting $k \to \infty$ in (2.40) we get $u = 0$ on K. Since K is arbitrary, it follows that $u = 0$, i.e. $v = s$. Therefore $v \in \mathrm{Dom}(H_{\bar{h}})$ and the theorem is proven. □

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