Title: On holomorphic families of Schrödinger-type operators with singular potentials on manifolds of bounded geometry

Author: Ognjen Milatovic

Address for correspondence: Department of Mathematics, University of Toledo, Toledo, OH 43606-3390, USA.

Phone: (419) 530-2425

E-mail address: Ognjen.Milatovic@utoledo.edu

ABSTRACT

We consider a family of Schrödinger-type differential expressions $L(\kappa) = D^2 + V + \kappa V^{(1)}$, where $\kappa \in \mathbb{C}$, and D is the Dirac operator associated with a Clifford bundle (E, ∇^E) of bounded geometry over a manifold of bounded geometry (M, g) with metric g, and V and $V^{(1)}$ are self-adjoint locally integrable sections of $\operatorname{End} E$. We also consider the family $I(\kappa) = (\nabla^F)^*\nabla^F + V + \kappa V^{(1)}$, where $\kappa \in \mathbb{C}$, and ∇^F is a Hermitian connection on a Hermitian vector bundle F of bonded geometry over a manifold of bounded geometry (M, g), and V and $V^{(1)}$ are self-adjoint locally integrable sections of $\operatorname{End} F$. We give sufficient conditions for $L(\kappa)$ and $I(\kappa)$ to have a realization in $L^2(E)$ and $L^2(F)$, respectively, as self-adjoint holomorphic families of type (B). In the proofs we use Kato's inequality for Bochner Laplacian operator and Weitzenböck formula.

Key words: Schrödinger operator, holomorphic family, manifold, bounded geometry, singular potential

2000 Mathematics Subject Classification: 58J50; 35P05

1. Introduction and the main results

1.1. **The setting.** Let (M,g) be a C^∞ Riemannian manifold without boundary, with metric $g=(g_{jk})$ and $\dim M=n$. We will assume that M is connected and oriented. By $d\mu$ we will denote the Riemannian volume element of M. In any local coordinates $x^1,\dots,x^n$, we have $d\mu=\sqrt{\det(g_{jk})}\,dx^1dx^2\dots dx^n$.

In what follows, by T_xM, TM and T^*M we will denote the tangent space of M at $x\in M$, tangent bundle of M and cotangent bundle of M respectively, and by ∇^g we will denote the Levi-Civita connection on TM.

Let E be a Hermitian vector bundle over M such that each fibre E_x at $x\in M$ is a module over the Clifford algebra $\mathcal{C}\ell(T_xM)$ and

$$\langle \xi\cdot u, v\rangle_{E_x}+\langle u,\xi\cdot v\rangle_{E_x}=0,\quad \text{for all } \xi\in T_xM \text{ and all } u,v\in E_x,$$

where $\langle\cdot,\cdot\rangle_{E_x}$ denotes the fibrewise inner product in E_x and $\cdot$ denotes the Clifford action.

Moreover, we assume that E is endowed with a Hermitian connection ∇^E satisfying the property

$$\nabla^E_X(Y\cdot s) \;=\; \left(\nabla^g_XY\right)\cdot s+Y\cdot\left(\nabla^E_Xs\right),\quad \text{for all } s\in C^\infty(E) \text{ and } Y,X\in C^\infty(TM).$$

Here $\cdot$ denotes the Clifford action, and $C^\infty(E)$ and $C^\infty(TM)$ denote smooth sections of E and TM respectively.

The pair (E,∇^E) satisfying the properties described in the preceding two paragraphs is called a Clifford bundle; see, for example, Definition 2.3 in [9].

The composition

$$C^\infty(E)\longrightarrow C^\infty(T^*M\otimes E)\longrightarrow C^\infty(TM\otimes E)\longrightarrow C^\infty(E),$$

where the first arrow is given by the connection ∇^E, the second—by the metric, and the third—by the Clifford action, defines a first order differential operator

$$D\colon C^\infty(E)\to C^\infty(E). \tag{1.1}$$

The operator D is called the Dirac operator corresponding to the Clifford bundle (E,∇^E); see, for example, Definition 2.4 in [9].

The operator D is formally self-adjoint:

$$(Du,v)=(u,Dv),\quad \text{for all } u\in C^\infty(E) \text{ and } v\in C_c^\infty(E), \tag{1.2}$$

where $C_c^\infty(E)$ denotes smooth compactly supported sections of E.

For the proof of (1.2) see, for example, Proposition 2.9 in [9].

We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u,v) \;=\; \int_M \langle u(x),v(x)\rangle\, d\mu(x). \tag{1.3}$$

Here $\langle\cdot,\cdot\rangle$ denotes the fibrewise inner product in E_x.

We will consider the following family of Schrödinger-type differential expressions in $L^2(E)$:

$$L(\kappa) := D^2 + V + \kappa V^{(1)},$$

where D is as in (1.1), $\kappa \in \mathbb{C}$, and $V \in L^1_{\text{loc}}(\operatorname{End} E)$ and $V^{(1)} \in L^1_{\text{loc}}(\operatorname{End} E)$ are linear self-adjoint bundle endomorphisms.

We want to give a sufficient condition for $L(\kappa)$ to have a realization in $L^2(E)$ as a self-adjoint holomorphic family of type (B).

1.2. **Self-adjoint holomorphic families of operators.** Here we review some terminology from Section VII.3.1 in [7]. Let $\mathcal{H}$ be a Hilbert space and let $T(\kappa)$ be a family of closed operators in $\mathcal{H}$, holomorphic in the sense of the definition in Section VII.1.2 of [7], for κ in a domain $U_0 \subset \mathbb{C}$ which is symmetric with respect to the real axis. Suppose also that for all $\kappa \in U_0$, the operator $T(\kappa)$ is densely defined and $(T(\kappa))^* = T(\bar{\kappa})$. We will then call $T(\kappa)$ a self-adjoint holomorphic family. Clearly, $T(\kappa)$ is a self-adjoint operator for all real $\kappa \in U_0$.

1.3. **Holomorphic families of operators of type (B).** Here we review some terminology from Section VII.4.2 of [7]. Let $t(\kappa)$ be a family of sesquilinear forms in a Hilbert space $\mathcal{H}$ defined for all $\kappa \in U_0$, where U_0 is a domain in $\mathbb{C}$. For each $\kappa \in U_0$, let $\operatorname{Dom}(t(\kappa))$ denote the domain of the form $t(\kappa)$. The family $t(\kappa)$ is called a holomorphic family of type (a) if

(i) each $t(\kappa)$ is sectorial and closed with $\operatorname{Dom}(t(\kappa)) = G$ independent of κ and dense in $\mathcal{H}$, and
(ii) for each fixed $u \in G$, $t(\kappa)(u)$ is holomorphic for $\kappa \in U_0$. Here $t(\kappa)(\cdot)$ denotes the quadratic form corresponding to the sesquilinear form $t(\kappa)(\cdot,\cdot)$.

Note that (ii) implies, by polarization, that $t(\kappa)(u, v)$ is holomorphic in $\kappa \in U_0$ for each fixed pair u, $v \in G$.

If $t(\kappa)$, $\kappa \in U_0$, is a holomorphic family of type (a), then by Theorem VI.2.7 in [7] it follows that for each $\kappa \in U_0$, one can associate to $t(\kappa)$ a unique m-sectorial operator $T(\kappa)$ such that $\operatorname{Dom}(T(\kappa)) \subset \operatorname{Dom}(t(\kappa))$ and

$$t(\kappa)(u, v) = (T(\kappa)u, v), \quad \text{for all } u \in \operatorname{Dom}(T(\kappa)) \text{ and } v \in \operatorname{Dom}(t(\kappa)),$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$.

By Theorem VII.4.2, it follows that $T(\kappa)$ form a holomorphic family of operators. A holomorphic family of m-sectorial operators associated with a holomorphic family of forms of type (a) in the above described way is called a holomorphic family of type (B); see Section VII.4.2 in [7].

In what follows, we will denote by ∇ Hermitian connections on all tensor bundles $T^p_q \otimes E$ induced by the Levi-Civita connection ∇^g and ∇^E.

We now make the assumptions on (M, g) and (E, ∇^E).

Assumption A.

(i) Assume that (M, g) has bounded geometry, i.e. $r_{\text{inj}} > 0$ and

$$\left|\nabla^i R\right| \leq C_i, \quad \text{for all } i = 1, 2, \dots,$$

where $C_i \geq 0$ are constants. Here r_{inj} denotes the injectivity radius of (M, g) and R denotes the curvature tensor associated to the Levi-Civita connection.

(ii) We also assume that

$$\left|\nabla^i R^E\right| \leq C_i, \quad \text{for all } i = 1, 2, \ldots,$$

where $C_i \geq 0$ are constants and R^E denotes the curvature tensor associated to the connection ∇^E on E.

We now make assumptions on V and $V^{(1)}$.

Assumption B.

(i) Assume that

$$V = V_1 + V_2 \quad \text{and} \quad V^{(1)} = V_1^{(1)} + V_2^{(1)}$$

where $0 \leq V_1 \in L^1_{\text{loc}}(\operatorname{End} E)$, $0 \leq V_1^{(1)} \in L^1_{\text{loc}}(\operatorname{End} E)$, $0 \geq V_2 \in L^1_{\text{loc}}(\operatorname{End} E)$ and $0 \geq V_2^{(2)} \in L^1_{\text{loc}}(\operatorname{End} E)$ are linear self-adjoint bundle endomorphisms (here the inequalities are understood in the sense of operators $E_x \to E_x$).

(ii) Assume that for all $x \in M$,

$$V_1^{(1)}(x) \leq \beta V_1(x), \tag{1.4}$$

where $\beta > 0$ is a constant, and the inequality (1.4) is understood in the sense of operators $E_x \to E_x$.

We will also make domination-type assumptions on V_2 and $V_2^{(1)}$. To do this, we will need some notations on Sobolev spaces and quadratic forms.

1.4. **Sobolev spaces.** By $W^{1,2}(E)$ we will denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_{W^{1,2}(E)}$ defined by the scalar product

$$(u, v)_{W^{1,2}(E)} := \int \langle u, v\rangle \, d\mu + \int \langle \nabla^E u, \nabla^E v\rangle \, d\mu \qquad u, v \in C_c^\infty(E). \tag{1.5}$$

By $H^{1,2}(E)$ we will denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_{H^{1,2}(E)}$ defined by the scalar product

$$(u, v)_{H^{1,2}(E)} := \int \langle u, v\rangle \, d\mu + \int \langle Du, Dv\rangle \, d\mu \qquad u, v \in C_c^\infty(E),$$

where D is as in (1.1).

Remark 1.5. If (M, g) and (E, ∇^E) satisfy Assumption A, it follows by Theorem 2.3 in [4] or by Theorem 3.5 in [10] that $W^{1,2}(E) = H^{1,2}(E)$. Moreover, with our assumptions on (M, g) and (E, ∇^E), by Lemma 3.2 in [10] it follows that $H^{1,2}(E) = \{u \in L^2(E) : Du \in L^2(E)\}$, and by Proposition 2.4 in [5], it follows that $W^{1,2}(E) = \{u \in L^2(E) : \nabla^E u \in L^2(T^*M \otimes E)\}$.

By Remark 1.5, from now on, we will use the same notation $W^{1,2}(E)$ for both Sobolev spaces defined in Sect. 1.4.

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

1.6. **Quadratic forms.** In what follows, all quadratic forms are considered in the Hilbert space $L^2(E)$.

1. By h_0 we denote the quadratic form

$$h_0(u) = \int \langle Du, Du\rangle \, d\mu \tag{1.6}$$

with the domain $\mathrm{Dom}(h_0) = W^{1,2}(E) \subset L^2(E)$. The quadratic form h_0 is non-negative, densely defined (since $C_c^\infty(E) \subset \mathrm{Dom}(h_0)$) and closed (see Sect. 1.4).

2. By h_1 we denote the quadratic form

$$h_1(u) = \int \langle V_1 u, u\rangle \, d\mu \tag{1.7}$$

with the domain

$$\mathrm{Dom}(h_1) = \Big\{u \in L^2(E) : \int \langle V_1 u, u\rangle \, d\mu < +\infty\Big\}. \tag{1.8}$$

Since $0 \le V_1 \in L^1_{\mathrm{loc}}(\mathrm{End}\, E)$, it follows that h_1 is non-negative and densely defined ($C_c^\infty(E) \subset \mathrm{Dom}(h_1)$). Moreover, the form h_1 is closed. Indeed, by Theorem VI.1.11 in [7], it suffices to show that the pre-Hilbert space $\mathrm{Dom}(h_1)$ with the inner product

$$(u, v)_{h_1} = h_1(u, v) + (u, v) = \int \langle V_1 u, v\rangle \, d\mu + (u, v),$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(E)$, is complete.

By (1.8) it follows that $\mathrm{Dom}(h_1)$ is the set of all $u \in L^2(E)$ such that $\|u\|^2_{h_1} < +\infty$, where $\|\cdot\|_{h_1}$ denotes the norm corresponding to the inner product $(\cdot, \cdot)_{h_1}$. By Example VI.1.15 in [7], it follows that $\mathrm{Dom}(h_1)$ is complete.

3. By h_2 we denote the quadratic form

$$h_2(u) = \int \langle V_2 u, u\rangle \, d\mu \tag{1.9}$$

with the domain

$$\mathrm{Dom}(h_2) = \Big\{u \in L^2(E) : \int |\langle V_2 u, u\rangle| \, d\mu < +\infty\Big\}. \tag{1.10}$$

Since $0 \ge V_2 \in L^1_{\mathrm{loc}}(\mathrm{End}\, E)$, it follows that $C_c^\infty(E) \subset \mathrm{Dom}(h_2)$; thus, h_2 is a densely defined form. Moreover, h_2 is symmetric (but not semi-bounded below).

4. By $h_1^{(1)}$ we denote the quadratic form (1.7) with V_1 replaced by $V_1^{(1)}$ with the domain as in (1.8) with V_1 replaced by $V_1^{(1)}$. As in 2) above, it follows that $h_1^{(1)}$ is a non-negative, densely defined and closed form.

5. By $h_2^{(1)}$ we denote the quadratic form (1.9) with V_2 replaced by $V_2^{(1)}$ with the domain as in (1.10) with V_2 replaced by $V_2^{(1)}$. As in 3) above, the form $h_2^{(1)}$ is densely defined and symmetric (but not semi-bounded below).

We make the following assumptions on h_2 and $h_2^{(1)}$.

Assumption C1. Assume that h_2 is h_0-bounded with relative bound $0 \le b < 1$, i.e.

(i) $\mathrm{Dom}(h_2) \supset \mathrm{Dom}(h_0)$

(ii) there exist constants $a \ge 0$ and $0 \le b < 1$ such that

$$|h_2(u)| \le a\|u\|^2 + b|h_0(u)|, \quad \text{for all } u \in \mathrm{Dom}(h_0), \tag{1.11}$$

where $\|\cdot\|$ denotes the norm in $L^2(E)$.

Assumption C2. Assume that $h_2^{(1)}$ is h_0-bounded with relative bound $\widetilde{b} \ge 0$, i.e. assume that (i) and (ii) of Assumption C1 hold with h_2 replaced by $h_2^{(1)}$, with a replaced by some constant $\widetilde{a} \ge 0$ and b replaced by some constant $\widetilde{b} \ge 0$ (we do not assume $\widetilde{b} < 1$).

Remark 1.7. With our assumptions on (M, g) and (E, ∇^E), Assumptions C1 and C2 hold if $V_2 \in L^p(\mathrm{End}\, E)$ and $V_2^{(1)} \in L^p(\mathrm{End}\, E)$, where $p = n/2$ for $n \ge 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$. The proof is given in Sec. 5.

We now state the main results.

Theorem 1.8. *Assume that (M, g) is a manifold of bounded geometry and (E, ∇^E) is a Clifford bundle over M satisfying Assumption A. Suppose that Assumptions B, C1 and C2 hold. Then there exists a self-adjoint holomorphic family $H(k)$ in $L^2(E)$ of type (B), defined for all κ in the disc $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$, such that $H(\kappa)u = L(\kappa)u$ for all $u \in \mathrm{Dom}(H(\kappa))$, where*

$$\mathrm{Dom}(H(\kappa)) = \left\{u \in W^{1,2}(E) : \int \langle V_1 u, u\rangle \, d\mu < +\infty \text{ and } L(\kappa)u \in L^2(E)\right\}. \tag{1.12}$$

In the next theorem, (M, g) is a manifold of bounded geometry, F is a Hermitian vector bundle over M and ∇^F is a Hermitian connection on F. We will consider the following family of Schrödinger-type differential expressions in $L^2(F)$:

$$I(\kappa) := \left(\nabla^F\right)^* \nabla^F + V + \kappa V^{(1)}, \tag{1.13}$$

where $\left(\nabla^F\right)^*$ is the formal adjoint of ∇^F with respect to the inner product (1.3) in $L^2(F)$, and $V \in L^1_{\mathrm{loc}}(\mathrm{End}\, F)$ and $V^{(1)} \in L^1_{\mathrm{loc}}(\mathrm{End}\, F)$ are linear self-adjoint bundle endomorphisms.

In the next Theorem, $W^{1,2}(F)$ denotes the completion of $C_c^\infty(F)$ with respect to the norm $\|\cdot\|_{W^{1,2}(F)}$ defined by the scalar product (1.5) with ∇^E replaced by ∇^F, and R^F denotes the curvature tensor corresponding to the connection ∇^F.

Theorem 1.9. *Assume that (M, g) is a manifold of bounded geometry and F is a Hermitian vector bundle over M with a Hermitian connection ∇^F satisfying Assumption A with R^E replaced by R^F. Suppose that Assumptions B, C1 and C2 hold with the bundle E replaced by F. Then there exists a self-adjoint holomorphic family $J(k)$ in $L^2(F)$ of type (B), defined for all κ in the disc $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$, such that $J(\kappa)u = I(\kappa)u$ for all $u \in \mathrm{Dom}(J(\kappa))$, where*

$$\mathrm{Dom}(J(\kappa)) = \left\{u \in W^{1,2}(F) : \int \langle V_1 u, u\rangle \, d\mu < +\infty \text{ and } I(\kappa)u \in L^2(F)\right\}. \tag{1.14}$$

Remark 1.10. Note that the domains $\mathrm{Dom}(H(\kappa))$ in (1.12) and $\mathrm{Dom}(J(\kappa))$ in (1.14) depend on κ through the conditions $L(\kappa)u \in L^2(E)$ and $I(\kappa)u \in L^2(F)$ respectively.

Remark 1.11. Theorem 1.8 covers an important example of operator D. Let $\Lambda^\bullet T^*M$ denote the exterior bundle of the cotangent bundle T^*M, and let $\Omega^\bullet(M)$ denote the space of smooth sections of $\Lambda^\bullet T^*M$. By Lemma 2.12 in [9], the bundle $\Lambda^\bullet T^*M$ equipped with its natural metric and its Levi-Civita connection is a Clifford bundle (with the Clifford action as in Lemma 2.11 in [9]). By Proposition 3.53 in [1] (or by the equation (2.13) in [9]), the Dirac operator corresponding to the Clifford bundle $\Lambda^\bullet T^*M$ and its Levi-Civita connection is the operator $D = d + d^*$, where $d\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is the exterior differential and $d^*\colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ is the formal adjoint of d. The operator $D^2 = (d+d^*)^2$ is the Laplace-Beltrami operator on differential forms.

Remark 1.12. Theorem 1.9 covers an important example of operator ∇^F. If we take $\nabla^F = d$, where $d\colon C^\infty(M) \to \Omega^1(M)$ is the standard differential, then $d^*d\colon C^\infty(M) \to C^\infty(M)$ is called the scalar Laplacian and, in what follows, it will be denoted by Δ_M.

Remark 1.13. Theorem 1.9 extends a result of T. Kato (see Section VII.4.8 in [7]) which was proven for the differential expression $-\Delta + V + \kappa V^{(1)}$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V^{(1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are as in Assumptions B, C1 and C2 above. Theorem 1.9 also extends Theorem 2.3 in [8] which establishes the self-adjointness of $\left(\nabla^F\right)^*\nabla^F + V$ on the domain (1.14) with $\kappa = 0$, where ∇^F is a C^∞-bounded Hermitian connection on a Hermitian vector bundle F of bounded geometry over a manifold of bounded geometry (M, g) (hence, the Assumption A of Theorem 1.9 is satisfied) and V satisfies the Assumptions B and C1.

2. Proof of Theorem 1.8

We adopt the arguments from Section VI.4 in [7] to our setting with the help of Weitzenböck formula and a more general version of Kato's inequality.

2.1. **Weitzenböck formula.**

Let (M, g) be a Riemannian manifold with metric g and let (E, ∇^E) be a Clifford bundle over M. Let D be the Dirac operator associated with (E, ∇^E) as in (1.1). Then the following holds for all $u \in C^\infty(E)$:

$$D^2 u \;=\; \left(\nabla^E\right)^*\nabla^E u + R^W u, \tag{2.1}$$

where $\left(\nabla^E\right)^*$ denotes the formal adjoint of ∇^E with respect to the inner product (1.3), and $R^W \in C^\infty(\mathrm{End}\,E)$.

More explicitly, if $\{e_j\}_{j=1}^n$ is a local orthonormal basis of sections of TM, then for all $u \in C^\infty(E)$,

$$R^W u = \frac{1}{2}\sum_{j,k=1}^{n} e_j e_k R^E(e_j, e_k)u,$$

where R^E is the curvature tensor corresponding to the connection ∇^E.

If (M,g) and (E,∇^E) satisfy the Assumption A, then by Lemma 3.3 in [10] R^W is a bounded smooth section of $\operatorname{End} E$; hence, there exists a constant $K\geq 0$ such that

$$\sup_{x\in M}\left|R^W(x)\right|\leq K, \tag{2.2}$$

where $\left|R^W(x)\right|$ denotes the norm of the linear operator $R^W(x)\colon E_x\to E_x$.

For the proof of Weitzenböck formula (2.1), see, for example, Proposition 4.1 in Section 10.4 of [12] or the argument preceding the equation (2.6) in [9]. For more on Clifford bundles and Dirac operators, see, for example, Chapter 3 in [1]. For more on manifolds of bounded geometry, see, for example, Section A1.1 in [11].

2.2. **Kato's inequality.** We will use the following variant of Kato's inequality for Bochner Laplacian (for the proof, see Theorem 5.7 in [2]).

Lemma 2.3. *Assume that (M,g) is a Riemannian manifold with metric g. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E. Assume that $w\in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla w\in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M|w| \;\leq\; \operatorname{Re}\langle\nabla^*\nabla w,\operatorname{sign} w\rangle, \tag{2.3}$$

where $\Delta_M=d^*d$ is the scalar Laplacian on M, and

$$\operatorname{sign} w(x) \;=\; \begin{cases}\frac{w(x)}{|w(x)|} & \text{if } w(x)\neq 0\ ,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 2.4. The original version of Kato's inequality was proven in Kato [6].

2.5. **Positivity.** In what follows, we will use the following Lemma whose proof is given in Appendix B of [2].

Lemma 2.6. *Assume that (M,g) is a manifold of bounded geometry with a smooth positive measure $d\mu$. Assume that*

$$\big(b+\Delta_M\big)\,u \;=\; \nu \;\geq\; 0,\quad u\in L^2(M),$$

where $b>0$, $\Delta_M=d^*d$ is the scalar Laplacian on M, and the inequality $\nu\geq 0$ means that ν is a positive distribution on M, i.e. $(\nu,\phi)\geq 0$ for any $0\leq\phi\in C_c^\infty(M)$.

Then $u\geq 0$ (almost everywhere or, equivalently, as a distribution).

Remark 2.7. It is not known whether Lemma 2.6 holds if M is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

From now on, we assume that all the hypotheses of Theorem 1.8 are satisfied.

We define the quadratic form $h(u):=h_0(u)+h_1(u)+h_2(u)$ with the domain

$$\operatorname{Dom}(h)=\operatorname{Dom}(h_0)\cap\operatorname{Dom}(h_1)\cap\operatorname{Dom}(h_2)=\operatorname{Dom}(h_0)\cap\operatorname{Dom}(h_1). \tag{2.4}$$

The last equality in (2.4) holds since, by Assumption C1, $\operatorname{Dom}(h_2)\supset\operatorname{Dom}(h_0)$.

Lemma 2.8. *The quadratic form h is densely defined, semi-bounded below and closed.*

Proof. Since h_0 and h_1 are non-negative and closed, it follows by Theorem VI.1.31 from [7] that $h_0 + h_1$ is non-negative and closed with $\mathrm{Dom}(h_0 + h_1) = \mathrm{Dom}(h_0) \cap \mathrm{Dom}(h_1)$. By (i) of Assumption C1 it follows that $\mathrm{Dom}(h_2) \supset \mathrm{Dom}(h_0) \cap \mathrm{Dom}(h_1)$, and by (1.11), (1.6) and (1.7), the following inequality holds:

$$|h_2(u)| \le a\|u\|^2 + b|h_0(u) + h_1(u)|, \quad \text{for all } u \in \mathrm{Dom}(h_0) \cap \mathrm{Dom}(h_1), \tag{2.5}$$

where $\|\cdot\|$ denotes the norm in $L^2(E)$, and $a \ge 0$ and $0 \le b < 1$ are as in (1.11). Thus the quadratic form h_2 is $(h_0 + h_1)$-bounded with relative bound $b < 1$. Since $h_0 + h_1$ is a closed non-negative form, by Theorem VI.1.33 from [7], it follows that $h = (h_0+h_1)+h_2$ is a closed semi-bounded below form with $\mathrm{Dom}(h) = \mathrm{Dom}(h_0)\cap\mathrm{Dom}(h_1)$. Since $C_c^\infty(E) \subset (\mathrm{Dom}(h_0)\cap\mathrm{Dom}(h_1))$, it follows that h is densely defined. □

In what follows, $t(\cdot,\cdot)$ will denote the corresponding sesquilinear form obtained from a quadratic form $t(\cdot)$ via polarization identity.

Lemma 2.9. *The following inequality holds for all $u \in \mathrm{Dom}(h)$:*

$$|h_2(u)| \le (1-b)^{-1}(a\|u\|^2 + b|h(u)|), \tag{2.6}$$

where $a \ge 0$ and $0 \le b < 1$ are as in (1.11) *and $\|\cdot\|$ is the norm in $L^2(E)$.*

Proof. Let u be an arbitrary element of $\mathrm{Dom}(h) = (\mathrm{Dom}(h_0) \cap \mathrm{Dom}(h_1)) \subset \mathrm{Dom}(h_2)$. Since $h(u) = h_0(u) + h_1(u) + h_2(u)$, we have

$$|h_0(u) + h_1(u)| - |h_2(u)| \le |h(u)|,$$

and, hence, by (2.5), we obtain

$$-a\|u\|^2 + (1-b)|h_0(u) + h_1(u)| \le |h(u)|,$$

where $a \ge 0$ and $0 \le b < 1$ are as in (1.11).

Therefore,

$$|h_0(u) + h_1(u)| \le (1-b)^{-1}(a\|u\|^2 + |h(u)|),$$

and, hence, by (2.5), we get

$$\begin{aligned} |h_2(u)| &\le a\|u\|^2 + b|h_0(u) + h_1(u)| \\ &\le a\|u\|^2 + b(1-b)^{-1}(a\|u\|^2 + |h(u)|) = (1-b)^{-1}(a\|u\|^2 + b|h(u)|). \end{aligned} \tag{2.7}$$

This concludes the proof of the Lemma. □

We now define the quadratic form $h^{(1)}(u) := h_1^{(1)}(u) + h_2^{(1)}(u)$ with the domain $\mathrm{Dom}(h^{(1)}) = \mathrm{Dom}(h_1^{(1)}) \cap \mathrm{Dom}(h_2^{(1)})$.

Lemma 2.10. *The form $h^{(1)}$ has the following properties:*

(i) $\mathrm{Dom}(h^{(1)}) \supset \mathrm{Dom}(h)$,

(ii) *For all* $u \in \operatorname{Dom}(h)$*, the following holds:*

$$\left|h^{(1)}(u)\right| \leq \left(\frac{(\beta+\widetilde{b})a}{1-b}+\widetilde{a}\right)\|u\|^2+\left(\frac{\beta+\widetilde{b}}{1-b}\right)|h(u)|, \tag{2.8}$$

where $a \geq 0$ *and* $0 \leq b < 1$ *are as in (1.11), the constants* $\widetilde{a} \geq 0$ *and* $\widetilde{b} \geq 0$ *are as in Assumption C2, and* $\beta > 0$ *is as in (ii) of Assumption B.*

Proof. By (2.4) we have $\operatorname{Dom}(h) = \operatorname{Dom}(h_0) \cap \operatorname{Dom}(h_1)$. By (ii) of Assumption B, it follows that $\operatorname{Dom}(h_1) \subset \operatorname{Dom}(h_1^{(1)})$ and by Assumption C2, it follows that $\operatorname{Dom}(h_0) \subset \operatorname{Dom}(h_2^{(1)})$. Therefore, $\operatorname{Dom}(h) \subset \operatorname{Dom}(h^{(1)})$, and the property (i) of the Lemma is proven.

We now prove the property (ii). For all $u \in \operatorname{Dom}(h)$, using property (i) of the Lemma, the inequality (1.4), the non-negativity of h_0 and h_1, Assumption C2 and (2.6), we have

$$\begin{aligned}
\left|h^{(1)}(u)\right| &\leq \beta h_1(u) + \left|h_2^{(1)}(u)\right| \leq \beta h_1(u) + \widetilde{a}\|u\|^2 + \widetilde{b}|h_0(u)| \\
&\leq \widetilde{a}\|u\|^2 + (\beta+\widetilde{b})(h_0(u)+h_1(u)) = \widetilde{a}\|u\|^2 + (\beta+\widetilde{b})(h(u)) - (\beta+\widetilde{b})(h_2(u)) \\
&\leq \widetilde{a}\|u\|^2 + (\beta+\widetilde{b})|h(u)| + \frac{(\beta+\widetilde{b})a}{1-b}\|u\|^2 + \frac{(\beta+\widetilde{b})b}{1-b}|h(u)| \\
&= \left(\frac{(\beta+\widetilde{b})a}{1-b}+\widetilde{a}\right)\|u\|^2+\left(\frac{\beta+\widetilde{b}}{1-b}\right)|h(u)|.
\end{aligned} \tag{2.9}$$

Here, in the fourth inequality, we used (2.6). This concludes the proof of the Lemma. □

Lemma 2.11. *The family of forms* $h(\kappa) = h + \kappa h^{(1)}$*, where* $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$*, is holomorphic of type (a).*

Proof. By Lemma 2.8, the form h is densely defined, semi-bounded below (hence, sectorial) and closed. By (i) of Lemma 2.10 and the inequality (2.8), for all $u \in \operatorname{Dom}(h)$, we have

$$\left|\kappa h^{(1)}(u)\right| \leq |\kappa|\left(\frac{(\beta+\widetilde{b})a}{1-b}+\widetilde{a}\right)\|u\|^2+|\kappa|\left(\frac{\beta+\widetilde{b}}{1-b}\right)|h(u)|. \tag{2.10}$$

Now by Theorem VI.1.33 from [7], it follows that for all $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$, the form

$$h(\kappa)u = h(u) + \kappa h^{(1)}(u), \quad \operatorname{Dom}(h(\kappa)) = \operatorname{Dom}(h) \cap \operatorname{Dom}(h^{(1)}) = \operatorname{Dom}(h), \tag{2.11}$$

is sectorial and closed. Since $\operatorname{Dom}(h(\kappa)) = \operatorname{Dom}(h)$, it follows that $h(\kappa)$ is densely defined. By the definition in Sect. 1.3 it follows that $h(\kappa)$, where $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$, is a holomorphic family of type (a). □

2.12. m**-sectorial operator** $H(\kappa)$ **associated to** $h(\kappa)$**.** Since $h(\kappa)$, with $|\kappa| < \frac{1-b}{\beta+\widetilde{b}}$, is a densely defined, closed and sectorial form in $L^2(E)$, by Theorem VI.2.1 from [7], there exists an m-sectorial operator $H(\kappa)$ in $L^2(E)$ such that

(i) $\mathrm{Dom}(H(\kappa)) \subset \mathrm{Dom}(h(\kappa))$ and

$$h(\kappa)(u,v) = (H(\kappa)u, v), \quad \text{for all } u \in \mathrm{Dom}(H(\kappa)) \text{ and } v \in \mathrm{Dom}(h(\kappa)).$$

(ii) $\mathrm{Dom}(H(\kappa))$ is a core of $h(\kappa)$.

(iii) If $u \in \mathrm{Dom}(h(\kappa))$, $w \in L^2(E)$, and

$$h(\kappa)(u,v) = (w,v)$$

holds for every v belonging to a core of $h(\kappa)$, then $u \in \mathrm{Dom}(H(\kappa))$ and $H(\kappa)u = w$. The operator $H(\kappa)$ is uniquely determined by the condition (i).

Lemma 2.13. *For all κ in the disc $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, the operators $H(\kappa)$ form a self-adjoint holomorphic family of type (B).*

Proof. Since by Lemma 2.11 the family $h(\kappa)$, with $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, is holomorphic of type (a), by Theorem VII.4.2 in [7] it follows that for all $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, the family of operators $H(\kappa)$ is holomorphic of type (B). By the definition of $h(\kappa)$, we have $h(\kappa)^* = h(\bar{\kappa})$, where $h(\kappa)^*(u,v) := \overline{h(\kappa)(v,u)}$ denotes the adjoint of the form $h(\kappa)$ (see, for example, the equation VI.1.6 in [7]). By Remark VII.4.7 in [7], it follows that $H(\kappa)^* = H(\bar{\kappa})$. Now by the definition in Sect. 1.2, it follows that $H(\kappa)$, where $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, is a self-adjoint holomorphic family of type (B). $\square$

It remains to show that $\mathrm{Dom}(H(\kappa))$ is the set on the right hand side of (1.12) and that $H(\kappa)u = L(\kappa)u$ for all $u \in \mathrm{Dom}(H(\kappa))$.

2.14. **A realization of $L(\kappa)$ in $L^2(E)$.** For $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, we define an operator $S(\kappa)$ in $L^2(E)$ by the formula $S(\kappa)u = L(\kappa)u$ on the domain

$$\mathrm{Dom}(S(\kappa)) = \Big\{u \in W^{1,2}(E) : \int \langle V_1 u, u\rangle \, d\mu < +\infty \text{ and } L(\kappa)u \in L^2(E)\Big\}. \tag{2.12}$$

Remark 2.15. For all $u \in \mathrm{Dom}(h_0) = W^{1,2}(E)$ we have $D^2 u \in W^{-1,2}(E)$, and from Corollary 2.18 below it follows that for all $u \in (W^{1,2}(E) \cap \mathrm{Dom}(h_1)) \subset \mathrm{Dom}(h^{(1)})$, we have $Vu \in L^1_{\mathrm{loc}}(E)$ and $\kappa V^{(1)} u \in L^1_{\mathrm{loc}}(E)$. Thus $L(\kappa)u$ in (2.12) is a distributional section of E, and the condition $L(\kappa)u \in L^2(E)$ makes sense.

To complete the proof of Theorem 1.8, it remains to show that $H(\kappa) = S(\kappa)$. In what follows, we will use the following well-known Lemma.

Lemma 2.16. *Assume that $0 \le T \in L^1_{\mathrm{loc}}(\mathrm{End}\, E)$ is a linear self-adjoint bundle endomorphism. Assume also that $u \in Q(T)$, where $Q(T) = \{u \in L^2(E) \colon \langle Tu, u\rangle \in L^1(M)\}$.*

Then $Tu \in L^1_{\mathrm{loc}}(E)$.

Proof. By adding a constant we can assume that $T \ge 1$ (in operator sense).

Assume that $u \in Q(T)$. We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $1 \le T(x) \in \mathrm{End}(E_x)$ to get $T(x) = \mathrm{diag}(c_1(x), c_2(x), \dots, c_m(x))$, where $0 < c_j \in L^1_{\mathrm{loc}}(M)$, $j = 1, 2, \dots, m$ and $m = \dim E_x$.

Let $u_j(x)$ $(j = 1, 2, \dots, m)$ be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of E_x. Then for all $x \in M$

$$\langle Tu, u\rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u\rangle \, d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j|u_j|^2 \in L^1(M)$, for all $j = 1, 2, \dots, m$.

Now, for all $x \in M$ and $j = 1, 2, \cdots, m$

$$2|c_j u_j| = 2|c_j||u_j| \leq |c_j| + |c_j||u_j|^2, \tag{2.13}$$

The right hand side of (2.13) is clearly in $L^1_{\text{loc}}(M)$. Therefore $c_j u_j \in L^1_{\text{loc}}(M)$.

But $(Tu)(x)$ has components $c_j(x)u_j(x)$ $(j = 1, 2, \dots, m)$ with respect to chosen bases of E_x. Therefore $Tu \in L^1_{\text{loc}}(E)$, and the Lemma is proven. □

The following corollary follows immediately from Lemma 2.16.

Corollary 2.17. *The following properties hold:*

(i) *If* $u \in \text{Dom}(h_1)$, *then* $V_1 u \in L^1_{\text{loc}}(E)$.
(ii) *If* $u \in \text{Dom}(h_1^{(1)})$, $V_1^{(1)} u \in L^1_{\text{loc}}(E)$.

Corollary 2.18. *The following properties hold:*

(i) *If* $u \in \text{Dom}(h)$, *then* $Vu \in L^1_{\text{loc}}(E)$.
(ii) *If* $u \in \text{Dom}(h^{(1)})$, *then* $V^{(1)} u \in L^1_{\text{loc}}(E)$.
(iii) *If* $u \in \text{Dom}(h(\kappa))$, *then* $(V + \kappa V^{(1)}) \in L^1_{\text{loc}}(E)$.

Proof. We will first prove the property (i). Assume that $u \in \text{Dom}(h) = \text{Dom}(h_0) \cap \text{Dom}(h_1)$. By Assumption B we have $V = V_1 + V_2$, where $0 \leq V_1 \in L^1_{\text{loc}}(\text{End}\, E)$ and $0 \geq V_2 \in L^1_{\text{loc}}(\text{End}\, E)$. By Corollary 2.17 it follows that $V_1 u \in L^1_{\text{loc}}(E)$ and since, by Assumption C1, $\text{Dom}(h) \subset \text{Dom}(h_2)$, by Lemma 2.16 we have $-V_2 u \in L^1_{\text{loc}}(E)$. Thus $Vu \in L^1_{\text{loc}}(E)$, and the property (i) is proven. We now prove the property (ii). Assume that $u \in \text{Dom}(h^{(1)}) = \text{Dom}(h_1^{(1)}) \cap \text{Dom}(h_2^{(1)})$. By Corollary 2.17 it follows that $V_1^{(1)} u \in L^1_{\text{loc}}(E)$, and by Lemma 2.16 we have $-V_2^{(1)} u \in L^1_{\text{loc}}(E)$. Therefore, $V^{(1)} u \in L^1_{\text{loc}}(E)$, and the property (ii) is proven. The property (iii) follows immediately from (2.11) and properties (i) and (ii). This concludes the proof of the Corollary. □

Lemma 2.19. *The operator relation* $H(\kappa) \subset S(\kappa)$ *holds for all* $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$.

Proof. We will show that for all $u \in \text{Dom}(H(\kappa))$, where $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, we have $H(\kappa)u = L(\kappa)u$.

Let $u \in \text{Dom}(H(\kappa))$ be arbitrary. By property (i) of Sect. 2.12 we have $u \in \text{Dom}(h(\kappa)) = \text{Dom}(h) \cap \text{Dom}(h^{(1)}) = \text{Dom}(h)$; hence, by Corollary 2.18 we get $Vu \in L^1_{\text{loc}}(E)$ and $\kappa V^{(1)} u \in$

$L^1_{\mathrm{loc}}(E)$. Then, for any $v \in C_c^\infty(E)$, we have

$$\begin{aligned}(H(\kappa)u, v) \;=\; h(\kappa)(u, v) \;&=\; (Du, Dv) + \int \langle Vu, v\rangle \, d\mu + \int \langle \kappa V^{(1)}u, v\rangle \, d\mu \\ &=\; (u, D^2 v) + \int \langle Vu, v\rangle \, d\mu + \int \langle \kappa V^{(1)}u, v\rangle \, d\mu \end{aligned} \quad (2.14)$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(E)$.

The first equality in (2.14) holds by property (i) from Sec. 2.12, and the second equality holds by definition of $h(\kappa)$. In the first term on the right hand side of the third equality we used integration by parts (see, for example, Lemma 8.8 in [2]) and the formal self-adjointness of D.

From (2.14) we get

$$(u, D^2 v) = \int \langle H(\kappa)u - Vu - \kappa V^{(1)}u, v\rangle \, d\mu, \qquad \text{for all} \quad v \in C_c^\infty(E). \quad (2.15)$$

Since $Vu \in L^1_{\mathrm{loc}}(E)$, $\kappa V^{(1)}u \in L^1_{\mathrm{loc}}(E)$ and $H(\kappa)u \in L^2(E)$, it follows that $(H(\kappa)u - Vu - \kappa V^{(1)}u) \in L^1_{\mathrm{loc}}(E)$, and (2.15) implies $D^2 u = H(\kappa)u - Vu - \kappa V^{(1)}u$ (as distributional sections of E). Therefore,

$$D^2 u + Vu + \kappa V^{(1)}u = H(\kappa)u,$$

and this shows that $H(\kappa)u = L(\kappa)u$ for all $u \in \mathrm{Dom}(H(\kappa))$.

Now by definition of $S(\kappa)$ it follows that $\mathrm{Dom}(H(\kappa)) \subset \mathrm{Dom}(S(\kappa))$ and $H(\kappa)u = S(\kappa)u$ for all $u \in \mathrm{Dom}(H(\kappa))$. Therefore $H(\kappa) \subset S(\kappa)$, and the Lemma is proven. □

Lemma 2.20. *The following equality of distributional sections of E holds for all $u \in W^{1,2}(E)$:*

$$D^2 u \;=\; \left(\nabla^E\right)^* \nabla^E u + R^W u, \quad (2.16)$$

where D^2, (E, ∇^E) and R^W are as in (2.1).

Proof. Let $u \in W^{1,2}(E)$. Then by Sect. 1.4, there exists a sequence $u_k \in C_c^\infty(E)$ such that $u_k \to u$ in the norm $\|\cdot\|_{W^{1,2}(E)}$, as $k \to \infty$. For all $v \in C_c^\infty(E)$, by (2.1) we have

$$(D^2 u_k, v) \;=\; (\left(\nabla^E\right)^* \nabla^E u_k, v) + (R^W u_k, v),$$

and, hence, using integration by parts, we get

$$(Du_k, Dv) \;=\; (\nabla^E u_k, \nabla^E v) + (R^W u_k, v). \quad (2.17)$$

Now, taking limits as $k \to \infty$ on both sides of (2.17) and using Sect. 1.4, Remark 1.5 and (2.2), we get

$$(Du, Dv) \;=\; (\nabla^E u, \nabla^E v) + (R^W u, v). \quad (2.18)$$

Using integration by parts on the left hand side and the first term on the right hand side of (2.18) (see, for example, Lemma 8.8 in [2]), we get

$$(D^2 u, v) \;=\; (\left(\nabla^E\right)^* \nabla^E u, v) + (R^W u, v), \quad \text{for all } u \in W^{1,2}(E) \text{ and all } v \in C_c^\infty(E), \quad (2.19)$$

where $(\cdot,\cdot)$ on the left hand side and the first term on the right hand side denotes the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$. Since (2.19) holds for all $v \in C_c^\infty(E)$, we get the equality of distributional sections (2.16), and the Lemma is proven. □

Lemma 2.21. *$C_c^\infty(E)$ is a core of the quadratic form $h_0 + h_1$.*

Proof. By Theorem VI.1.21 in [7], it suffices to show that $C_c^\infty(E)$ is dense in the Hilbert space $\mathrm{Dom}(h_0 + h_1) = \mathrm{Dom}(h_0) \cap \mathrm{Dom}(h_1)$ with the inner product

$$(u, v)_{h_0+h_1} := h_0(u, v) + h_1(u, v) + (K + 1)(u, v),$$

where $K \geq 0$ is as in (2.2), and $h_0(\cdot, \cdot)$ and $h_1(\cdot, \cdot)$ denote the sesquilinear forms corresponding to the quadratic forms h_0 and h_1 respectively via polarization identity.

Let $u \in \mathrm{Dom}(h_0 + h_1)$ and $(u, v)_{h_0+h_1} = 0$ for all $v \in C_c^\infty(E)$. We will show that $u = 0$.

We have

$$0 \;=\; h_0(u, v) + h_1(u, v) + (K + 1)(u, v) \;=\; (u, D^2v) + \int \langle V_1u, v\rangle \, d\mu + (K + 1)(u, v). \tag{2.20}$$

In the first term on the right hand side of the second equality, we used integration by parts (see, for example, Lemma 8.8 in [2]).

By Corollary 2.17 it follows that $V_1u \in L^1_{\mathrm{loc}}(E)$, and from (2.20) we get the following equality of distributional sections of E:

$$D^2u = (-V_1 - K - 1)u. \tag{2.21}$$

Since $u \in W^{1,2}(E)$, by (2.16) and (2.21) we obtain

$$\big(\nabla^E\big)^*\nabla^E u = (-V_1 - K - 1 - R^W)u. \tag{2.22}$$

Since by (2.2) the section $R^W \in C^\infty(\mathrm{End}\, E)$ is bounded and since $V_1u \in L^1_{\mathrm{loc}}(E)$, by (2.22) we have $\big(\nabla^E\big)^*\nabla^E u \in L^1_{\mathrm{loc}}(E)$. By Lemma 2.3 and by (2.22) we obtain

$$\begin{aligned}\Delta_M|u| \;&\leq\; \mathrm{Re}\langle\big(\nabla^E\big)^*\nabla^E u, \mathrm{sign}\, u\rangle \;=\; \mathrm{Re}\langle-(V_1 + K + 1 + R^W)u, \mathrm{sign}\, u\rangle \\ &=\; \langle-(V_1 + K + 1 + R^W)u, \mathrm{sign}\, u\rangle \leq\; -(K + 1)|u| + \langle-R^W u, \mathrm{sign}\, u\rangle \\ &\leq\; -(K + 1)|u| + K|u| \;=\; -|u|.\end{aligned} \tag{2.23}$$

The second equality in (2.23) holds since V_1 and R^W are self-adjoint bundle endomorphisms, the second inequality holds since $V_1 \geq 0$ (as an operator $E_x \to E_x$), and the third inequality follows from (2.2).

From (2.23) we get

$$(\Delta_M + 1)|u| \leq 0. \tag{2.24}$$

By Lemma 2.6, it follows that $|u| \leq 0$. So $u = 0$, and the Lemma is proven. □

In what follows, by $t^\sim$ we denote the closure of a closable form t in $L^2(E)$.

Lemma 2.22. *$C_c^\infty(E)$ is a core of the quadratic form $h = (h_0 + h_1) + h_2$.*

Proof. By the proof of Lemma 2.8, the form $h_0 + h_1$ is closed and non-negative, and by (2.5), the quadratic form h_2 is $(h_0 + h_1)$-bounded with relative bound $0 \leq b < 1$. By $(h_0 + h_1)|_{C_c^\infty(E)}$, $(h_2)|_{C_c^\infty(E)}$ and $h|_{C_c^\infty(E)}$, we will denote the restriction of $h_0 + h_1$, h_2 and h to $C_c^\infty(E)$ respectively. By Lemma 2.21, $C_c^\infty(E)$ is a core of the form $h_0 + h_1$. Hence, by the remark preceding Theorem VI.1.21 in [7], the form $(h_0 + h_1)|_{C_c^\infty(E)}$ is closable and $((h_0 + h_1)|_{C_c^\infty(E)})^\sim = h_0 + h_1$.

Since $(h_2)|_{C_c^\infty(E)}$ is relatively bounded by $(h_0+h_1)|_{C_c^\infty(E)}$ with relative bound $0 \le b < 1$, by Theorem VI.1.33 in [7], it follows that $(h_0+h_1)|_{C_c^\infty(E)} + (h_2)|_{C_c^\infty(E)} = h|_{C_c^\infty(E)}$ is closable and $\mathrm{Dom}((h|_{C_c^\infty(E)})^\sim) = \mathrm{Dom}(((h_0+h_1)|_{C_c^\infty(E)})^\sim) = \mathrm{Dom}(h_0+h_1)$. By Lemma 2.8 and Assumption C1, the form h is closed with the domain $\mathrm{Dom}(h) = \mathrm{Dom}(h_0+h_1)$. Since h is a closed extension of $h|_{C_c^\infty(E)}$, by Theorem VI.1.17 in [7] it follows that h is a closed extension of $(h|_{C_c^\infty(E)})^\sim$. Since $\mathrm{Dom}((h|_{C_c^\infty(E)})^\sim) = \mathrm{Dom}(h)$, it follows that $h = (h|_{C_c^\infty(E)})^\sim$. Thus $C_c^\infty(E)$ is a core of the form h, and the Lemma is proven. □

Lemma 2.23. *For all $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, the space $C_c^\infty(E)$ is a core of the quadratic form $h(\kappa) = h + \kappa h^{(1)}$.*

Proof. By Lemma 2.8, the form h is densely defined, semi-bounded below (hence, sectorial) and closed. By (2.10) it follows that for all $|\kappa| < \frac{1-b}{\beta+\tilde{b}}$, the form $\kappa h^{(1)}$ is h-bounded with relative bound

$$|\kappa|\left(\frac{\beta+\widetilde{b}}{1-b}\right) < 1.$$

By $(h(\kappa))|_{C_c^\infty(E)}$ we denote the restriction of $h(\kappa)$ to $C_c^\infty(E)$. By Lemma 2.22, it follows that $C_c^\infty(E)$ is a core of the form h. To prove that $((h(\kappa))|_{C_c^\infty(E)})^\sim = h(\kappa)$, we use the same argument as in the proof of Lemma 2.22, and we will not repeat it here. □

3. Proof of Theorem 1.8

We will show that $S(\kappa) = H(\kappa)$. By Lemma 2.19 we have $H(\kappa) \subset S(\kappa)$, so it is enough to show that $\mathrm{Dom}(S(\kappa)) \subset \mathrm{Dom}(H(\kappa))$.

Let $u \in \mathrm{Dom}(S(\kappa))$. By definition of $\mathrm{Dom}(S(\kappa))$ in (2.12), we have $u \in \mathrm{Dom}(h_0) \subset \mathrm{Dom}(h_2)$ and $u \in \mathrm{Dom}(h_1)$. Hence $u \in \mathrm{Dom}(h)$, and, thus, by (2.11) we have $u \in \mathrm{Dom}(h(\kappa))$.

Since $u \in \mathrm{Dom}(S(\kappa))$, it follows that $u \in W^{1,2}(E)$ and $(D^2u + Vu + \kappa V^{(1)}u) \in L^2(E)$. Thus, $D^2u \in W^{-1,2}(E)$ and, hence, $(Vu + \kappa V^{(1)}u) \in W^{-1,2}(E)$.

For all $v \in C_c^\infty(E)$ we have

$$\begin{aligned} h(\kappa)(u,v) &= h_0(u,v) + h_1(u,v) + h_2(u,v) + \kappa h_1^{(1)}(u,v) + \kappa h_2^{(2)}(u,v) \\ &= (Du, Dv) + \int \langle (V+\kappa V^{(1)})u, v\rangle\, d\mu \;=\; (D^2u, v) + ((V+\kappa V^{(1)})u, v) \;=\; (L(\kappa)u, v), \end{aligned}$$

where on the right hand side of the third equality, $(\cdot,\cdot)$ denotes the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$. In first term on the right hand side of the third equality we used the used the integration by parts (see, for example Lemma 8.8 in [2]). Since $L(\kappa)u = S(\kappa)u \in L^2(E)$, the last equality holds, and, on the right hand side of the last equality, $(\cdot,\cdot)$ denotes the inner product in $L^2(E)$. By Lemma 2.23 it follows that $C_c^\infty(E)$ is a form core of $h(\kappa)$. Now from property (iii) of Sect. 2.12 we have $u \in \mathrm{Dom}(H(\kappa))$ with $H(\kappa)u = L(\kappa)u$. This concludes the proof of the Theorem. □

4. Proof of Theorem 1.9

The proof is the same as the proof of Theorem 1.8 with E replaced by F, the operator D replaced by ∇^F (or, where appropriate, by $(\nabla^F)^*$), the differential expression $L(\kappa)$ replaced by $B(\kappa)$, and with $R^W = 0$. $\square$

5. Proof of Remark 1.7

We will give the proof for V_2; the proof of the Remark for $V_2^{(1)}$ proceeds in the same way. Let p be as in Remark 1.7. We may assume that $\|V_2\|_{L^p(\operatorname{End} E)}$ is arbitrarily small because there exists a sequence a sequence $(V_2)_k \in L^\infty(\operatorname{End} E) \cap L^p(\operatorname{End} E)$, $k \in \mathbb{Z}_+$, such that

$$\|(V_2)_k - V_2\|_{L^p(\operatorname{End} E)} \to 0, \quad \text{as } k \to \infty,$$

and $(V_2)_k$, $k \in \mathbb{Z}_+$, contributes to h_2 only a bounded form.

From now on, we will assume that $\|V_2\|_{L^p(\operatorname{End} E)}$ is arbitrarily small.

By Cauchy-Schwartz inequality and Hölder's inequality we have

$$\left|\int \langle V_2 u, u\rangle \, d\mu\right| \leq \int |\langle V_2 u, u\rangle| \, d\mu \leq \int |V_2||u|^2 \, d\mu \leq \|V_2\|_{L^p(\operatorname{End} E)} \|u\|^2_{L^t(E)}, \tag{5.1}$$

where $|V_2|$ denotes the norm of the operator $V_2(x)\colon E_x \to E_x$ and

$$\frac{1}{p} + \frac{2}{t} = 1. \tag{5.2}$$

With our assumptions on (M, g) and (E, ∇^E), by Theorem 3.2 (a) in [5] we have the continuous embedding $W^{1,2}(E) \subset L^t(E)$ for

$$1 - \frac{n}{2} \geq -\frac{n}{t} \quad \text{and} \quad t \geq 2.$$

For $n \geq 3$, we know by hypothesis that $p = n/2$, so from (5.2) we get $1/t = 1/2 - 1/n$. By Theorem 3.2 (a) in [5], we have

$$\|u\|^2_{L^t(E)} \leq C(\|Du\|^2_{L^2(E)} + \|u\|^2_{L^2(E)}), \qquad \text{for all } u \in W^{1,2}(E), \tag{5.3}$$

where $C > 0$ is a positive constant.

For $n = 2$, we know by hypothesis that $p > 1$, so from (5.2) we get $2 < t < \infty$. By Theorem 3.2 (a) in [5] we get (5.3).

For $n = 1$, we know by hypothesis that $p = 1$, so from (5.2) we get $t = \infty$. For $n = 1$, by Theorem 3.2 (b) in [5] or by Theorem 1 in [3], we have the continuous embedding $W^{1,2}(E) \subset C_b(E)$, where $C_b(E)$ denotes bounded continuous sections of E. Therefore,

$$\|u\|^2_{L^\infty(E)} \leq C(\|Du\|^2_{L^2(E)} + \|u\|^2_{L^2(E)}), \qquad \text{for all } u \in W^{1,2}(E), \tag{5.4}$$

where $C > 0$ is a positive constant.

Combining (5.3) and (5.4) with (5.1), we get (1.11) (with constant $b < 1$ because $\|V_2\|_{L^p(\operatorname{End} E)}$ is arbitrarily small). $\square$

References

[1] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Grundlehren der Mathematischen Wissenschaften, 298, Springer-Verlag, Berlin, 1992.

[2] M. Braverman, O. Milatovic, M. Shubin, Essential self-adjointness of Schrödinger type operators on manifolds, Russian Math. Surveys 57(4) (2002) 641–692.

[3] M. Cantor, Sobolev inequalities for Riemannian bundles, in: Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, Amer. Math. Soc., Providence, R.I., 1975, pp. 171–184.

[4] J. Eichhorn, Relative Zeta Functions, Determinants, Torsion, Index Theorems and Invariants for Open Manifolds, arXiv:math.DG/0111301, to appear in Comment. Math. Helv.

[5] J. Eichhorn, J. Fricke, The module structure theorem for Sobolev spaces on open manifolds, Math. Nachr. 194 (1998) 35–47.

[6] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972) 135–148.

[7] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.

[8] O. Milatovic, Self-adjointness of Schrödinger-type operators with singular potentials on manifolds of bounded geometry, Electron. J. Differential Equations, No. 64 (2003) 1–8 (electronic).

[9] J. Roe, Elliptic Operators, Topology and Asymptotic Methods, Pitman Research Notes in Mathematics Series, 179, Longman Scientific and Technical, Harlow, copublished in the United States with John Wiley and Sons, Inc., New York, 1988.

[10] G. Salomonsen, Equivalence of Sobolev spaces, Results Math. 39(1-2) (2001) 115–130.

[11] M. A. Shubin, Spectral theory of elliptic operators on noncompact manifolds, Astérisque, No. 207 (1992) 35–108.

[12] M. Taylor, Partial Differential Equations II: Qualitative Studies of Linear Equations, Springer-Verlag, New York e.a., 1996.