THE FORM SUM AND THE FRIEDRICHS EXTENSION OF SCHRÖDINGER-TYPE OPERATORS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider $H_V = \Delta_M + V$, where (M, g) is a Riemannian manifold (not necessarily complete), and Δ_M is the scalar Laplacian on M. We assume that $V = V_0 + V_1$, where $V_0 \in L^2_{\rm loc}(M)$ and $-C \leq V_1 \in L^1_{\rm loc}(M)$ (C is a constant) are real-valued, and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$. Let T_0 be the Friedrichs extension of $(\Delta_M + V_0)|_{C_c^\infty(M)}$. We prove that the form sum $T_0 \tilde{+} V_1$, coincides with the self-adjoint operator T_F associated to the closure of the restriction to $C_c^\infty(M) \times C_c^\infty(M)$ of the sum of two closed quadratic forms of T_0 and V_1. This is an extension of a result of Cycon. The proof adopts the scheme of Cycon, but it requires the use of a more general version of Kato's inequality for operators on Riemannian manifolds.

1. Introduction and the main result

Let (M, g) be a Riemannian manifold (i.e. M is a C^∞-manifold, (g_{jk}) is a Riemannian metric on M), $\dim M = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$. We do not assume that (M, g) is complete.

We will consider a Schrödinger type operator of the form

$$H_V = \Delta_M + V.$$

Here $\Delta_M := d^*d$, where $d\colon C^\infty(M) \to \Omega^1(M)$, and $V \in L^1_{\rm loc}(M)$ is real-valued.

1.1. **Maximal operator.** We define the maximal operator $H_{V,\max}$ associated to H_V as an operator in $L^2(M)$ given by $H_{V,\max}u = H_V u$ with domain

$$\mathrm{Dom}(H_{V,\max}) = \{u \in L^2(M) : Vu \in L^1_{\rm loc}(M),\ H_V u \in L^2(M)\}. \tag{1.1}$$

Here $\Delta_M u$ in $H_V u = \Delta_M u + Vu$ is understood in distributional sense.

We make the following assumptions on V.

Assumption A. Assume $V = V_0 + V_1$, where

(i) $V_0 \in L^2_{\rm loc}(M)$ and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$.
(ii) $V_1 \in L^1_{\rm loc}(M)$ and $V_1 \geq -C$, where $C > 0$ is a constant.

2000 *Mathematics Subject Classification.* Primary 35P05, 58G25; Secondary 47B25, 81Q10.

1.2. **Quadratic forms.** For any self-adjoint operator $T\colon \mathrm{Dom}(T) \subset L^2(M) \to L^2(M)$ such that $T \geq -\alpha$, we will denote by $Q(T)$ the domain of the quadratic form t associated to T. By Theorem 2.1 in [3], t is a closed semibounded below form, i.e. $Q(T)$ is a Hilbert space with the inner product

$$(u, v)_t = t(u, v) + (1 + \alpha)(u, v)_{L^2(M)}, \tag{1.2}$$

where $t(\cdot, \cdot)$ is the sesquilinear form obtained by polarization of t.

1.3. **Form sum.** By (i) of Assumption A, $\Delta_M + V_0$ is symmetric and semibounded below on $C_c^\infty(M)$, so we can associate to it a semibounded below self-adjoint operator T_0 (Friedrichs extension, cf. Theorem 14.1 in [3]).

We will denote by $T_0 \tilde{+} V_1$ the form sum of T_0 and V_1. By Theorem 4.1 in [3], this is the self-adjoint operator associated to the semibounded below closed quadratic form t_q given by the sum of two semibounded below closed quadratic forms corresponding to T_0 and V_1. By the same theorem, the following is true: $Q(T_0 \tilde{+} V_1) = Q(T_0) \bigcap Q(V_1)$. Clearly, $T_0 \tilde{+} V_1$ is a self-adjoint restriction of $H_{V,\max}$.

1.4. **Operator T_F.** Denote by $t_{\min}$ the restriction of t_q to $C_c^\infty(M) \times C_c^\infty(M)$. Denote by T_F the self-adjoint operator associated to the closure of $t_{\min}$ in the sense of the norm in $Q(T_0 \tilde{+} V_1)$. Clearly, T_F is a self-adjoint restriction of $H_{V,\max}$.

We will give a sufficient condition for $T_F = T_0 \tilde{+} V_1$.

Theorem 1.5. *Suppose that the assumption A holds.*

Then $T_F = T_0 \tilde{+} V_1$.

Remark 1.6. Theorem 1.5 was proven by Cycon [2] in case of the operator $-\Delta + V$ in an open set $M \subset \mathbb{R}^n$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric. In case $V_0 = 0$ and $M = \mathbb{R}^n$ with standard metric, Theorem 1.5 was proven in Simon [13].

2. Operators with a positive form core

Definition 2.1. Let $T\colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$ be a symmetric semibounded below operator. Let T_F denote its Friedrichs extension and $Q(T_F)^+$ the set of a.e. positive elements of $Q(T_F)$. We say that T_F has a positive form core if for every $u \in Q(T_F)^+$, there exists a sequence $u_k \in C_c^\infty(M)^+$ such that

$$\|u_k - u\|_t \to 0 \qquad \text{as } k \to \infty,$$

where $\|\cdot\|_t$ is the norm associated to the closure of quadratic form $t(v, w) := (Tv, w)$ ($v, w \in C_c^\infty(M)$) via (1.2).

The main result of this section is

Theorem 2.2. *Suppose that* $\Delta_M + V_0$ *is as in (i) of Assumption A. Let* T_0 *be the Friedrichs extension of* $(\Delta_M + V_0)|_{C_c^\infty(M)}$.

Then T_0 *has a positive form core.*

Remark 2.3. In case of the operator $-\Delta + V_0$ in an open set $M \subset \mathbb{R}^n$, Theorem 2.2 was proven in [2, Th. 1].

We will first prove the following special case of Theorem 2.2

Proposition 2.4. *Suppose that* $-C \leq V_0 \in L^2_{\text{loc}}(M)$, *where* $C > 0$ *is a constant. Let* T_b *be the Friedrichs extension of* $(\Delta_M + V_0)|_{C_c^\infty(M)}$.

Then T_b *has a positive form core.*

We begin with a few preliminary lemmas.

In what follows T_b is as in the hypothesis of Proposition 2.4, and t_b is the closed quadratic form associated with T_b. Without the loss of generality, we may and we will assume that $V_0 \geq 0$ so that T_b is a positive self-adjoint operator.

We will denote $W^{1,2}(M) := \{u \in L^2(M) \colon du \in L^2(T^*M)\}$. By $W_0^{1,2}(M)$ we will denote the closure of $C_c^\infty(M)$ in the norm $\|u\|^2_{W^{1,2}} := \|du\|^2 + \|u\|^2$, where $\|\cdot\|$ is the L^2 norm. By $Q(V_0)$ we will denote the set $\{u \in L^2(M) \colon V_0^{1/2}u \in L^2(M)\}$. Clearly, $Q(V_0)$ is the closure of $C_c^\infty(M)$ in the norm

$$\|u\|^2_{V_0} := \|V_0^{1/2}u\|^2 + \|u\|^2, \tag{2.1}$$

where $\|\cdot\|$ is the norm in $L^2(M)$.

In proofs of the following three lemmas, we will use the arguments from the proof of Lemma 1 in [5].

Lemma 2.5. $Q(T_b) = W_0^{1,2}(M) \bigcap Q(V_0)$.

Proof. Denote by $\mathcal{H}_1 := W_0^{1,2}(M) \bigcap Q(V_0)$. Consider a sesquilinear form $S \colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ given by

$$S(u,v) := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v),$$

where $(\cdot,\cdot)$ is the inner product in L^2.

This sesquilinear form is closed, so the pre-Hilbert space $\mathcal{H}_1$ is complete in the norm

$$(u,v)_{t_b} := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v) + (u,v). \tag{2.2}$$

By definition of $W_0^{1,2}(M)$ and $Q(V_0)$, it follows that $\mathcal{H}_1$ is the closure of $C_c^\infty(M)$ in the norm $\|\cdot\|_{t_b}$ corresponding to (2.2).

For all $u, v \in C_c^\infty(M)$, $(u,v)_{t_b} = (u,v) + (T_b u, v)$. By Theorem 14.1 in [3], $Q(T_b)$ is the closure of $C_c^\infty(M)$ in the norm $\|\cdot\|_{t_b}$ corresponding to (2.2), so $Q(T_b) = W_0^{1,2}(M) \bigcap Q(V_0)$. □

Lemma 2.6. *Assume that* $u \in C_c^\infty(M)$. *Then there exists a sequence* $\phi_k \in C_c^\infty(M)^+$ *such that* $\|\phi_k - |u|\|_{t_b} \to 0$ *as* $k \to \infty$, *where* $\|\cdot\|_{t_b}$ *is the norm corresponding to* (2.2).

Proof. Let $u \in C_c^\infty(M)$. Then $|u| \in W^{1,2}_{\text{comp}}(M)$. Using a partition of unity we may assume that u is supported in a coordinate neighborhood. Let $|u|^\rho = J^\rho |u|$, where J^ρ is the Friedrichs mollifying operator, cf. Sect. 5.11 in [1]. Then $|u|^\rho \in C_c^\infty(M)$. It is well-known that $|u|^\rho \to |u|$

as $\rho \to 0+$ both in the space $W^{1,2}_{\text{comp}}(M)$ and in the space $L^2_{\text{comp}}(M)$. Also, since $|u|$ is continuous compactly supported on M and $V_0 \in L^2_{\text{loc}}(M)$, we have

$$\int V_0(|u|^\rho)^2\, d\mu \to \int V_0|u|^2\, d\mu \qquad \text{as } \rho \to 0+\,. \tag{2.3}$$

Therefore,

$$\||u|^\rho - |u|\|_{t_b} \to 0 \qquad \text{as } \rho \to 0+, \tag{2.4}$$

where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2). □

Lemma 2.7. *Suppose that $u \in Q(T_b)$. Then $|u| \in Q(T_b)$.*

Proof. Let $u \in Q(T_b)$. By Lemma 2.5, we get $u \in W^{1,2}_0(M) \bigcap Q(V_0)$. Since $u \in W^{1,2}_0(M)$, Lemma 7.6 from [4] gives $|u| \in W^{1,2}_0(M)$. From $u \in Q(V_0)$, we immediately get $|u| \in Q(V_0)$. Therefore, $|u| \in W^{1,2}_0(M) \bigcap Q(V_0)$, so by Lemma 2.5, we obtain $|u| \in Q(T_b)$. □

2.8. **Proof of Proposition 2.4.** We will follow the proof of Lemma 2 in [2].

Suppose that $u \in Q(T_b)^+$. By Lemma 2.5, there exists a sequence $\phi_j \in C_c^\infty(M)$ such that

$$\|\phi_j - u\|_{t_b} \to 0 \qquad \text{as } j \to \infty, \tag{2.5}$$

where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

In what follows, we will denote $(\operatorname{sign} w)(x) := \frac{w(x)}{|w(x)|}$ when $w(x) \neq 0$, and 0 otherwise.

We have

$$\begin{aligned}
\||\phi_j| - u\|^2_{t_b} &= \||\phi_j| - u\|^2 + \|d|\phi_j| - du\|^2 + \|V_0^{1/2}(|\phi_j| - u)\|^2 \\
&\le \|\phi_j - u\|^2 + \|d|\phi_j| - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2 \\
&= \|\phi_j - u\|^2 + \|\operatorname{Re}((\operatorname{sign}\bar{\phi_j})d\phi_j) - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2, \qquad (2.6)
\end{aligned}$$

where $\|\cdot\|$ denotes the norm L^2.

From (2.6) we obtain

$$\begin{aligned}
\||\phi_j| - u\|^2_{t_b} &\le \|\phi_j - u\|^2 + [\|(\operatorname{sign}\bar{\phi_j})(d\phi_j - du)\| + \|(\operatorname{sign}\bar{\phi_j} - 1)du\|]^2 + \|V_0^{1/2}(\phi_j - u)\|^2 \\
&\le \|\phi_j - u\|^2 + [\|d\phi_j - du\| + \|(\operatorname{sign}\bar{\phi_j} - 1)du\|]^2 + \|V_0^{1/2}(\phi_j - u)\|^2, \qquad (2.7)
\end{aligned}$$

where $\|\cdot\|$ denotes the norm in L^2.

By (2.5), the first, second and fourth term on the right hand side of (2.7) go to 0 as $j \to \infty$.

It remains to show that

$$\|(\operatorname{sign}\bar{\phi_j} - 1)du\| \to 0 \qquad \text{as } j \to \infty. \tag{2.8}$$

Since $\phi_j \to u$ in $L^2(M)$, a lemma of Riesz shows that there exists a subsequence ϕ_{j_k} such that $\phi_{j_k} \to u$ a.e $d\mu$, as $k \to \infty$. By Lemma 7.7 from [4], it follows that $du = 0$ almost everywhere on $\{x \in M : u(x) = 0\}$. Hence, as $k \to \infty$, $\operatorname{sign}\bar{\phi_{j_k}} \to 1$ a.e. on M. Since $du \in L^2(T^*M)$, dominated convergence theorem immediately implies (2.8) (after passing to the chosen subsequence ϕ_{j_k}).

This shows that

$$\||\phi_{j_k}| - u\|_{t_b} \to 0 \qquad \text{as } k \to \infty. \tag{2.9}$$

By (2.9) and Lemma 2.6, there exists a sequence ψ_l in $C_c^\infty(M)^+$ such that $\|\psi_l - u\|_{t_b} \to 0$ as $l \to \infty$. By Definition 2.1 it follows that T_b has a positive form core. □

In what follows, we will use a version of Kato's inequality. For the proof of this inequality in general setting, cf. Theorem 5.6 in [1].

Theorem 2.9. *Let E be a Hermitian vector bundle on M, and let $\nabla\colon C^\infty(E) \to C^\infty(T^*M \otimes E)$ be a Hermitian connection on E. Let $\nabla^*\colon C^\infty(T^*M \otimes E) \to C^\infty(E)$ be formal adjoint of ∇ with respect to the usual inner product on $L^2(E)$. Assume that $u \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla u \in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M|u| \;\leq\; \operatorname{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle, \tag{2.10}$$

where

$$\operatorname{sign} u(x) \;=\; \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } \; u(x) \neq 0 \;, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10. Let (X, μ) be a measure space. A bounded linear operator $A\colon L^2(X,\mu) \to L^2(X,\mu)$ is said to be *positivity preserving* if for every $0 \leq u \in L^2(X,\mu)$ we have $Au \geq 0$.

We will also use the following abstract theorem due to Simon, cf. Theorem 2.1 in [11].

Theorem 2.11 (Simon [11])**.** *Suppose that (X,μ) is a measure space. Suppose that H is a positive self-adjoint operator in $L^2(X,\mu)$. Then $(H+1)^{-1}$ is positivity preserving if and only if the following two conditions are satisfied*

(i) *For every $u \in Q(H)$, we have $|u| \in Q(H)$.*
(ii) *For every $u \in \operatorname{Dom}(H)$ and $0 \leq v \in Q(H)$, the following is true*

$$\operatorname{Re}[h(|u|, v)] \leq \operatorname{Re}((\operatorname{sign} u)v, Hu),$$

where h is the quadratic form associated to H, and $(\operatorname{sign} u)(x) = \frac{u(x)}{|u|}$ whenever $u(x) \neq 0$, and 0 otherwise.

The following lemma extends Lemma 1 from [5] to the case of Riemannian manifolds.

Lemma 2.12. *The operator $(T_b + 1)^{-1}$ is positivity preserving.*

Proof. Let t_b be the quadratic form associated to T_b. By Theorem 2.11, it suffices to check the following conditions

(i) For every $u \in Q(T_b)$, we have $|u| \in Q(T_b)$ and
(ii) For every $u \in \operatorname{Dom}(T_b)$ and $0 \leq v \in Q(T_b)$, the following is true

$$\operatorname{Re}[t_b(|u|, v)] \leq \operatorname{Re}((\operatorname{sign} u)v, T_b u)$$

Condition (i) follows immediately by Lemma 2.7.

We now prove that the condition (ii) holds. Let $u \in \mathrm{Dom}(T_b)$. Then $(\Delta_M + V_0)u \in L^2(M)$ and hence $\Delta_M u \in L^1_{\mathrm{loc}}(M)$.

For $u \in \mathrm{Dom}(T_b)$ and $0 \le \phi \in C_c^\infty(M)$ we have,

$$\begin{aligned} \mathrm{Re}[t_b(|u|,\phi)] &= \mathrm{Re}(|u|,(\Delta_M + V_0)\phi) = (|u|,\Delta_M\phi) + (|u|,V_0\phi) = (\Delta_M|u|,\phi) + (V_0|u|,\phi) \\ &\le \mathrm{Re}((\operatorname{sign}\bar{u})\Delta_M u,\phi) + ((\operatorname{sign}\bar{u})V_0 u,\phi) = \mathrm{Re}((\operatorname{sign}\bar{u})T_b u,\phi) = \mathrm{Re}((\operatorname{sign} u)\phi, T_b u). \end{aligned} \tag{2.11}$$

Here we used integration by parts and the special case of Kato inequality (2.10) for Δ_M.

Let $0 \le v \in Q(T_b)$. By Proposition 2.4, there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\|\phi_j - v\|_{t_b} \to 0$ as $j \to \infty$, where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

From (2.11), we obtain

$$\mathrm{Re}[t_b(|u|,v)] = \lim_{j\to\infty} \mathrm{Re}[t_b(|u|,\phi_j)] \le \lim_{j\to\infty} \mathrm{Re}((\operatorname{sign} u)\phi_j, T_b u) = \mathrm{Re}((\operatorname{sign} u)v, T_b u).$$

This proves condition (ii) and the lemma. □

In what follows T_0 is as in hypothesis of Theorem 2.2. Without the loss of generality we may and we will assume that T_0 is a positive self-adjoint operator.

We will also use the following notation $\mathbb{Z}_+ := \{1,2,3,\dots\}$.

Proposition 2.13. $(T_0+1)^{-1}$ *is positivity preserving.*

Proof. We will adopt the arguments from the proof of Lemma 2 in [5] to our setting.

For every $k \in \mathbb{Z}_+$ and $x \in M$, define

$$Q_k(x) = \begin{cases} V_0(x) & \text{if } V_0(x) \ge -k, \\ -k & \text{if } V_0(x) < -k. \end{cases}$$

Let T_k be the Friedrichs extension of $(\Delta_M + Q_k)|C_c^\infty(M)$.

Then for all $k \in \mathbb{Z}_+$ and $u \in C_c^\infty(M)$, we have

$$(u, T_k u) \ge (u, T_0 u) \ge 0, \tag{2.12}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(M)$.

From (2.12) it follows that

$$T_0 \le T_k \qquad \text{for all } k \in \mathbb{Z}_+, \tag{2.13}$$

i.e. $Q(T_k) \subset Q(T_0)$, and for all $u \in Q(T_k)$, $t_0(u,u) \le t_k(u,u)$, where t_0 and t_k are the quadratic forms associated to T_0 and T_k respectively.

Furthermore, for all $u \in C_c^\infty(M)$, the following is true

$$(u, T_k u) \to (u, T_0 u) \qquad \text{as } k \to \infty. \tag{2.14}$$

Clearly, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of Friedrichs extension, it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

This, (2.13) and (2.14) show that the hypotheses of abstract Theorem 7.9 from [3] are satisfied.

Therefore, as $k \to \infty$, $T_k \to T_0$ in the strong resolvent sense.

By Lemma 2.12, $(T_k + 1)^{-1}$ is positivity preserving for all $k \in \mathbb{Z}_+$. Therefore, $(T_0 + 1)^{-1}$ is also positivity preserving. □

Corollary 2.14. *Assume that $u \in Q(T_0)$. Then $|u| \in Q(T_0)$.*

Proof. T_0 is a positive self-adjoint operator in $L^2(M)$. By Proposition 2.13, the operator $(T_0 + 1)^{-1}$ is positivity preserving. Now the corollary follows immediately from Theorem 2.11. □

2.15. **Truncation operators corresponding to T_0.** Let T_0 be as in hypothesis of Theorem 2.2.

Define $V_0^+ := \max\{V_0, 0\}$, $V_0^- := \max\{-V_0, 0\}$, and for each $k \in \mathbb{Z}_+$, let $V_0^k := \min\{k, V_0^-\}$.

Denote by T_+ and T_k the Friedrichs extension of $(\Delta_M + V_0^+)|_{C_c^\infty(M)}$ and $(\Delta_M + V_0^+ - V_0^k)|_{C_c^\infty(M)}$ respectively.

Let t_0, t_+ and t_k ($k \in \mathbb{Z}_+$) be the quadratic forms associated to T_0, T_+ and T_k respectively.

The following lemma is analogous to Lemma 3 in [2].

Lemma 2.16. *With the notations of Sect. 2.15,*

(i) *$T_k \to T_0$ in the strong resolvent sense as $k \to \infty$.*

(ii) *$Q(T_+) \subset Q(T_0)$.*

Proof. For all $k \in \mathbb{Z}_+$, we clearly have $T_0 \leq T_k$. Also, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of T_0 it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

Clearly, for all $w \in C_c^\infty(M)$

$$(w, T_k w) \to (w, T_0 w) \qquad \text{as } k \to \infty.$$

We now apply Theorem 7.9 in [3] to conclude the proof of (i).

Property (ii) follows immediately since $T_+ \geq T_0$. □

2.17. **Proof of Theorem 2.2.** We will adopt the arguments from the proof of Theorem 1 in [2].

By the proof of Lemma 2.16 it follows that

$$Q(T_+) \subset Q(T_k) \subset Q(T_0) \tag{2.15}$$

and

$$\|\cdot\|_{t_0} \leq \|\cdot\|_{t_k} \leq \|\cdot\|_{t_+}, \tag{2.16}$$

where $\|\cdot\|_{t_0}$, $\|\cdot\|_{t_k}$ and $\|\cdot\|_{t_+}$ are the norms associated to t_0, t_k and t_+ respectively, cf. (1.2).

In fact, $Q(T_k) = Q(T_+)$ since the norms $\|\cdot\|_{t_k}$ and $\|\cdot\|_{t_+}$ are equivalent, because $V_0^+ - V_0^k$ and V_0^+ differ by a bounded function.

By Proposition 2.4, T_+ has a positive form core, i.e. for every $u \in Q(T_+)^+$, there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\|\phi_j - u\|_{t_+} \to 0$ as $j \to \infty$. By (2.16) it follows that

$$\|\phi_j - u\|_{t_0} \to 0 \qquad \text{as } j \to \infty.$$

To prove the theorem, it remains to show that for every $w \in Q(T_0)^+$, there exists a sequence $w_j \in Q(T_+)^+$ such that

$$\|w_j - w\|_{t_0} \to 0 \qquad \text{as } j \to \infty. \tag{2.17}$$

Let $w \in Q(T_0)^+$. For every $k, l \in \mathbb{Z}_+$ define

$$w_l := \left(\frac{1}{l}T_0 + 1\right)^{-1} w$$

and

$$w_l^k := \left(\frac{1}{l}T_k + 1\right)^{-1} w.$$

This makes sense since $0 \le T_0 \le T_k$ are self-adjoint operators.

By Lemma 2.12, the operator $(T_k + 1)^{-1}$ is positivity preserving. Hence $w_l^k \in \mathrm{Dom}(T_k)^+ \subset Q(T_k)^+ = Q(T_+)^+$.

Since the operators $(T_0 + 1)^{1/2}$ and $(T_0/l + 1)^{-1}$ commute, we have

$$\|w_l - w\|_{t_0} = \left\| \left(\left(\frac{1}{l}T_0 + 1\right)^{-1} - 1 \right) (T_0 + 1)^{1/2} w \right\|, \tag{2.18}$$

where $\|\cdot\|$ denotes $L^2(M)$ norm.

Clearly,

$$\left(\frac{1}{l}T_0 + 1\right)^{-1} \to 1 \qquad \text{strongly as } l \to \infty.$$

This and (2.18) show that

$$\|w_l - w\|_{t_0} \to 0 \qquad \text{as } l \to \infty. \tag{2.19}$$

Fix $l \in \mathbb{Z}_+$. For each $k \in \mathbb{Z}_+$, let $t_0 + l$ and $t_k + l$ denote the quadratic forms corresponding to (positive self-adjoint) operators $T_0 + l$ and $T_k + l$ respectively. Let $\|\cdot\|_{t_0+l}$ and $\|\cdot\|_{t_k+l}$ denote the norms in $Q(T_0 + l)$ and $Q(T_k + l)$ respectively, cf. (1.2). The corresponding inner products will be denoted by $(\cdot,\cdot)_{t_0+l}$ and $(\cdot,\cdot)_{t_k+l}$.

Using (2.15), (2.16) and Cauchy-Schwarz inequality we have for all $w \in Q(T_0)^+$

$$\begin{aligned}
&\|(T_k + l)^{-1}w - (T_0 + l)^{-1}w\|^2_{t_0+l} \\
&\quad = \|(T_k + l)^{-1}w\|^2_{t_0+l} + \|(T_0 + l)^{-1}w\|^2_{t_0+l} - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)_{t_0+l} \\
&\quad \le \|(T_k + l)^{-1}w\|^2_{t_k+l} + \|(T_0 + l)^{-1}w\|^2_{t_0+l} - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)_{t_0+l} \\
&\quad = ((T_0 + l)^{-1}w, w) - ((T_k + l)^{-1}w, w) \\
&\quad + (1 - l)[\|(T_k + l)^{-1}w\|^2 + \|(T_0 + l)^{-1}w\|^2 - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)] \\
&\quad \le ((T_0 + l)^{-1}w, w) - ((T_k + l)^{-1}w, w) \le \|(T_0 + l)^{-1}w - (T_k + l)^{-1}w\|\|w\|,
\end{aligned} \tag{2.20}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(M)$ and $\|\cdot\|$ is the norm in $L^2(M)$.

By Lemma 2.16, it follows that for fixed $l \in \mathbb{Z}_+$, $T_k + l \to T_0 + l$ in the strong resolvent sense as $k \to \infty$.

Clearly, for any positive self-adjoint operator A, $(A/l+1)^{-1} = l(A+l)^{-1}$. Therefore by (2.20), for a fixed $l \in \mathbb{Z}_+$

$$\|w_l^k - w_l\|_{t_0+l} \to 0 \qquad \text{as } k \to \infty.$$

This is equivalent to

$$\|w_l^k - w_l\|_{t_0} \to 0 \qquad \text{as } k \to \infty. \tag{2.21}$$

Since $w_l^k \in Q(T_+)^+$, we can use (2.19) and (2.21) to choose a subsequence $\{w_j\}$ from $\{w_l^k\}$ so that (2.17) holds.

This concludes the proof of the theorem. □

3. Proof of Theorem 1.5

We essentially follow the proof of Theorem 2 in [2]; however, we need to use Kato inequality (2.10) for operators on manifolds.

Without the loss of generality, we may and we will assume that $\Delta_M + V_0 \geq 0$ and $V_1 \geq 0$.

Let us denote $T_q := T_0 \tilde{+} V_1$ and let $t_{\min}$ and t_q be as in Sect. 1.4 and Sect. 1.3. Since $t_{\min}$ and t_q coincide on $C_c^\infty(M)$, it is sufficient to show that $C_c^\infty(M)$ is dense in the Hilbert space $Q(T_q) = Q(T_0) \bigcap Q(V_1)$ with the inner product

$$(\cdot,\cdot)_{t_q} := t_q(\cdot,\cdot) + (\cdot,\cdot)_{L^2(M)},$$

where $t_q(\cdot,\cdot)$ is the sesquilinear form obtained by polarization of t_q.

Let $v \in Q(T_q)$ be orthogonal to $C_c^\infty(M)$ in $(\cdot,\cdot)_{t_q}$. This means that for all $w \in C_c^\infty(M)$,

$$((\Delta_M + V_0 + V_1)v, w)_{L^2(M)} + (v, w)_{L^2(M)} = 0.$$

This leads to the following distributional equality

$$\Delta_M v = -(V_0 + V_1 + 1)v. \tag{3.1}$$

Since $V_1 \in L^1_{\text{loc}}(M)$ and $v \in Q(V_1)$, we have

$$2|V_1 v| = 2|V_1||v| \leq |V_1| + |V_1||v|^2$$

which immediately gives $V_1 v \in L^1_{\text{loc}}(M)$.

Since $V_0 \in L^2_{\text{loc}}(M)$, it follows that $V_0 v \in L^1_{\text{loc}}(M)$. From (3.1) we obtain $\Delta_M v \in L^1_{\text{loc}}(M)$.

Using Kato inequality (2.10) in case $\nabla = d$ and the equation (3.1), we get

$$\Delta_M|v| \leq \operatorname{Re}(\operatorname{sign}\bar{v}\Delta_M v) = -V_0|v| - V_1|v| - |v| \leq -(V_0+1)|v|. \tag{3.2}$$

The last inequality in (3.2) holds since $V_1 \geq 0$.

From (3.2), we obtain the following distributional inequality

$$(\Delta_M + V_0 + 1)|v| \leq 0. \tag{3.3}$$

Let T_0 be as in hypothesis, and let t_0 denote the closed quadratic form associated to T_0.

Using (3.3), we get

$$((T_0 + 1)w, |v|)_{L^2(M)} \leq 0 \qquad \text{for all } w \in C_c^\infty(M)^+. \tag{3.4}$$

Since $v \in Q(T_0)$, Corollary 2.14 gives $|v| \in Q(T_0)$. Therefore, we can write (3.4) as

$$(w, |v|)_{t_0} \leq 0 \qquad \text{for all } w \in C_c^\infty(M)^+, \tag{3.5}$$

where $(\cdot,\cdot)_{t_0} = t_0(\cdot,\cdot) + (\cdot,\cdot)_{L^2(M)}$ denotes the inner product in $Q(T_0)$.

Let $f := (T_0 + 1)^{-1}|v|$. By Proposition 2.13, $(T_0 + 1)^{-1}$ is positivity preserving, so $f \in \mathrm{Dom}(T_0)^+ \subset Q(T_0)^+$.

By Theorem 2.2, T_0 has a positive form core. Therefore, there exists a sequence $f_k \in C_c^\infty(M)^+$ such that

$$\lim_{k\to\infty}(f_k, |v|)_{t_0} = (f, |v|)_{t_0} = ((T_0 + 1)^{-1}|v|, |v|)_{t_0} = \|v\|^2, \tag{3.6}$$

where v and $(\cdot,\cdot)_{t_0}$ are as in (3.5), and $\|\cdot\|$ is the norm in $L^2(M)$.

From (3.5) and (3.6) we obtain $\|v\|^2 \le 0$, i.e. $v = 0$.

This shows that $C_c^\infty(M)$ is dense in $Q(T_q)$, and the theorem is proven. □

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