THE FORM SUM AND THE FRIEDRICHS EXTENSION OF SCHRÖDINGER-TYPE OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract. We consider $H_V = \Delta_M + V$, where $(M, g)$ is a Riemannian manifold (not necessarily complete), and $\Delta_M$ is the scalar Laplacian on $M$. We assume that $V = V_0 + V_1$, where $V_0 \in L^2_{\text{loc}}(M)$ and $-C \leq V_1 \in L^1_{\text{loc}}(M)$ ($C$ is a constant) are real-valued, and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$. Let $T_0$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C_c^\infty(M)}$. We prove that the form sum $T_0 \tilde{+} V_1$ coincides with the self-adjoint operator $T_F$ associated to the closure of the restriction to $C_c^\infty(M) \times C_c^\infty(M)$ of the sum of two closed quadratic forms of $T_0$ and $V_1$. This is an extension of a result of Cycon. The proof adopts the scheme of Cycon, but it requires the use of a more general version of Kato’s inequality for operators on Riemannian manifolds.

1. Introduction and the main result

Let $(M, g)$ be a Riemannian manifold (i.e. $M$ is a $C^\infty$-manifold, $(g_{jk})$ is a Riemannian metric on $M$), dim $M = n$. We will assume that $M$ is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive $C^\infty$-density $\rho(x)$ such that $d\mu = \rho(x) dx^1 dx^2 \ldots dx^n$. We do not assume that $(M, g)$ is complete.

We will consider a Schrödinger type operator of the form

$$H_V = \Delta_M + V.$$ 

Here $\Delta_M := d^*d$, where $d: C^\infty(M) \to \Omega^1(M)$, and $V \in L^1_{\text{loc}}(M)$ is real-valued.

1.1. Maximal operator. We define the maximal operator $H_{V,\text{max}}$ associated to $H_V$ as an operator in $L^2(M)$ given by $H_{V,\text{max}} u = H_V u$ with domain

$$\text{Dom}(H_{V,\text{max}}) = \{ u \in L^2(M) : Vu \in L^1_{\text{loc}}(M), \ H_V u \in L^2(M) \}. \quad (1.1)$$

Here $\Delta_M u$ in $H_V u = \Delta_M u + Vu$ is understood in distributional sense.

We make the following assumptions on $V$.

Assumption A. Assume $V = V_0 + V_1$, where

(i) $V_0 \in L^2_{\text{loc}}(M)$ and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$.

(ii) $V_1 \in L^1_{\text{loc}}(M)$ and $V_1 \geq -C$, where $C > 0$ is a constant.

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1.2. **Quadratic forms.** For any self-adjoint operator $T$: $\text{Dom}(T) \subset L^2(M) \to L^2(M)$ such that $T \geq -\alpha$, we will denote by $Q(T)$ the domain of the quadratic form $t$ associated to $T$. By Theorem 2.1 in [3], $t$ is a closed semibounded below form, i.e. $Q(T)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_t = t(u, v) + (1 + \alpha)(u, v)_{L^2(M)}, \quad (1.2)$$

where $t(\cdot, \cdot)$ is the sesquilinear form obtained by polarization of $t$.

1.3. **Form sum.** By (i) of Assumption A, $\Delta_M + V_0$ is symmetric and semibounded below on $C_c^\infty(M)$, so we can associate to it a semibounded below self-adjoint operator $T_0$ (Friedrichs extension, cf. Theorem 14.1 in [3]).

We will denote by $T_0 + V_1$ the form sum of $T_0$ and $V_1$. By Theorem 4.1 in [3], this is the self-adjoint operator associated to the semibounded below closed quadratic form $t_q$ given by the sum of two semibounded below closed quadratic forms corresponding to $T_0$ and $V_1$. By the same theorem, the following is true: $Q(T_0 + V_1) = Q(T_0) \cap Q(V_1)$. Clearly, $T_0 + V_1$ is a self-adjoint restriction of $H_{V,\max}$.

1.4. **Operator $T_F$.** Denote by $t_{\text{min}}$ the restriction of $t_q$ to $C_c^\infty(M) \times C_c^\infty(M)$. Denote by $T_F$ the self-adjoint operator associated to the closure of $t_{\text{min}}$ in the sense of the norm in $Q(T_0 + V_1)$.

Clearly, $T_F$ is a self-adjoint restriction of $H_{V,\max}$.

We will give a sufficient condition for $T_F = T_0 + V_1$.

**Theorem 1.5.** Suppose that the assumption A holds.

Then $T_F = T_0 + V_1$.

**Remark 1.6.** Theorem 1.5 was proven by Cycon [2] in case of the operator $-\Delta + V$ in an open set $M \subset \mathbb{R}^n$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric. In case $V_0 = 0$ and $M = \mathbb{R}^n$ with standard metric, Theorem 1.5 was proven in Simon [13].

2. **Operators with a positive form core**

**Definition 2.1.** Let $T: C_c^\infty(M) \subset L^2(M) \to L^2(M)$ be a symmetric semibounded below operator. Let $T_F$ denote its Friedrichs extension and $Q(T_F)^+$ the set of a.e. positive elements of $Q(T_F)$. We say that $T_F$ has a positive form core if for every $u \in Q(T_F)^+$, there exists a sequence $u_k \in C_c^\infty(M)^+$ such that

$$\|u_k - u\|_t \to 0 \quad \text{as } k \to \infty,$$

where $\| \cdot \|_t$ is the norm associated to the closure of quadratic form $t(v, w) := (Tv, w)$ $(v, w \in C_c^\infty(M))$ via (1.2).

The main result of this section is

**Theorem 2.2.** Suppose that $\Delta_M + V_0$ is as in (i) of Assumption A. Let $T_0$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C_c^\infty(M)}$.

Then $T_0$ has a positive form core.
Remark 2.3. In case of the operator $-\Delta + V_0$ in an open set $M \subset \mathbb{R}^n$, Theorem 2.2 was proven in [2, Th. 1].

We will first prove the following special case of Theorem 2.2

Proposition 2.4. Suppose that $-C \leq V_0 \in L^2_{\text{loc}}(M)$, where $C > 0$ is a constant. Let $T_b$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C_c^\infty(M)}$.

Then $T_b$ has a positive form core.

We begin with a few preliminary lemmas.

In what follows $T_b$ is as in the hypothesis of Proposition 2.4, and $t_b$ is the closed quadratic form associated with $T_b$. Without the loss of generality, we may and we will assume that $V_0 \geq 0$ so that $T_b$ is a positive self-adjoint operator.

We will denote $W^{1,2}(M) := \{ u \in L^2(M) : du \in L^2(T^*M) \}$. By $W^{1,2}_0(M)$ we will denote the closure of $C_c^\infty(M)$ in the norm $\|u\|^2_{W^{1,2}} := \|du\|^2 + \|u\|^2$, where $\| \cdot \|$ is the $L^2$ norm. By $Q(V_0)$ we will denote the set $\{ u \in L^2(M) : V_0^{1/2}u \in L^2(M) \}$. Clearly, $Q(V_0)$ is the closure of $C_c^\infty(M)$ in the norm

$$\|u\|^2_{T_b} := \|V_0^{1/2}u\|^2 + \|u\|^2,$$

(2.1)

where $\| \cdot \|$ is the norm in $L^2(M)$.

In proofs of the following three lemmas, we will use the arguments from the proof of Lemma 1 in [5].

Lemma 2.5. $Q(T_b) = W^{1,2}_0(M) \cap Q(V_0)$.

Proof. Denote by $\mathcal{H}_1 := W^{1,2}_0(M) \cap Q(V_0)$. Consider a sesquilinear form $S : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ given by

$$S(u, v) := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v),$$

where $(\cdot, \cdot)$ is the inner product in $L^2$.

This sesquilinear form is closed, so the pre-Hilbert space $\mathcal{H}_1$ is complete in the norm

$$(u, v)_{T_b} := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v) + (u, v).$$

(2.2)

By definition of $W^{1,2}_0(M)$ and $Q(V_0)$, it follows that $\mathcal{H}_1$ is the closure of $C_c^\infty(M)$ in the norm $\| \cdot \|_{T_b}$ corresponding to (2.2).

For all $u, v \in C_c^\infty(M)$, $(u, v)_{T_b} = (u, v) + (T_bu, v)$. By Theorem 14.1 in [3], $Q(T_b)$ is the closure of $C_c^\infty(M)$ in the norm $\| \cdot \|_{T_b}$ corresponding to (2.2), so $Q(T_b) = W^{1,2}_0(M) \cap Q(V_0)$. \hfill $\square$

Lemma 2.6. Assume that $u \in C_c^\infty(M)$. Then there exists a sequence $\phi_k \in C_c^\infty(M)^+$ such that $\|\phi_k - |u|\|_{T_b} \to 0$ as $k \to \infty$, where $\| \cdot \|_{T_b}$ is the norm corresponding to (2.2).

Proof. Let $u \in C_c^\infty(M)$. Then $|u| \in W^{1,2}_{\text{comp}}(M)$. Using a partition of unity we may assume that $u$ is supported in a coordinate neighborhood. Let $|u|^\rho = J^\rho|u|$, where $J^\rho$ is the Friedrichs mollifying operator, cf. Sect. 5.11 in [1]. Then $|u|^\rho \in C_c^\infty(M)$. It is well-known that $|u|^\rho \to |u|$
as $\rho \to 0+$ both in the space $W^{1,2}_\text{comp}(M)$ and in the space $L^2_\text{comp}(M)$. Also, since $|u|$ is continuous compactly supported on $M$ and $V_0 \in L^2_\text{loc}(M)$, we have
\[
\int V_0(|u|^\rho)^2 \, d\mu \to \int V_0|u|^2 \, d\mu \quad \text{as } \rho \to 0.+.
\]
Therefore,
\[
\|\|u|^\rho - |u|\|_{t_b} \to 0 \quad \text{as } \rho \to 0+,
\tag{2.4}
\]
where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2). \hfill \Box

**Lemma 2.7.** Suppose that $u \in Q(T_b)$. Then $|u| \in Q(T_b)$.

**Proof.** Let $u \in Q(T_b)$. By Lemma 2.5, we get $u \in W^{1,2}_0(M) \cap Q(V_0)$. Since $u \in W^{1,2}_0(M)$, Lemma 7.6 from [4] gives $|u| \in W^{1,2}_0(M)$. From $u \in Q(V_0)$, we immediately get $|u| \in Q(V_0)$. Therefore, $|u| \in W^{1,2}_0(M) \cap Q(V_0)$, so by Lemma 2.5, we obtain $|u| \in Q(T_b)$ \hfill \Box

### 2.8. Proof of Proposition 2.4.
We will follow the proof of Lemma 2 in [2]. Suppose that $u \in Q(T_b^+)$. By Lemma 2.5, there exists a sequence $\phi_j \in C_\infty(M)$ such that
\[
\|\phi_j - u\|_{t_b} \to 0 \quad \text{as } j \to \infty,
\tag{2.5}
\]
where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

In what follows, we will denote $(\text{sign } w)(x) := \frac{w(x)}{|w(x)|}$ when $w(x) \neq 0$, and 0 otherwise. We have
\[
\|\phi_j - u\|^2_{t_b} = \|\phi_j - u\|^2 + \|\text{sign } \phi_j \cdot (\phi_j - u)\|^2
\leq \|\phi_j - u\|^2 + \|\text{sign } \phi_j \cdot (\phi_j - u)\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2
\leq 2\|\phi_j - u\|^2 + \|\text{sign } \phi_j \cdot (\phi_j - u)\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2
\tag{2.6}
\]
where $\|\cdot\|$ denotes the norm $L^2$.

From (2.6) we obtain
\[
\|\phi_j - u\|^2_{t_b} \leq \|\phi_j - u\|^2 + \|(\text{sign } \phi_j) (\phi_j - u)\|^2 + \|(\text{sign } \phi_j - 1) du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2
\leq \|\phi_j - u\|^2 + \|\phi_j - u\|^2 + \|(\text{sign } \phi_j - 1) du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2,
\tag{2.7}
\]
where $\|\cdot\|$ denotes the norm in $L^2$.

By (2.5), the first, second and fourth term on the right hand side of (2.7) go to 0 as $j \to \infty$.

It remains to show that
\[
\|(\text{sign } \phi_j - 1) du\| \to 0 \quad \text{as } j \to \infty.
\tag{2.8}
\]

Since $\phi_j \to u$ in $L^2(M)$, a lemma of Riesz shows that there exists a subsequence $\phi_{j_k}$ such that $\phi_{j_k} \to u$ a.e. $du$, as $k \to \infty$. By Lemma 7.7 from [4], it follows that $du = 0$ almost everywhere on $\{x \in M : u(x) = 0\}$. Hence, as $k \to \infty$, sign $\phi_{j_k} \to 1$ a.e. on $M$. Since $du \in L^2(T^*M)$, dominated convergence theorem immediately implies (2.8) (after passing to the chosen subsequence $\phi_{j_k}$).
This shows that
\[ \| \phi_{j_k} - u \|_{t_b} \to 0 \quad \text{as } k \to \infty. \] (2.9)

By (2.9) and Lemma 2.6, there exists a sequence \( \psi_l \) in \( C_c^\infty(M)^+ \) such that \( \| \psi_l - u \|_{t_b} \to 0 \) as \( l \to \infty \). By Definition 2.1 it follows that \( T_b \) has a positive form core. \( \square \)

In what follows, we will use a version of Kato’s inequality. For the proof of this inequality in general setting, cf. Theorem 5.6 in [1].

**Theorem 2.9.** Let \( E \) be a Hermitian vector bundle on \( M \), and let \( \nabla : C^\infty(E) \to C^\infty(T^*M \otimes E) \) be a Hermitian connection on \( E \). Let \( \nabla^* : C^\infty(T^*M \otimes E) \to C^\infty(E) \) be formal adjoint of \( \nabla \) with respect to the usual inner product on \( L^2(E) \). Assume that \( u \in L^1_{\text{loc}}(E) \) and \( \nabla^* \nabla u \in L^1_{\text{loc}}(E) \). Then
\[ \Delta_M |u| \leq \text{Re}(\nabla^* \nabla u, \text{sign } u), \] (2.10)

where
\[ \text{sign } u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]

**Definition 2.10.** Let \( (X, \mu) \) be a measure space. A bounded linear operator \( A : L^2(X, \mu) \to L^2(X, \mu) \) is said to be positivity preserving if for every \( 0 \leq u \in L^2(X, \mu) \) we have \( Au \geq 0 \).

We will also use the following abstract theorem due to Simon, cf. Theorem 2.1 in [11].

**Theorem 2.11** (Simon [11]). Suppose that \( (X, \mu) \) is a measure space. Suppose that \( H \) is a positive self-adjoint operator in \( L^2(X, \mu) \). Then \( (H + 1)^{-1} \) is positivity preserving if and only if the following two conditions are satisfied

(i) For every \( u \in Q(H) \), we have \( |u| \in Q(H) \).
(ii) For every \( u \in \text{Dom}(H) \) and \( 0 \leq v \in Q(H) \), the following is true
\[ \text{Re}[h(|u|, v)] \leq \text{Re}((\text{sign } u)v, Hu), \]
where \( h \) is the quadratic form associated to \( H \), and \( (\text{sign } u)(x) = \frac{u(x)}{|u(x)|} \) whenever \( u(x) \neq 0 \), and 0 otherwise.

The following lemma extends Lemma 1 from [5] to the case of Riemannian manifolds.

**Lemma 2.12.** The operator \( (T_b + 1)^{-1} \) is positivity preserving.

**Proof.** Let \( t_b \) be the quadratic form associated to \( T_b \). By Theorem 2.11, it suffices to check the following conditions

(i) For every \( u \in Q(T_b) \), we have \( |u| \in Q(T_b) \) and
(ii) For every \( u \in \text{Dom}(T_b) \) and \( 0 \leq v \in Q(T_b) \), the following is true
\[ \text{Re}[t_b(|u|, v)] \leq \text{Re}((\text{sign } u)v, T_b u) \]
Condition (i) follows immediately by Lemma 2.7. We now prove that the condition (ii) holds. Let $u \in \text{Dom}(T_0)$. Then $(\Delta_M + V_0)u \in L^2(M)$ and hence $\Delta_M u \in L^1_{\text{loc}}(M)$. For $u \in \text{Dom}(T_0)$ and $0 \leq \phi \in C^\infty_c(M)$ we have,

$$\text{Re}[t_b(|u|, \phi)] = \text{Re}(|u|, (\Delta_M + V_0)\phi) = (|u|, \Delta_M \phi) + (|u|, V_0\phi) = (\Delta_M|u|, \phi) + (V_0|u|, \phi) \leq \text{Re}((\text{sign } \bar{u})\Delta_M u, \phi) + ((\text{sign } \bar{u})V_0 u, \phi) = \text{Re}((\text{sign } u)\Delta_M u, \phi) = \text{Re}((\text{sign } u)\phi, T_b u).$$

(2.11)

Here we used integration by parts and the special case of Kato inequality (2.10) for $\Delta_M$.

Let $0 \leq v \in Q(T_0)$. By Proposition 2.4, there exists a sequence $\phi_j \in C^\infty_c(M)^+$ such that $\|\phi_j - v\|_{b} \to 0$ as $j \to \infty$, where $\| \cdot \|_{b}$ is the norm corresponding to (2.2).

From (2.11), we obtain

$$\text{Re}[t_b(|u|, v)] = \lim_{j \to \infty} \text{Re}[t_b(|u|, \phi_j)] \leq \lim_{j \to \infty} \text{Re}((\text{sign } u)\phi_j, T_b u) = \text{Re}((\text{sign } u)v, T_b u).$$

This proves condition (ii) and the lemma.

In what follows $T_0$ is as in hypothesis of Theorem 2.2. Without the loss of generality we may and we will assume that $T_0$ is a positive self-adjoint operator.

We will also use the following notation $Z_+ := \{1, 2, 3, \ldots \}$.

**Proposition 2.13.** $(T_0 + 1)^{-1}$ is positivity preserving.

**Proof.** We will adopt the arguments from the proof of Lemma 2 in [5] to our setting.

For every $k \in Z_+$ and $x \in M$, define

$$Q_k(x) = \begin{cases} V_0(x) & \text{if } V_0(x) \geq -k, \\ -k & \text{if } V_0(x) < -k. \end{cases}$$

Let $T_k$ be the Friedrichs extension of $(\Delta_M + Q_k)|C^\infty_c(M)$. Then for all $k \in Z_+$ and $u \in C^\infty_c(M)$, we have

$$(u, T_k u) \geq (u, T_0 u) \geq 0,$$

(2.12)

where $(\cdot, \cdot)$ is the inner product in $L^2(M)$.

From (2.12) it follows that

$$T_0 \leq T_k \quad \text{for all } k \in Z_+,$$

(2.13)

i.e. $Q(T_k) \subset Q(T_0)$, and for all $u \in Q(T_k)$, $t_0(u, u) \leq t_k(u, u)$, where $t_0$ and $t_k$ are the quadratic forms associated to $T_0$ and $T_k$ respectively.

Furthermore, for all $u \in C^\infty_c(M)$, the following is true

$$(u, T_k u) \to (u, T_0 u) \quad \text{as } k \to \infty.$$

(2.14)

Clearly, $C^\infty_c(M) \subset Q(T_k)$ for all $k \in Z_+$. By definition of Friedrichs extension, it follows that $C^\infty_c(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

This, (2.13) and (2.14) show that the hypotheses of abstract Theorem 7.9 from [3] are satisfied. Therefore, as $k \to \infty$, $T_k \to T_0$ in the strong resolvent sense.
By Lemma 2.12, \((T_k + 1)^{-1}\) is positivity preserving for all \(k \in \mathbb{Z}_+\). Therefore, \((T_0 + 1)^{-1}\) is also positivity preserving.

\[\square\]

**Corollary 2.14.** Assume that \(u \in Q(T_0)\). Then \(|u| \in Q(T_0)\).

**Proof.** \(T_0\) is a positive self-adjoint operator in \(L^2(M)\). By Proposition 2.13, the operator \((T_0 + 1)^{-1}\) is positivity preserving. Now the corollary follows immediately from Theorem 2.11.

\[\square\]

2.15. Truncation operators corresponding to \(T_0\). Let \(T_0\) be as in hypothesis of Theorem 2.2.

Define \(V_0^+ := \max\{V_0, 0\}\), \(V_0^- := \max\{-V_0, 0\}\), and for each \(k \in \mathbb{Z}_+\), let \(V_0^k := \min\{k, V_0^-\}\).

Denote by \(T_+\) and \(T_k\) the Friedrichs extension of \((\Delta_M + V_0^+)|_{C_0^\infty(M)}\) and \((\Delta_M + V_0^+ - V_0^k)|_{C_0^\infty(M)}\) respectively.

Let \(t_0\), \(t_+\), and \(t_k\) \((k \in \mathbb{Z}_+)\) be the quadratic forms associated to \(T_0\), \(T_+\) and \(T_k\) respectively.

The following lemma is analogous to Lemma 3 in [2].

**Lemma 2.16.** With the notations of Sect. 2.15,

(i) \(T_k \to T_0\) in the strong resolvent sense as \(k \to \infty\).

(ii) \(Q(T_+) \subset Q(T_0)\).

**Proof.** For all \(k \in \mathbb{Z}_+\), we clearly have \(T_0 \leq T_k\). Also, \(C_0^\infty(M) \subset Q(T_k)\) for all \(k \in \mathbb{Z}_+\). By definition of \(T_0\) it follows that \(C_0^\infty(M)\) is dense in \(Q(T_0)\) (in the norm of \(Q(T_0)\)).

Clearly, for all \(w \in C_0^\infty(M)\)

\[(w, T_k w) \to (w, T_0 w) \quad \text{as} \quad k \to \infty.\]

We now apply Theorem 7.9 in [3] to conclude the proof of (i).

Property (ii) follows immediately since \(T_+ \geq T_0\).

\[\square\]

2.17. **Proof of Theorem 2.2.** We will adopt the arguments from the proof of Theorem 1 in [2].

By the proof of Lemma 2.16 it follows that

\[Q(T_+) \subset Q(T_k) \subset Q(T_0)\]

and

\[\|\cdot\|_{t_0} \leq \|\cdot\|_{t_k} \leq \|\cdot\|_{t_+},\]

where \(\|\cdot\|_{t_0}, \|\cdot\|_{t_k}\) and \(\|\cdot\|_{t_+}\) are the norms associated to \(t_0\), \(t_k\), and \(t_+\) respectively, cf. (1.2).

In fact, \(Q(T_k) = Q(T_+)\) since the norms \(\|\cdot\|_{t_k}\) and \(\|\cdot\|_{t_+}\) are equivalent, because \(V_0^+ - V_0^k\) and \(V_0^+\) differ by a bounded function.

By Proposition 2.4, \(T_+\) has a positive form core, i.e. for every \(u \in Q(T_+)^+\), there exists a sequence \(\phi_j \in C_0^\infty(M)^+\) such that \(\|\phi_j - u\|_{t_+} \to 0\) as \(j \to \infty\). By (2.16) it follows that

\[\|\phi_j - u\|_{t_0} \to 0 \quad \text{as} \quad j \to \infty.\]

To prove the theorem, it remains to show that for every \(w \in Q(T_0)^+\), there exists a sequence \(w_j \in Q(T_+)^+\) such that

\[\|w_j - w\|_{t_0} \to 0 \quad \text{as} \quad j \to \infty.\]
Let \( w \in Q(T_0)^+ \). For every \( k, l \in \mathbb{Z}_+ \) define
\[
w_l := \left( \frac{1}{l} T_0 + 1 \right)^{-1} w
\]
and
\[
w_l^k := \left( \frac{1}{l} T_k + 1 \right)^{-1} w.
\]
This makes sense since \( 0 \leq T_0 \leq T_k \) are self-adjoint operators.

By Lemma 2.12, the operator \( (T_k + 1)^{-1} \) is positivity preserving. Hence \( w_l^k \in \text{Dom}(T_k)^+ \subset Q(T_k)^+ = Q(T_+)^+ \).

Since the operators \( (T_0 + 1)^{1/2} \) and \( (T_0/l + 1)^{-1} \) commute, we have
\[
\|w_l - w\|_{t_0} = \left\| \left( \frac{1}{l} T_0 + 1 \right)^{-1} - 1 \right\| (T_0 + 1)^{1/2} w, \quad (2.18)
\]
where \( \| \cdot \| \) denotes \( L^2(M) \) norm.

Clearly,
\[
\left( \frac{1}{l} T_0 + 1 \right)^{-1} \to 1 \quad \text{strongly as } l \to \infty.
\]
This and (2.18) show that
\[
\|w_l - w\|_{t_0} \to 0 \quad \text{as } l \to \infty. \quad (2.19)
\]

Fix \( l \in \mathbb{Z}_+ \). For each \( k \in \mathbb{Z}_+ \), let \( t_0 + l \) and \( t_k + l \) denote the quadratic forms corresponding to (positive self-adjoint) operators \( T_0 + l \) and \( T_k + l \) respectively. Let \( \| \cdot \|_{t_0+l} \) and \( \| \cdot \|_{t_k+l} \) denote the norms in \( Q(T_0 + l) \) and \( Q(T_k + l) \) respectively, cf. (1.2). The corresponding inner products will be denoted by \( \langle \cdot, \cdot \rangle_{t_0+l} \) and \( \langle \cdot, \cdot \rangle_{t_k+l} \).

Using (2.15), (2.16) and Cauchy-Schwarz inequality we have for all \( w \in Q(T_0)^+ \)
\[
\|(T_k + l)^{-1} w - (T_0 + l)^{-1} w\|_{t_0+l}^2
= \|(T_k + l)^{-1} w\|^2_{t_0+l} + \|(T_0 + l)^{-1} w\|^2_{t_0+l} - 2 \langle (T_k + l)^{-1} w, (T_0 + l)^{-1} w \rangle_{t_0+l}
\leq \|(T_k + l)^{-1} w\|_{t_0+l}^2 + \|(T_0 + l)^{-1} w\|_{t_0+l}^2 - 2 \langle (T_k + l)^{-1} w, (T_0 + l)^{-1} w \rangle_{t_0+l}
\leq \langle (T_0 + l)^{-1} w, (T_k + l)^{-1} w \rangle - \langle (T_k + l)^{-1} w, w \rangle + (1 - l) \|(T_k + l)^{-1} w\|^2 + \|(T_0 + l)^{-1} w\|^2 - 2 \langle (T_k + l)^{-1} w, (T_0 + l)^{-1} w \rangle
\leq \langle (T_0 + l)^{-1} w, w \rangle - \langle (T_k + l)^{-1} w, w \rangle \leq \|(T_0 + l)^{-1} w - (T_k + l)^{-1} w\| \| w \|, \quad (2.20)
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2(M) \) and \( \| \cdot \| \) is the norm in \( L^2(M) \).

By Lemma 2.16, it follows that for fixed \( l \in \mathbb{Z}_+ \), \( T_k + l \to T_0 + l \) in the strong resolvent sense as \( k \to \infty \).

Clearly, for any positive self-adjoint operator \( A, (A+l)^{-1} = l(A+l)^{-1} \). Therefore by (2.20), for a fixed \( l \in \mathbb{Z}_+ \)
\[
\|w_l^k - w_l\|_{t_0+l} \to 0 \quad \text{as } k \to \infty.
\]
This is equivalent to
\[ \|w_i^k - w_i\|_{t_0} \to 0 \quad \text{as } k \to \infty. \] (2.21)

Since \( w_i^k \in Q(T_+)^+ \), we can use (2.19) and (2.21) to choose a subsequence \( \{w_j\} \) from \( \{w_i^k\} \)
so that (2.17) holds.
This concludes the proof of the theorem.

3. Proof of Theorem 1.5

We essentially follow the proof of Theorem 2 in [2]; however, we need to use Kato inequality (2.10) for operators on manifolds.

Without the loss of generality, we may and we will assume that \( \Delta_M + V_0 \geq 0 \) and \( V_1 \geq 0 \).

Let us denote \( T_q := T_0 + V_1 \) and let \( t_{\text{min}} \) and \( t_q \) be as in Sect. 1.4 and Sect. 1.3. Since \( t_{\text{min}} \) and \( t_q \) coincide on \( C_c^\infty(M) \), it is sufficient to show that \( C_c^\infty(M) \) is dense in the Hilbert space
\[ Q(T_q) = Q(T_0) \cap Q(V_1) \]
with the inner product
\[ (\cdot, \cdot)_{t_q} := (\cdot, \cdot) + (\cdot, \cdot)_{L^2(M)}, \]
where \( t_q(\cdot, \cdot) \) is the sesquilinear form obtained by polarization of \( t_q \).

Let \( v \in Q(T_q) \) be orthogonal to \( C_c^\infty(M) \) in \( (\cdot, \cdot)_{t_q} \). This means that for all \( w \in C_c^\infty(M) \),
\[ ((\Delta_M + V_0 + V_1)v, w)_{L^2(M)} + (v, w)_{L^2(M)} = 0. \]
This leads to the following distributional equality
\[ \Delta_M v = -(V_0 + V_1 + 1)v. \] (3.1)

Since \( V_1 \in L^1_{\text{loc}}(M) \) and \( v \in Q(V_1) \), we have
\[ 2|V_1|v = 2|V_1||v| \leq |V_1| + |V_1||v|^2 \]
which immediately gives \( V_1 v \in L^1_{\text{loc}}(M) \).

Since \( V_0 \in L^2_{\text{loc}}(M) \), it follows that \( V_0 v \in L^1_{\text{loc}}(M) \). From (3.1) we obtain \( \Delta_M v \in L^1_{\text{loc}}(M) \).

Using Kato inequality (2.10) in case \( \nabla = d \) and the equation (3.1), we get
\[ \Delta_M |v| \leq \text{Re}(\text{sign} \hat{v} \Delta_M v) = -V_0|v| - V_1|v| - |v| \leq -(V_0 + 1)|v|. \] (3.2)
The last inequality in (3.2) holds since \( V_1 \geq 0 \).

From (3.2), we obtain the following distributional inequality
\[ (\Delta_M + V_0 + 1)|v| \leq 0. \] (3.3)

Let \( T_0 \) be as in hypothesis, and let \( t_0 \) denote the closed quadratic form associated to \( T_0 \).

Using (3.3), we get
\[ ((T_0 + 1)w, |v|)_{L^2(M)} \leq 0 \quad \text{for all } w \in C_c^\infty(M)^+. \] (3.4)

Since \( v \in Q(T_0) \), Corollary 2.14 gives \( |v| \in Q(T_0) \). Therefore, we can write (3.4) as
\[ (w, |v|)_{t_0} \leq 0 \quad \text{for all } w \in C_c^\infty(M)^+, \] (3.5)
where \((\cdot, \cdot)_{t_0} = t_0(\cdot, \cdot) + (\cdot, \cdot)_{L^2(M)} \) denotes the inner product in \( Q(T_0) \).
Let \( f := (T_0 + 1)^{-1}|v| \). By Proposition 2.13, \((T_0 + 1)^{-1}\) is positivity preserving, so \( f \in \text{Dom}(T_0)^+ \subset Q(T_0)^+ \).

By Theorem 2.2, \( T_0 \) has a positive form core. Therefore, there exists a sequence \( f_k \in C_c^{\infty}(M)^+ \) such that

\[
\lim_{k \to \infty} (f_k, |v|)_{t_0} = (f, |v|)_{t_0} = ((T_0 + 1)^{-1}|v|, |v|)_{t_0} = \|v\|^2, \tag{3.6}
\]

where \( v \) and \((\cdot, \cdot)_{t_0}\) are as in (3.5), and \( \| \cdot \| \) is the norm in \( L^2(M) \).

From (3.5) and (3.6) we obtain \( \|v\|^2 \leq 0 \), i.e. \( v = 0 \).

This shows that \( C_c^{\infty}(M) \) is dense in \( Q(T_0) \), and the theorem is proven. \(\square\)

References


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