On m-accretive Schrödinger operators in L^p-spaces on manifolds of bounded geometry

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Abstract

Let (M, g) be a manifold of bounded geometry with metric g. We consider a Schrödinger-type differential expression $H = \Delta_M + V$, where Δ_M is the scalar Laplacian on M and V is a non-negative locally integrable function on M. We give a sufficient condition for H to have an m-accretive realization in the space $L^p(M)$, where $1 < p < +\infty$. The proof uses Kato's inequality and L^p-theory of elliptic operators on Riemannian manifolds.

Key words: manifold of bounded geometry, m-accretive, Schrödinger operator,
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1 Introduction and the main results

Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected and oriented. By $d\mu$ we will denote the Riemannian volume element of M.

In what follows, by T_x^*M and T^*M we will denote the cotangent space of M at $x \in M$ and cotangent bundle of M respectively. By $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$—the space of smooth compactly supported functions on M, by $\Omega^1(M)$—the space of smooth 1-forms on M, and by $\mathcal{D}'(M)$—the distributions on M.

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Let $1 \le p < +\infty$. By $L^p(M)$ we denote the completion of $C_c^\infty(M)$ with respect to the norm

$$\|u\|_p := \left(\int_M |u|^p \, d\mu \right)^{1/p}.$$

By $\langle \cdot, \cdot \rangle$ we will denote the anti-duality of the pair $(L^p(M), L^{p'}(M))$, where $1 \le p < +\infty$ and $1/p + 1/p' = 1$, and the anti-duality of the pair $(\mathcal{D}'(M), C_c^\infty(M))$.

In the sequel, $d \colon C^\infty(M) \to \Omega^1(M)$ is the standard differential, $d^* \colon \Omega^1(M) \to C^\infty(M)$ is the formal adjoint of d with respect to the usual inner product in $L^2(M)$, and $\Delta_M := d^* d$ is the scalar Laplacian on M; see [2, Sec. 1].

We will consider a Schrödinger type differential expression of the form

$$H = \Delta_M + V,$$

where $V \in L^1_{\text{loc}}(M)$ is real-valued.

1.1 Operators associated to H

Let $1 < p < +\infty$. We define the maximal operator $H_{p,\max}$ in $L^p(M)$ associated to H by the formula $H_{p,\max} u = Hu$ with domain

$$\text{Dom}(H_{p,\max}) = \{u \in L^p(M) : Vu \in L^1_{\text{loc}}(M), \ \Delta_M u + Vu \in L^p(M)\}. \quad (1)$$

Here, the term $\Delta_M u$ in $\Delta_M u + Vu$ is understood in distributional sense.

In general, $\text{Dom}(H_{p,\max})$ does not contain $C_c^\infty(M)$, but it does if $V \in L^p_{\text{loc}}(M)$. In this case, we can define $H_{p,\min} := H_{p,\max}|_{C_c^\infty(M)}$.

Remark 1 *Using the same definitions as in Sec. 1.1, we can also define $H_{p,\max}$ and $H_{p,\min}$ for $p = 1$ and $p = \infty$. However, we will not use those operators in this paper.*

1.2 Operators associated to Δ_M

Let $1 < p < +\infty$. We define the maximal operator $A_{p,\max}$ in $L^p(M)$ associated to Δ_M by the formula $A_{p,\max} u = \Delta_M u$ with the domain

$$\text{Dom}(A_{p,\max}) = \{u \in L^p(M) : \Delta_M u \in L^p(M)\}. \quad (2)$$

We define $A_{p,\min} := A_{p,\max}|_{C_c^\infty(M)}$.

Throughout this paper, we will use the terminology of contraction semigroups, accretive and m-accretive operators on a Banach space; see Sec. 1.4 below for a brief review.

1.3 Domination and positivity

Suppose that B and C are bounded linear operators on $L^p(M)$. In what follows, the notation $B \leq C$ means that for all $0 \leq f \in L^p(M)$ we have $(C - B)f \geq 0$. We will use the notation $B << C$ to indicate that B is dominated by C, i.e. $|Bf| \leq C|f|$ for all $f \in L^p(M)$.

Assumption (A1). Assume that (M, g) has bounded geometry.

Remark 2 *In this paper, the term "bounded geometry" is the same as in [8, Sec. A.1.1] or [2, Sec. 1]. In particular, a manifold of bounded geometry is complete.*

In the sequel, by $\overline{A}$ we denote the closure of a closable operator A.

We now state the main results.

Theorem 3 *Assume that (M, g) is a connected C^∞ Riemannian manifold without boundary. Assume that the Assumption (A1) is satisfied. Assume that $1 < p < +\infty$ and $0 \leq V \in L^p_{\mathrm{loc}}(M)$. Then the following properties hold:*

(1) the operator $H_{p,\max}$ generates a contraction semigroup on $L^p(M)$. In particular, $H_{p,\max}$ is an m-accretive operator.
(2) the set $C_c^\infty(M)$ is a core for $H_{p,\max}$ (i.e. $\overline{H_{p,\min}} = H_{p,\max}$.).

Theorem 4 *Under the same hypotheses as in Theorem 3, the following properties hold (with the notations as in Sec. 1.2 and Sec. 1.3):*

(1) $0 \leq (\lambda + H_{p,\max})^{-1} \leq (\lambda + A_{p,\max})^{-1}$, for all $\lambda > 0$;
(2) $(\lambda + H_{p,\max})^{-1} << (\gamma + A_{p,\max})^{-1}$, for all $\lambda \in \mathbb{C}$ such that $\gamma := \operatorname{Re}\lambda > 0$.

Remark 5 *T. Kato [6, Part A] considered the differential expression $-\Delta + V$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with standard metric and measure and $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assuming $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Kato proved the property (1) of Theorem 3 and Theorem 4 for all $1 \leq p \leq +\infty$. Assuming $0 \leq V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, Kato proved the property (2) of Theorem 3 for $1 \leq p < +\infty$. In his proof, Kato used certain properties (specific to the $\mathbb{R}^n$ setting) of $(-\Delta_{2,\max} + \gamma)^{-1}$, where $\gamma > 0$ and $-\Delta_{2,\max}$ is the self-adjoint closure of $-\Delta|_{C_c^\infty(\mathbb{R}^n)}$ in*

$L^2(\mathbb{R}^n)$, which enabled him to handle the cases $p = 1$ and $p = \infty$. In the context of non-compact Riemannian manifolds, we use L^p-theory of elliptic differential operators, which works well for $1 < p < +\infty$.

1.4 Accretive operators and contraction semigroups

Here we briefly review some terms and facts concerning accretive operators, m-accretive operators and contraction semigroups. For more details, see, for example, [7, Sec. X.8] or [5, Sec. IX.1].

A densely defined linear operator T on a Banach space Y is said to be accretive if for each $u \in \mathrm{Dom}(T)$ the following property holds: $\mathrm{Re}(f(Tu)) \geq 0$ for some normalized tangent functional $f \in Y^*$ to u. An operator T is said to be m-accretive if T is accretive and has no proper accretive extension.

Let Y be a Banach space and let $T\colon \mathrm{Dom}(T) \subset Y \to Y$ be a closed linear operator. Let $\rho(T)$ denote the resolvent set of T. By [7, Theorem X.47(a)], the operator T generates a contraction semigroup on Y if and only if the following two conditions are satisfied:

(1) $(-\infty, 0) \subset \rho(T)$;
(2) for all $\lambda > 0$,

$$\|(T + \lambda)^{-1}\| \leq \frac{1}{\lambda},$$

where $\|\cdot\|$ denotes the operator norm.

By [7, Theorem X.48], a closed linear operator T on a Banach space Y is the generator of a contraction semigroup if and only if T is accretive and $\mathrm{Ran}(\lambda_0 + T) = Y$ for some $\lambda_0 > 0$.

By the Remark preceding Theorem X.49 in [7], the following holds: if a closed linear operator T on a Banach space Y is the generator of a contraction semigroup, then T is m-accretive.

2 Preliminary Lemmas

In what follows, we will use the following notations for Sobolev spaces on Riemannian manifolds (M, g).

2.1 Sobolev space notations

Let $k \geq 0$ be an integer, and let $1 \leq p < +\infty$ be a real number. By $W^{k,p}(M)$ we will denote the completion of $C_c^\infty(M)$ in the norm

$$\|u\|_{W^{k,p}}^p := \sum_{l=0}^{k} \|\nabla^l u\|_{L^p}^p, \tag{3}$$

where $\nabla^l u$ is the l-th covariant derivative of u; see [2, Sec. 1] or [8, Sec. A.1.1].

Remark 6 *If (M, g) has bounded geometry, then by [2, Proposition 1.6] it follows that $W^{k,p}(M) = \{u \in L^p(M) : \nabla^l u \in L^p$, for all $1 \leq l \leq k\}$.*

Remark 7 *Under the assumption that (M, g) is a complete Riemannian manifold (not necessarily of bounded geometry) and $1 < p < +\infty$, the first and the second paragraph in the proof of [9, Theorem 3.5] give the proofs of the following properties:*

(1) the operator $A_{p,\min} = \Delta_M|_{C_c^\infty(M)}$ is accretive (hence, closable);
(2) the closure $\overline{A_{p,\min}}$ generates a contraction semigroup on $L^p(M)$;
(3) $(-\infty, 0) \subset \rho(\overline{A_{p,\min}})$ and

$$\|(\lambda + \overline{A_{p,\min}})^{-1}\| \leq \frac{1}{\lambda}, \qquad \text{for all } \lambda > 0, \tag{4}$$

where $\|\cdot\|$ denotes the operator norm (for a bounded linear operator $L^p(M) \to L^p(M)$);
(4) the resolvents $(\lambda + \overline{A_{p,\min}})^{-1}$ and $(\lambda + \overline{A_{2,\min}})^{-1}$, where $\lambda > 0$, are equal on $L^2(M) \cap L^p(M)$.

Remark 8 *By an abstract fact, the properties (2) and (3) in Remark 7 are equivalent; see [7, Theorem X.47(a)].*

Remark 9 *Assume that (M, g) has bounded geometry and $1 < p < +\infty$. Then by [8, Proposition 4.1] it follows that $A_{p,\max} = \overline{A_{p,\min}}$. Thus, the properties (2), (3) and (4) of Remark 7 hold with $\overline{A_{p,\min}}$ replaced by $A_{p,\max}$.*

2.2 Distributional inequality

Assume that $1 < p < +\infty$ and $\lambda > 0$, and consider the following distributional inequality:

$$(\Delta_M + \lambda)\, u \;=\; \nu \;\geq\; 0, \qquad u \in L^p(M), \tag{5}$$

where the inequality $\nu \geq 0$ means that ν is a positive distribution, i.e. $\langle \nu, \phi \rangle \geq 0$ for any $0 \leq \phi \in C_c^\infty(M)$.

Remark 10 *It follows that ν is in fact a positive Radon measure (see e. g. [3], Theorem 1 in Sec. 2, Ch. II).*

Lemma 11 *Assume that (M, g) is a manifold of bounded geometry. Assume that $1 < p < +\infty$. Assume that $u \in L^p(M)$ satisfies (5). Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).*

For the proof of Lemma 11 in the case $p = 2$, see the proof of Proposition B.3 in Appendix B of [1]. In the proof of Lemma 11, which we give in Sec. 4 below, we adopt the scheme of proof for the case $p = 2$ from [1, Appendix B].

2.3 Kato's inequality

In what follows, we will use a version of Kato's inequality. For the proof of a more general version of this inequality, see [1, Theorem 5.7].

Lemma 12 *Assume that (M, g) is an arbitrary Riemannian manifold. Assume that $u \in L^1_{\text{loc}}(M)$ and $\Delta_M u \in L^1_{\text{loc}}(M)$. Then the following distributional inequality holds:*

$$\Delta_M |u| \;\leq\; \operatorname{Re}((\Delta_M u) \operatorname{sign} \bar{u}), \tag{6}$$

where

$$\operatorname{sign} u(x) \;=\; \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } \; u(x) \neq 0 \;, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 13 *For the original version of Kato's inequality, see Kato [4, Lemma A].*

In the sequel, with the help of Lemma 11, we will adopt certain arguments of Kato [6, Part A] to our setting.

Lemma 14 *Let (M, g) be a Riemannian manifold. Assume that $1 < p < +\infty$. Assume that $0 \leq V \in L^1_{\text{loc}}(M)$, $u \in \operatorname{Dom}(H_{p,\max})$ and $\lambda \in \mathbb{C}$. Let $f := (H_{p,\max} + \lambda)u$.*

Then the following distributional inequality holds:

$$(\operatorname{Re} \lambda + \Delta_M + V)|u| \;\leq\; |f|. \tag{7}$$

Proof Since $u \in \mathrm{Dom}(H_{p,\max})$ it follows that $Vu \in L^1_{\mathrm{loc}}(M)$ and $H_{p,\max}u \in L^p(M) \subset L^1_{\mathrm{loc}}(M)$. Thus $u \in L^1_{\mathrm{loc}}(M)$ and $\Delta_M u \in L^1_{\mathrm{loc}}(M)$. By Kato's inequality (6) we have

$$\begin{aligned}(\mathrm{Re}\,\lambda + \Delta_M + V)|u| \ &\leq \mathrm{Re}[((\lambda + \Delta_M + V)u)\,\mathrm{sign}\,\bar{u}] \ = \\ = \ \mathrm{Re}(f\,\mathrm{sign}\,\bar{u}) \ &\leq \ |f|,\end{aligned}$$

and the Lemma is proven. □

In the sequel, we will always assume that (M, g) is a manifold of bounded geometry.

Lemma 15 *Assume that $1 < p < +\infty$ and $0 \leq V \in L^1_{\mathrm{loc}}(M)$. Assume that $\lambda \in \mathbb{C}$ and $\gamma := \mathrm{Re}\,\lambda > 0$. Then the following properties hold:*

(1) for all $u \in \mathrm{Dom}(H_{p,\max})$, we have

$$\gamma \|u\|_p \leq \|(\lambda + H_{p,\max})u\|_p; \tag{8}$$

(2) the operator $\lambda + H_{p,\max}\colon \mathrm{Dom}(H_{p,\max}) \subset L^p(M) \to L^p(M)$ is injective;
(3) for all $u \in \mathrm{Dom}(H_{p,\max})$ such that $(\lambda + H_{p,\max})u \geq 0$ (where $\lambda > 0$), we have $u \geq 0$.

Proof We first prove property (1). Let $u \in \mathrm{Dom}(H_{p,\max})$ and $f := (\lambda + H_{p,\max})u$. By the definition of $\mathrm{Dom}(H_{p,\max})$, we have $f \in L^p(M)$, where $1 < p < +\infty$. Since $V \geq 0$ and since $Vu \in L^1_{\mathrm{loc}}(M)$, from (7) we get the following distributional inequality:

$$(\gamma + \Delta_M)|u| \ \leq \ |f|. \tag{9}$$

By property (3) of Remark 7 and by Remark 9, it follows that

$$(\gamma + A_{p,\max})^{-1}\colon L^p(M) \to L^p(M)$$

is a bounded linear operator.

Let us rewrite (9) as

$$(\gamma + \Delta_M)[(\gamma + A_{p,\max})^{-1}|f| - |u|] \geq 0.$$

Note that

$$(\gamma + A_{p,\max})^{-1}|f| \in L^p(M) \qquad \text{and} \qquad |u| \in L^p(M).$$

Thus, $((\gamma + A_{p,\max})^{-1}|f| - |u|) \in L^p(M)$, and, hence, by Lemma 11 we have $(\gamma + A_{p,\max})^{-1}|f| - |u| \geq 0$, i.e.

$$|u| \leq (\gamma + A_{p,\max})^{-1}|f|. \tag{10}$$

By (4) and by Remark 9 it follows that

$$\|(\gamma + A_{p,\max})^{-1}|f|\|_p \leq \frac{1}{\gamma}\|f\|_p. \tag{11}$$

By (10) and (11) we have

$$\|u\|_p \leq \|(\gamma + A_{p,\max})^{-1}|f|\|_p \leq \frac{1}{\gamma}\|f\|_p,$$

and (8) is proven.

We now prove property (2). Assume that $u \in \mathrm{Dom}(H_{p,\max})$ and $(\lambda + H_{p,\max})u = 0$. Using (8) with $f = 0$, we get $\|u\|_p = 0$, and hence $u = 0$. This shows that $\lambda + H_{p,\max}$ is injective.

We now prove property (3). Let $\lambda > 0$ and assume that $u \in \mathrm{Dom}(H_{p,\max})$ satisfies

$$f := (H_{p,\max} + \lambda)u \geq 0.$$

We claim that u is real. Indeed, since $(H_{p,\max} + \lambda)\bar{u} = f$, we have $(H_{p,\max} + \lambda)(u - \bar{u}) = 0$. By property (2) of this lemma we have $u = \bar{u}$. Since $f \geq 0$ and $\lambda > 0$, by (7) we have

$$(\lambda + \Delta_M + V)|u| \ \leq \ f. \tag{12}$$

Subtracting $f = (\lambda + H_{p,\max})u$ from both sides of (12) we get

$$(\lambda + \Delta_M + V)v \leq 0, \qquad \text{where } v := |u| - u \geq 0.$$

Since $V \geq 0$, it follows that $(\lambda + \Delta_M)v \leq 0$. By Lemma 11 we get $v \leq 0$. Thus $v = 0$, i.e. $u = |u| \geq 0$. This concludes the proof. □

Lemma 16 *Assume that* $1 < p < +\infty$ *and* $0 \leq V \in L^1_{\mathrm{loc}}(M)$. *Then the following properties hold:*

(1) the operator $H_{p,\max}$ *is closed;*

(2) *the operator* $\lambda + H_{p,\max}$, *where* $\operatorname{Re}\lambda > 0$, *has a closed range.*

Proof We first prove the property (1). Let $u_k \in \operatorname{Dom}(H_{p,\max})$ be a sequence such that, as $k \to +\infty$,

$$u_k \to u, \quad f_k := H_{p,\max} u_k = \Delta_M u_k + V u_k \to f \quad \text{in } L^p(M). \tag{13}$$

We need to show that $u \in \operatorname{Dom}(H_{p,\max})$ and $H_{p,\max} u = f$.

By passing to subsequences, we may assume that the convergence in (13) is also pointwise almost everywhere.

The distributional inequality (7) holds if we replace u by $u_k - u_l$, f by $f_k - f_l$ and λ by 0. With these replacements, we apply a test function $0 \leq \phi \in C_c^\infty(M)$ to (7) and get

$$0 \leq \langle V|u_k - u_l|, \phi\rangle \leq \langle |f_k - f_l|, \phi\rangle - \langle \Delta_M |u_k - u_l|, \phi\rangle, \tag{14}$$

where $\langle\cdot,\cdot\rangle$ denotes the anti-duality of the pair $(\mathcal{D}'(M), C_c^\infty(M))$.

Using integration by parts in the second term on the right hand side of the second inequality in (14), we get

$$0 \leq \langle V|u_k - u_l|, \phi\rangle \leq \langle |f_k - f_l|, \phi\rangle - \langle |u_k - u_l|, \Delta_M \phi\rangle. \tag{15}$$

Letting $k, l \to +\infty$, the right hand side of the second inequality in (15) tends to 0 by (13). Thus $V u_k \phi$ is a Cauchy sequence in $L^1(M)$, and its limit must be equal to $V u \phi$. Since $\phi \in C_c^\infty(M)$ may have an arbitrarily large support, it follows that $Vu \in L^1_{\mathrm{loc}}(M)$. Thus $V u_k \to V u$ in $L^1_{\mathrm{loc}}(M)$ and hence in $\mathcal{D}'(M)$. Since $u_k \to u$ in $L^p(M)$ (and, hence in $L^1_{\mathrm{loc}}(M)$), we get $\Delta_M u_k \to \Delta_M u$ in $\mathcal{D}'(M)$. Thus, $f_k = \Delta_M u_k + V u_k \to \Delta_M u + V u$ in $\mathcal{D}'(M)$. Since $f_k \to f$ in $L^p(M) \subset \mathcal{D}'(M)$, we obtain $\Delta_M u + V u = f \in L^p(M)$. This shows that $u \in \operatorname{Dom}(H_{p,\max})$ and $H_{p,\max} u = f$. This proves that $H_{p,\max}$ is closed.

We now prove the property (2). Since $H_{p,\max}$ is closed, it immediately follows from (8) that $\lambda + H_{p,\max}$ has a closed range for $\operatorname{Re}\lambda > 0$. □

Lemma 17 *Assume that* $1 < p < +\infty$ *and* $0 \leq V \in L^p_{\mathrm{loc}}(M)$. *Let* $\lambda \in \mathbb{C}$ *and* $\gamma := \operatorname{Re}\lambda > 0$. *Then the following properties hold:*

(1) *the operator* $\lambda + H_{p,\max}\colon \operatorname{Dom}(H_{p,\max}) \subset L^p(M) \to L^p(M)$ *is surjective;*
(2) *the operator* $(\lambda + H_{p,\max})^{-1}\colon L^p(M) \to L^p(M)$ *is a bounded linear operator with the operator norm*

$$\|(\lambda + H_{p,\max})^{-1}\| \leq 1/\gamma. \tag{16}$$

Proof We first prove the property (1). Since $\lambda + H_{p,\max}$ has a closed range by Lemma 16, it is enough to show that $(\lambda + H_{p,\min})C_c^\infty(M)$ is dense in $L^p(M)$. Let $v \in (L^p(M))^* = L^{p'}(M)$, where $1/p + 1/p' = 1$, be a continuous linear functional annihilating $(\lambda + H_{p,\min})C_c^\infty(M)$:

$$\langle(\lambda + H_{p,\min})\phi, v\rangle = 0, \qquad \text{for all } \phi \in C_c^\infty(M), \tag{17}$$

where $\langle\cdot,\cdot\rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$.

From (17) we get the following distributional equality:

$$(\bar{\lambda} + \Delta_M + V)v = 0.$$

Since by hypothesis $V \in L^p_{\mathrm{loc}}(M)$ and since $v \in L^{p'}(M)$, by Hölder's inequality we have $Vv \in L^1_{\mathrm{loc}}(M)$. Since $\Delta_M v = -Vv - \bar{\lambda}v$, we get $\Delta_M v \in L^1_{\mathrm{loc}}(M)$. By Kato's inequality and since $V \geq 0$, we have

$$\Delta_M|v| \leq \mathrm{Re}((\Delta_M v)\operatorname{sign}\bar{v}) = \mathrm{Re}((-\bar{\lambda}v - Vv)\operatorname{sign}\bar{v}) \leq -(\mathrm{Re}\,\bar{\lambda})|v|,$$

and, hence,

$$(\Delta_M + \mathrm{Re}\,\bar{\lambda})|v| \leq 0.$$

Since $v \in L^{p'}(M)$ (with $1 < p' < +\infty$) and since $\mathrm{Re}\,\bar{\lambda} = \mathrm{Re}\,\lambda > 0$, by Lemma 11 we get $|v| \leq 0$. Thus $v = 0$, and the surjectivity of $\lambda + H_{p,\max}$ is proven.

We now prove the property (2). Assume that $\lambda \in \mathbb{C}$ satisfies $\gamma := \mathrm{Re}\,\lambda > 0$. Since $\lambda + H_{p,\max} : \mathrm{Dom}(H_{p,\max}) \subset L^p(M) \to L^p(M)$ is injective and surjective, the inverse $(\lambda + H_{p,\max})^{-1}$ is defined on the whole $L^p(M)$. The inequality (16) follows immediately from (8). This concludes the proof of the Lemma. □

3 Proofs of main results

3.1 Proof of Theorem 3

We first prove the property (1). By Lemma 17 it follows that $(-\infty, 0) \subset \rho(H_{p,\max})$ and

$$\|(\lambda + H_{p,\max})^{-1}\| \leq \frac{1}{\lambda}, \qquad \text{for all } \lambda > 0.$$

Thus, by [7, Theorem X.47(a)] it follows that $H_{p,\max}$ generates a contraction semigroup on $L^p(M)$. In particular, the operator $H_{p,\max}$ is m-accretive; see Sec. 1.4 above.

We now prove the property (2). By property (1) of this lemma it follows that $H_{p,\max}$ is m-accretive; hence, $H_{p,\min} = H_{p,\max}|_{C_c^\infty(M)}$ is accretive. By an abstract fact (see the remark preceding Theorem X.48 in [7]), the operator $H_{p,\min}$ is closable and $\overline{H_{p,\min}}$ is accretive. Let $\lambda > 0$. By the proof of property (1) in Lemma 17 it follows that $\operatorname{Ran}(\lambda + H_{p,\min})$ dense in $L^p(M)$. Using (8) and the definition of the closure of an operator (see, for example, definition below equation (5.6) in [5, Sec. III.5.3]), it follows that $\operatorname{Ran}(\lambda + \overline{H_{p,\min}}) = L^p(M)$. Now by [7, Theorem X.48] the operator $\overline{H_{p,\min}}$ generates a contraction semigroup on $L^p(M)$. Thus, by the remark preceding Theorem X.49 in [7], the operator $\overline{H_{p,\min}}$ is m-accretive. Since $\overline{H_{p,\min}} \subset H_{p,\max}$ and since $\overline{H_{p,\min}}$ and $H_{p,\max}$ are m-accretive, it follows that $\overline{H_{p,\min}} = H_{p,\max}$.

This concludes the proof of the Theorem. □

3.2 Proof of Theorem 4

We first prove the property (1). Let $\lambda > 0$, let $0 \leq f \in L^p(M)$ be arbitrary, and let $u := (H_{p,\max} + \lambda)^{-1} f$. Then $(H_{p,\max} + \lambda)u = f \geq 0$, and, hence, by the property (3) of Lemma 15, we have $u \geq 0$. This proves the inequality $0 \leq (H_{p,\max} + \lambda)^{-1}$.

We now prove

$$(H_{p,\max} + \lambda)^{-1} \leq (A_{p,\max} + \lambda)^{-1}. \tag{18}$$

Let $0 \leq f \in L^p(M)$ be arbitrary and let $u := (H_{p,\max} + \lambda)^{-1} f$. Then $0 \leq u \in \operatorname{Dom}(H_{p,\max})$, and, hence, using (10) with $u \geq 0$ and $f \geq 0$ we get

$$u \leq (A_{p,\max} + \lambda)^{-1} f. \tag{19}$$

By (19) with $u = (H_{p,\max} + \lambda)^{-1} f$, we immediately get (18). This concludes the proof of property (1).

We now prove the property (2). Let $\gamma := \operatorname{Re} \lambda > 0$. Let $f \in L^p(M)$ be arbitrary and let $u := (H_{p,\max} + \lambda)^{-1} f$. By (10) we have

$$\left|(H_{p,\max} + \lambda)^{-1} f\right| \leq (A_{p,\max} + \gamma)^{-1} |f|,$$

and property (2) is proven.

This concludes the proof of the Theorem. □

4 Proof of Lemma 11

We begin by introducing some additional notations and definitions.

4.1 Sobolev spaces $\widetilde{W}^{-2,p}(M)$ and $\widetilde{W}^{2,p}(M)$

Let $1 < p < +\infty$ and let $\lambda > 0$. Define

$$\widetilde{W}^{-2,p}(M) := (\Delta_M + \lambda)L^p(M)$$

and

$$\widetilde{W}^{2,p}(M) := \{u \in L^p(M) : \Delta_M u \in L^p(M)\}.$$

The norms in $\widetilde{W}^{2,p}(M)$ and $\widetilde{W}^{-2,p}(M)$ are given respectively by the formulas

$$\|v\|_{2,p} \;=\; \|\,(\Delta_M + \lambda)v\,\|_p, \qquad \|\,(\Delta_M + \lambda)\,f\,\|_{-2,p} \;=\; \|f\|_p, \tag{20}$$

where $\|\cdot\|_p$ is the norm in $L^p(M)$.

Let $1 < p < +\infty$ and $1/p + 1/p' = 1$. By $\langle\cdot,\cdot\rangle_S$ we denote the anti-duality

$$\langle\cdot,\cdot\rangle_S\colon \widetilde{W}^{-2,p}(M) \times \widetilde{W}^{2,p'}(M) \;\to \mathbb{C}, \tag{21}$$

of the spaces $\widetilde{W}^{-2,p}(M)$ and $\widetilde{W}^{2,p'}(M)$ obtained by extending the anti-duality of the pair $(L^p(M), L^{p'}(M))$ by continuity from $C_c^\infty(M) \times C_c^\infty(M)$.

The extension of the anti-duality (21) from $C_c^\infty(M) \times C_c^\infty(M)$ is well defined because $C_c^\infty(M)$ is dense in both spaces $\widetilde{W}^{2,p'}(M)$ and $\widetilde{W}^{-2,p}(M)$ in the corresponding norms (20). Indeed, density of $C_c^\infty(M)$ in $\widetilde{W}^{2,p'}(M)$ means simply that $(\Delta_M + \lambda)C_c^\infty(M)$ is dense in $L^{p'}(M)$. To establish this, let us take $f \in (L^{p'}(M))^* = L^p(M)$ which annihilates $(\Delta_M + \lambda)C_c^\infty(M)$:

$$\langle f, (\Delta_M + \lambda)\phi\rangle = 0, \qquad \text{for all } \phi \in C_c^\infty(M),$$

where $\langle\cdot,\cdot\rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$.

This implies that $(\Delta_M + \lambda)f = 0$ in the sense of distributions, i.e. f is in the null-space of $A_{p,\max} + \lambda$, where $A_{p,\max}$ is as in Sec. 1.2. By Remark 7 and Remark 9 it follows that $f = 0$.

Similarly, density of $C_c^\infty(M)$ in $\widetilde{W}^{-2,p}(M)$ means that $(A_{p,\max} + \lambda)^{-1} C_c^\infty(M)$ is dense in $L^p(M)$. To prove this, consider $h \in (L^p(M))^* = L^{p'}(M)$ such that h annihilates $(A_{p,\max} + \lambda)^{-1} C_c^\infty(M)$:

$$\langle (A_{p,\max} + \lambda)^{-1}\phi, h\rangle = 0, \qquad \text{for all } \phi \in C_c^\infty(M),$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$.

This implies that

$$\langle \phi, ((A_{p,\max} + \lambda)^{-1})^* h\rangle = 0, \qquad \text{for all } \phi \in C_c^\infty(M). \tag{22}$$

We will now show that

$$((A_{p,\max} + \lambda)^{-1})^* = (A_{p',\max} + \lambda)^{-1}. \tag{23}$$

By Remark 7 and Remark 9 it follows that $(A_{p,\max} + \lambda)^{-1}$ and $(A_{p',\max} + \lambda)^{-1}$ are bounded linear operators on $L^p(M)$ and $L^{p'}(M)$ respectively. Thus, since $C_c^\infty(M)$ is dense in $L^p(M)$ and $L^{p'}(M)$, it suffices to show that

$$\langle (A_{p,\max} + \lambda)^{-1}\phi, \psi\rangle = \langle \phi, (A_{p',\max} + \lambda)^{-1}\psi\rangle, \qquad \text{for all } \phi, \psi \in C_c^\infty(M).$$

By property (4) of Remark 7 and by Remark 9 we have for all $\phi, \psi \in C_c^\infty(M)$:

$$\begin{aligned} \langle (A_{p,\max} + \lambda)^{-1}\phi, \psi\rangle &= \langle (A_{2,\max} + \lambda)^{-1}\phi, \psi\rangle = \\ = \langle \phi, (A_{2,\max} + \lambda)^{-1}\psi\rangle &= \langle \phi, (A_{p',\max} + \lambda)^{-1}\psi\rangle. \end{aligned} \tag{24}$$

The second equality in (24) holds since $(A_{2,\max}+\lambda)^{-1}$ is a bounded self-adjoint operator on $L^2(M)$ (it is well known that, for a complete Riemannian manifold (M, g), the operator $A_{2,\max}$ is a non-negative self-adjoint operator in $L^2(M)$; see, for example, [2, Theorem 3.5]).

Thus, from (22) and (23) we get $((A_{p,\max} + \lambda)^{-1})^* h = (A_{p',\max} + \lambda)^{-1} h = 0$. But this means that $h = 0$.

In the sequel, we will use the following lemma.

Lemma 18 *Assume that (M, g) is a manifold of bounded geometry. Assume that $1 < p < +\infty$. Assume that $0 \le \phi \in C_c^\infty(M)$ and $\lambda > 0$. Then there exists* *a unique* $u \in L^p(M)$ *such that*

$$(\Delta_M + \lambda)\, u \;=\; \phi, \tag{25}$$

and $u \geq 0$.

Proof The existence and uniqueness of solution $u \in L^p(M)$ to (25) follows by property (3) in Remark 7 and by Remark 9; just take

$$u = (\lambda + A_{p,\max})^{-1}\phi.$$

To show that $u \geq 0$, we first note that $\phi \in L^2(M) \cap L^p(M)$. By the property (4) of Remark 7 and by Remark 9 it follows that

$$(\lambda + A_{p,\max})^{-1}\phi = (\lambda + A_{2,\max})^{-1}\phi. \tag{26}$$

By the proof of [1, Theorem B.1] it follows that

$$w := (\lambda + A_{2,\max})^{-1}\phi$$

satisfies (25) (with u replaced by w) and $w \geq 0$.

Now by (26) we have $w = u$, and, hence, $u \geq 0$. This concludes the proof of the Lemma. □

Remark 19 *Let* $1 < p < +\infty$. *If* $u \in L^p(M)$ *satisfies (25) with* $\phi \in C_c^\infty(M)$, *then it is well known (by using standard elliptic regularity and Sobolev imbedding theorems) that* $u \in C^\infty(M)$.

4.2 Proof of Lemma 11

We will adopt to the L^p-setting the arguments from [1, Appendix B] that were used in the L^2-setting. Let $1 < p < +\infty$ and let $\lambda > 0$.

Take a test function $\phi \in C_c^\infty(M)$ such that $\phi \geq 0$. We need to prove that

$$\int u\phi \, d\mu \geq 0.$$

Let p' satisfy $1/p + 1/p' = 1$. Let us solve the equation

$$(\Delta_M + \lambda)\psi = \phi, \qquad \psi \in L^{p'}(M).$$

By Lemma 18 and Remark 19 it follows that $\psi \in C^\infty(M)$ and $\psi \geq 0$. So we can write

$$\int u\phi \, d\mu \;=\; \int u(\Delta_M \psi + \lambda\psi)\, d\mu. \tag{27}$$

Now the right hand side can be rewritten as

$$\int u(\Delta_M \psi + \lambda\psi)\, d\mu \;=\; \langle\, (\Delta_M + \lambda)\, u, \psi\, \rangle_S \;=\; \langle \nu, \psi \rangle_S, \tag{28}$$

where $\langle \cdot, \cdot \rangle_S$ is the anti-duality of $\widetilde{W}^{-2,p}(M)$ and $\widetilde{W}^{2,p'}(M)$ as in Sec. 4.1.

Next, will show that

$$\langle \nu, \psi \rangle_S = \int_M \psi \nu \tag{29}$$

(the integral in the right hand side makes sense as the integral of a positive measure (see Remark 10), though it can be infinite.) Then, we will be done because the integral is obviously non-negative.

We will establish (29) by presenting the function ψ as a limit

$$\psi \;=\; \lim_{k \to \infty} \psi_k, \tag{30}$$

where $\psi_k \in C_c^\infty(M)$, $\psi_k \geq 0$, $\psi_k \leq \psi_{k+1}$, and the limit is taken in the norm $\| \cdot \|_{2,p'}$. Then the equality (29) obviously holds if we replace ψ by ψ_k, so in the limit we obtain the equality for ψ.

We take $\psi_k = \chi_k \psi$, where $\chi_k \in C_c^\infty(M)$, $0 \leq \chi_k \leq 1$, $\chi_k \leq \chi_{k+1}$, and for every compact $L \subset M$ there exists k such that $\chi_k|_L = 1$.

Since $\psi \in L^{p'}(M)$, we obviously have $\psi_k \to \psi$ in $L^{p'}(M)$, as $k \to +\infty$. We also want to have $\Delta_M \psi_k \to \Delta_M \psi$ in $L^{p'}(M)$. Clearly,

$$\Delta_M \psi_k \;=\; \chi_k \Delta_M \psi - 2\langle d\chi_k, d\psi \rangle_{T_x^* M} + (\Delta_M \chi_k)\psi, \tag{31}$$

where $\langle \cdot, \cdot \rangle_{T_x^* M}$ denotes pointwise scalar product of 1-forms $d\chi_k$ and $d\psi$.

Since $(\Delta_M + \lambda)\psi = \phi \in C_c^\infty(M)$ and since $\psi \in L^{p'}(M)$, it follows that $\psi \in \mathrm{Dom}(A_{p',\max})$. Hence, we have

$$\chi_k \Delta_M \psi \to \Delta_M \psi \qquad \text{in } L^{p'}(M).$$

Since $1 < p < +\infty$, it follows that $1 < p' < +\infty$, so by [8, Proposition 4.1], we have $\mathrm{Dom}(A_{p',\max}) = W^{2,p'}(M)$, where $W^{2,p'}(M)$ is as in Sec. 2.1. Thus, by Remark 6 we have $d\psi \in L^{p'}(\Lambda^1 T^*M)$, where $L^{p'}(\Lambda^1 T^*M)$ denotes the space of p'-integrable 1-forms on M. On the other hand, $d\chi_k \to 0$ and $\Delta_M \chi_k \to 0$ in $C^\infty(M)$.

To conclude the proof, it remains to construct χ_k in such a way that

$$\sup_{x\in M} |d\chi_k(x)| \;\leq\; C, \qquad \sup_{x\in M} |\Delta_M \chi_k(x)| \leq C, \tag{32}$$

where $C > 0$ does not depend on k.

In the case of any manifold of bounded geometry (M, g), the construction of χ_k satisfying all the necessary properties can be found in [8, Sec. 1.4]. $\square$

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