On $m$-accretive Schrödinger operators in $L^p$-spaces on manifolds of bounded geometry

Ognjen Milatovic

Department of Mathematics and Statistics
University of North Florida
Jacksonville, FL 32224
USA.

Abstract

Let $(M, g)$ be a manifold of bounded geometry with metric $g$. We consider a Schrödinger-type differential expression $H = \Delta_M + V$, where $\Delta_M$ is the scalar Laplacian on $M$ and $V$ is a non-negative locally integrable function on $M$. We give a sufficient condition for $H$ to have an $m$-accretive realization in the space $L^p(M)$, where $1 < p < +\infty$. The proof uses Kato’s inequality and $L^p$-theory of elliptic operators on Riemannian manifolds.

Key words: manifold of bounded geometry, $m$-accretive, Schrödinger operator, PACS: 58J50, 35P05

1 Introduction and the main results

Let $(M, g)$ be a $C^\infty$ Riemannian manifold without boundary, with metric $g = (g_{jk})$ and dim $M = n$. We will assume that $M$ is connected and oriented. By $d\mu$ we will denote the Riemannian volume element of $M$.

In what follows, by $T^*_x M$ and $T^* M$ we will denote the cotangent space of $M$ at $x \in M$ and cotangent bundle of $M$ respectively. By $C^\infty(M)$ we denote the space of smooth functions on $M$, by $C^\infty_c(M)$—the space of smooth compactly supported functions on $M$, by $\Omega^1(M)$—the space of smooth 1-forms on $M$, and by $D'(M)$—the distributions on $M$.

Email address: omilatov@unf.edu (Ognjen Milatovic).
Let $1 \leq p < +\infty$. By $L^p(M)$ we denote the completion of $C_c^\infty(M)$ with respect to the norm

$$
\|u\|_p := \left( \int_M |u|^p \, d\mu \right)^{1/p}.
$$

By $\langle \cdot, \cdot \rangle$ we will denote the anti-duality of the pair $(L^p(M), L^p(M))$, where $1 \leq p < +\infty$ and $1/p + 1/p' = 1$, and the anti-duality of the pair $(D'(M), C_c^\infty(M))$.

In the sequel, $d: C^\infty(M) \to \Omega^1(M)$ is the standard differential, $d^*: \Omega^1(M) \to C^\infty(M)$ is the formal adjoint of $d$ with respect to the usual inner product in $L^2(M)$, and $\Delta_M := d^*d$ is the scalar Laplacian on $M$; see [2, Sec. 1].

We will consider a Schrödinger type differential expression of the form

$$
H = \Delta_M + V,
$$

where $V \in L^1_{\text{loc}}(M)$ is real-valued.

### 1.1 Operators associated to $H$

Let $1 < p < +\infty$. We define the maximal operator $H_{p,\text{max}}$ in $L^p(M)$ associated to $H$ by the formula $H_{p,\text{max}}u = Hu$ with domain

$$
\text{Dom}(H_{p,\text{max}}) = \{u \in L^p(M) : Vu \in L^1_{\text{loc}}(M), \Delta_M u + Vu \in L^p(M)\}. \tag{1}
$$

Here, the term $\Delta_M u$ in $\Delta_M u + Vu$ is understood in distributional sense.

In general, $\text{Dom}(H_{p,\text{max}})$ does not contain $C_c^\infty(M)$, but it does if $V \in L^p_{\text{loc}}(M)$. In this case, we can define $H_{p,\text{min}} := H_{p,\text{max}}|C_c^\infty(M)$.

**Remark 1** Using the same definitions as in Sec. 1.1, we can also define $H_{p,\text{max}}$ and $H_{p,\text{min}}$ for $p = 1$ and $p = \infty$. However, we will not use those operators in this paper.

### 1.2 Operators associated to $\Delta_M$

Let $1 < p < +\infty$. We define the maximal operator $A_{p,\text{max}}$ in $L^p(M)$ associated to $\Delta_M$ by the formula $A_{p,\text{max}}u = \Delta_M u$ with the domain

$$
\text{Dom}(A_{p,\text{max}}) = \{u \in L^p(M) : \Delta_M u \in L^p(M)\}. \tag{2}
$$
We define $A_{p, \text{min}} := A_{p, \text{max}}|_{C_c^\infty(M)}$.

Throughout this paper, we will use the terminology of contraction semigroups, accretive and $m$-accretive operators on a Banach space; see Sec. 1.4 below for a brief review.

1.3 Domination and positivity

Suppose that $B$ and $C$ are bounded linear operators on $L^p(M)$. In what follows, the notation $B \leq C$ means that for all $0 \leq f \in L^p(M)$ we have $(C - B)f \geq 0$. We will use the notation $B << C$ to indicate that $B$ is dominated by $C$, i.e. $|Bf| \leq C|f|$ for all $f \in L^p(M)$.

Assumption (A1). Assume that $(M, g)$ has bounded geometry.

Remark 2 In this paper, the term “bounded geometry” is the same as in [8, Sec. A.1.1] or [2, Sec. 1]. In particular, a manifold of bounded geometry is complete.

In the sequel, by $\overline{A}$ we denote the closure of a closable operator $A$.

We now state the main results.

Theorem 3 Assume that $(M, g)$ is a connected $C^\infty$ Riemannian manifold without boundary. Assume that the Assumption (A1) is satisfied. Assume that $1 < p < +\infty$ and $0 \leq V \in L^p_{\text{loc}}(M)$. Then the following properties hold:

1. the operator $H_{p, \text{max}}$ generates a contraction semigroup on $L^p(M)$. In particular, $H_{p, \text{max}}$ is an $m$-accretive operator.
2. the set $C_c^\infty(M)$ is a core for $H_{p, \text{max}}$ (i.e. $\overline{H_{p, \text{min}}} = H_{p, \text{max}}$).

Theorem 4 Under the same hypotheses as in Theorem 3, the following properties hold (with the notations as in Sec. 1.2 and Sec. 1.3):

1. $0 \leq (\lambda + H_{p, \text{max}})^{-1} \leq (\lambda + A_{p, \text{max}})^{-1}$, for all $\lambda > 0$;
2. $(\lambda + H_{p, \text{max}})^{-1} << (\gamma + A_{p, \text{max}})^{-1}$, for all $\lambda \in \mathbb{C}$ such that $\gamma := \Re \lambda > 0$.

Remark 5 T. Kato [6, Part A] considered the differential expression $-\Delta + V$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with standard metric and measure and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Assuming $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, Kato proved the property (1) of Theorem 3 and Theorem 4 for all $1 \leq p \leq +\infty$. Assuming $0 \leq V \in L^p_{\text{loc}}(\mathbb{R}^n)$, Kato proved the property (2) of Theorem 3 for $1 \leq p < +\infty$. In his proof, Kato used certain properties (specific to the $\mathbb{R}^n$ setting) of $(-\Delta_{2, \text{max}} + \gamma)^{-1}$, where $\gamma > 0$ and $-\Delta_{2, \text{max}}$ is the self-adjoint closure of $-\Delta|_{C_c^\infty(\mathbb{R}^n)}$ in
L^2(\mathbb{R}^n), which enabled him to handle the cases p = 1 and p = \infty. In the context of non-compact Riemannian manifolds, we use L^p-theory of elliptic differential operators, which works well for 1 < p < +\infty.

1.4 Accretive operators and contraction semigroups

Here we briefly review some terms and facts concerning accretive operators, m-accretive operators and contraction semigroups. For more details, see, for example, [7, Sec. X.8] or [5, Sec. IX.1].

A densely defined linear operator T on a Banach space Y is said to be accretive if for each u \in \text{Dom}(T) the following property holds: Re(f(Tu)) \geq 0 for some normalized tangent functional f \in Y^* to u. An operator T is said to be m-accretive if T is accretive and has no proper accretive extension.

Let Y be a Banach space and let T: \text{Dom}(T) \subset Y \to Y be a closed linear operator. Let \rho(T) denote the resolvent set of T. By [7, Theorem X.47(a)], the operator T generates a contraction semigroup on Y if and only if the following two conditions are satisfied:

1. \((-\infty, 0) \subset \rho(T);
2. for all \lambda > 0,

\| (T + \lambda)^{-1} \| \leq \frac{1}{\lambda},

where \| \cdot \| denotes the operator norm.

By [7, Theorem X.48], a closed linear operator T on a Banach space Y is the generator of a contraction semigroup if and only if T is accretive and \text{Ran}(\lambda_0 + T) = Y for some \lambda_0 > 0.

By the Remark preceding Theorem X.49 in [7], the following holds: if a closed linear operator T on a Banach space Y is the generator of a contraction semigroup, then T is m-accretive.

2 Preliminary Lemmas

In what follows, we will use the following notations for Sobolev spaces on Riemannian manifolds (M, g).
2.1 Sobolev space notations

Let $k \geq 0$ be an integer, and let $1 \leq p < +\infty$ be a real number. By $W^{k,p}(M)$ we will denote the completion of $C^\infty_0(M)$ in the norm

$$\|u\|_{W^{k,p}}^p := \sum_{l=0}^{k} \|\nabla^l u\|_{L^p}^p,$$

where $\nabla^l u$ is the $l$-th covariant derivative of $u$; see [2, Sec. 1] or [8, Sec. A.1.1].

**Remark 6** If $(M, g)$ has bounded geometry, then by [2, Proposition 1.6] it follows that $W^{k,p}(M) = \{u \in L^p(M) : \nabla^l u \in L^p, \text{ for all } 1 \leq l \leq k\}$.

**Remark 7** Under the assumption that $(M, g)$ is a complete Riemannian manifold (not necessarily of bounded geometry) and $1 < p < +\infty$, the first and the second paragraph in the proof of [9, Theorem 3.5] give the proofs of the following properties:

1. the operator $A_{p,\min} = \Delta_M|_{C^\infty_0(M)}$ is accretive (hence, closable);
2. the closure $\overline{A_{p,\min}}$ generates a contraction semigroup on $L^p(M)$;
3. $(-\infty, 0) \subset \rho(\overline{A_{p,\min}})$ and
   $$\|(\lambda + A_{p,\min})^{-1}\| \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0,$$
   where $\| \cdot \|$ denotes the operator norm (for a bounded linear operator $L^p(M) \to L^p(M)$);
4. the resolvents $(\lambda + A_{p,\min})^{-1}$ and $(\lambda + A_{2,\min})^{-1}$, where $\lambda > 0$, are equal on $L^2(M) \cap L^p(M)$.

**Remark 8** By an abstract fact, the properties (2) and (3) in Remark 7 are equivalent; see [7, Theorem X.47(a)].

**Remark 9** Assume that $(M, g)$ has bounded geometry and $1 < p < +\infty$. Then by [8, Proposition 4.1] it follows that $A_{p,\max} = \overline{A_{p,\min}}$. Thus, the properties (2), (3) and (4) of Remark 7 hold with $\overline{A_{p,\min}}$ replaced by $A_{p,\max}$.

2.2 Distributional inequality

Assume that $1 < p < +\infty$ and $\lambda > 0$, and consider the following distributional inequality:

$$(\Delta_M + \lambda) u = \nu \geq 0, \quad u \in L^p(M),$$

(5)
where the inequality $\nu \geq 0$ means that $\nu$ is a positive distribution, i.e. $\langle \nu, \phi \rangle \geq 0$ for any $0 \leq \phi \in C_0^\infty(M)$.

**Remark 10** It follows that $\nu$ is in fact a positive Radon measure (see e. g. [3], Theorem 1 in Sec. 2, Ch. II).

**Lemma 11** Assume that $(M, g)$ is a manifold of bounded geometry. Assume that $1 < p < +\infty$. Assume that $u \in L^p(M)$ satisfies (5). Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).

For the proof of Lemma 11 in the case $p = 2$, see the proof of Proposition B.3 in Appendix B of [1]. In the proof of Lemma 11, which we give in Sec. 4 below, we adopt the scheme of proof for the case $p = 2$ from [1, Appendix B].

**2.3 Kato’s inequality**

In what follows, we will use a version of Kato’s inequality. For the proof of a more general version of this inequality, see [1, Theorem 5.7].

**Lemma 12** Assume that $(M, g)$ is an arbitrary Riemannian manifold. Assume that $u \in L^1_{\text{loc}}(M)$ and $\Delta_M u \in L^1_{\text{loc}}(M)$. Then the following distributional inequality holds:

$$
\Delta_M |u| \leq \text{Re}(\Delta_M u \text{ sign } \bar{u}),
$$

(6)

where

$$
\text{sign } u(x) = \begin{cases} 
\frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
$$

**Remark 13** For the original version of Kato’s inequality, see Kato [4, Lemma A].

In the sequel, with the help of Lemma 11, we will adopt certain arguments of Kato [6, Part A] to our setting.

**Lemma 14** Let $(M, g)$ be a Riemannian manifold. Assume that $1 < p < +\infty$. Assume that $0 \leq V \in L^1_{\text{loc}}(M)$, $u \in \text{Dom}(H_{p,\text{max}})$ and $\lambda \in \mathbb{C}$. Let $f := (H_{p,\text{max}} + \lambda)u$.

Then the following distributional inequality holds:

$$
(\text{Re } \lambda + \Delta_M + V)|u| \leq |f|.
$$

(7)
Proof Since \( u \in \text{Dom}(H_{p,\text{max}}) \) it follows that \( Vu \in L^1_{\text{loc}}(M) \) and \( H_{p,\text{max}}u \in L^p(M) \subset L^1_{\text{loc}}(M) \). Thus \( u \in L^1_{\text{loc}}(M) \) and \( \Delta_M u \in L^1_{\text{loc}}(M) \). By Kato’s inequality (6) we have

\[
(\text{Re}\lambda + \Delta_M + V)|u| \leq \text{Re}[(\lambda + \Delta_M + V)u \text{ sign } \bar{u}] = \text{Re}(f \text{ sign } \bar{u}) \leq |f|,
\]

and the Lemma is proven. \( \square \)

In the sequel, we will always assume that \((M, g)\) is a manifold of bounded geometry.

**Lemma 15** Assume that \( 1 < p < +\infty \) and \( 0 \leq V \in L^1_{\text{loc}}(M) \). Assume that \( \lambda \in \mathbb{C} \) and \( \gamma := \text{Re}\lambda > 0 \). Then the following properties hold:

1. for all \( u \in \text{Dom}(H_{p,\text{max}}) \), we have

\[
\gamma\|u\|_p \leq \|(\lambda + H_{p,\text{max}})u\|_p; \tag{8}
\]

2. the operator \( \lambda + H_{p,\text{max}}: \text{Dom}(H_{p,\text{max}}) \subset L^p(M) \to L^p(M) \) is injective;

3. for all \( u \in \text{Dom}(H_{p,\text{max}}) \) such that \( (\lambda + H_{p,\text{max}})u \geq 0 \) (where \( \lambda > 0 \)), we have \( u \geq 0 \).

Proof We first prove property (1). Let \( u \in \text{Dom}(H_{p,\text{max}}) \) and \( f := (\lambda + H_{p,\text{max}})u \). By the definition of \( \text{Dom}(H_{p,\text{max}}) \), we have \( f \in L^p(M) \), where \( 1 < p < +\infty \). Since \( V \geq 0 \) and since \( Vu \in L^1_{\text{loc}}(M) \), from (7) we get the following distributional inequality:

\[
(\gamma + \Delta_M)|u| \leq |f|. \tag{9}
\]

By property (3) of Remark 7 and by Remark 9, it follows that

\[
(\gamma + A_{p,\text{max}})^{-1}: L^p(M) \to L^p(M)
\]

is a bounded linear operator.

Let us rewrite (9) as

\[
(\gamma + \Delta_M)[(\gamma + A_{p,\text{max}})^{-1}|f| - |u|] \geq 0.
\]

Note that

\[
(\gamma + A_{p,\text{max}})^{-1}|f| \in L^p(M) \quad \text{and} \quad |u| \in L^p(M).
\]
Thus, $(\gamma + A_{p,\text{max}})^{-1}|f| - |u| \in L^p(M)$, and, hence, by Lemma 11 we have $(\gamma + A_{p,\text{max}})^{-1}|f| - |u| \geq 0$, i.e.

$$|u| \leq (\gamma + A_{p,\text{max}})^{-1}|f|. \quad (10)$$

By (4) and by Remark 9 it follows that

$$\| (\gamma + A_{p,\text{max}})^{-1}|f| \|_p \leq \frac{1}{\gamma} \| f \|_p. \quad (11)$$

By (10) and (11) we have

$$\| u \|_p \leq \| (\gamma + A_{p,\text{max}})^{-1}|f| \|_p \leq \frac{1}{\gamma} \| f \|_p,$$

and (8) is proven.

We now prove property (2). Assume that $u \in \text{Dom}(H_{p,\text{max}})$ and $(\lambda + H_{p,\text{max}})u = 0$. Using (8) with $f = 0$, we get $\|u\|_p = 0$, and hence $u = 0$. This shows that $\lambda + H_{p,\text{max}}$ is injective.

We now prove property (3). Let $\lambda > 0$ and assume that $u \in \text{Dom}(H_{p,\text{max}})$ satisfies

$$f := (H_{p,\text{max}} + \lambda)u \geq 0.$$

We claim that $u$ is real. Indeed, since $(H_{p,\text{max}} + \lambda)\bar{u} = f$, we have $(H_{p,\text{max}} + \lambda)(u - \bar{u}) = 0$. By property (2) of this lemma we have $u = \bar{u}$. Since $f \geq 0$ and $\lambda > 0$, by (7) we have

$$(\lambda + \Delta_M + V)|u| \leq f. \quad (12)$$

Subtracting $f = (\lambda + H_{p,\text{max}})u$ from both sides of (12) we get

$$(\lambda + \Delta_M + V)v \leq 0,$$

where $v := |u| - u \geq 0$.

Since $V \geq 0$, it follows that $(\lambda + \Delta_M)v \leq 0$. By Lemma 11 we get $v \leq 0$. Thus $v = 0$, i.e. $u = |u| \geq 0$. This concludes the proof. \hfill \Box

**Lemma 16** Assume that $1 < p < +\infty$ and $0 \leq V \in L^1_{\text{loc}}(M)$. Then the following properties hold:

1. the operator $H_{p,\text{max}}$ is closed;
(2) the operator $\lambda + H_{p,\text{max}}$, where $\text{Re} \lambda > 0$, has a closed range.

**Proof** We first prove the property (1). Let $u_k \in \text{Dom}(H_{p,\text{max}})$ be a sequence such that, as $k \to +\infty$,

$$u_k \to u, \quad f_k := H_{p,\text{max}}u_k = \Delta_M u_k + V u_k \to f \quad \text{in } L^p(M).$$

(13)

We need to show that $u \in \text{Dom}(H_{p,\text{max}})$ and $H_{p,\text{max}} u = f$.

By passing to subsequences, we may assume that the convergence in (13) is also pointwise almost everywhere.

The distributional inequality (7) holds if we replace $u$ by $u_k - u_l$, $f$ by $f_k - f_l$ and $\lambda$ by 0. With these replacements, we apply a test function $0 \leq \phi \in C_c^\infty(M)$ to (7) and get

$$0 \leq \langle V |u_k - u_l|, \phi \rangle \leq \langle |f_k - f_l|, \phi \rangle - \langle \Delta_M |u_k - u_l|, \phi \rangle,$$

(14)

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(\mathcal{D}'(M), C_c^\infty(M))$.

Using integration by parts in the second term on the right hand side of the second inequality in (14), we get

$$0 \leq \langle V |u_k - u_l|, \phi \rangle \leq \langle |f_k - f_l|, \phi \rangle - \langle |u_k - u_l|, \Delta_M \phi \rangle.$$

(15)

Letting $k, l \to +\infty$, the right hand side of the second inequality in (15) tends to 0 by (13). Thus $V u_k \phi$ is a Cauchy sequence in $L^1(M)$, and its limit must be equal to $V u \phi$. Since $\phi \in C_c^\infty(M)$ may have an arbitrarily large support, it follows that $Vu \in L^1_{\text{loc}}(M)$. Thus $Vu_k \to Vu$ in $L^1_{\text{loc}}(M)$ and hence in $\mathcal{D}'(M)$. Since $u_k \to u$ in $L^p(M)$ (and, hence in $L^1_{\text{loc}}(M)$), we get $\Delta_M u_k \to \Delta_M u$ in $\mathcal{D}'(M)$. Thus, $f_k = \Delta_M u_k + V u_k \to \Delta_M u + V u$ in $\mathcal{D}'(M)$. Since $f_k \to f$ in $L^p(M) \subset \mathcal{D}'(M)$, we obtain $\Delta_M u + V u = f \in L^p(M)$. This shows that $u \in \text{Dom}(H_{p,\text{max}})$ and $H_{p,\text{max}} u = f$. This proves that $H_{p,\text{max}}$ is closed.

We now prove the property (2). Since $H_{p,\text{max}}$ is closed, it immediately follows from (8) that $\lambda + H_{p,\text{max}}$ has a closed range for $\text{Re} \lambda > 0$. □

**Lemma 17** Assume that $1 < p < +\infty$ and $0 \leq V \in L^p_{\text{loc}}(M)$. Let $\lambda \in \mathbb{C}$ and $\gamma := \text{Re} \lambda > 0$. Then the following properties hold:

(1) the operator $\lambda + H_{p,\text{max}}$: $\text{Dom}(H_{p,\text{max}}) \subset L^p(M) \to L^p(M)$ is surjective;

(2) the operator $(\lambda + H_{p,\text{max}})^{-1}$: $L^p(M) \to L^p(M)$ is a bounded linear operator with the operator norm

$$\| (\lambda + H_{p,\text{max}})^{-1} \| \leq 1/\gamma.$$

(16)
Proof We first prove the property (1). Since $\lambda + H_{p,\max}$ has a closed range by Lemma 16, it is enough to show that $(\lambda + H_{p,\min})C^\infty_c(M)$ is dense in $L^p(M)$. Let $v \in (L^p(M))^* = L^{p'}(M)$, where $1/p + 1/p' = 1$, be a continuous linear functional annihilating $(\lambda + H_{p,\min})C^\infty_c(M)$:

$$\langle (\lambda + H_{p,\min})\phi, v \rangle = 0, \quad \text{for all } \phi \in C^\infty_c(M),$$

(17)

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$.

From (17) we get the following distributional equality:

$$(\bar{\lambda} + \Delta_M + V)v = 0.$$

Since by hypothesis $V \in L^p_{\text{loc}}(M)$ and since $v \in L^{p'}(M)$, by Hölder’s inequality we have $Vv \in L^1_{\text{loc}}(M)$. Since $\Delta_M v = -Vv - \lambda v$, we get $\Delta_M v \in L^1_{\text{loc}}(M)$. By Kato’s inequality and since $V \geq 0$, we have

$$\Delta_M |v| \leq \text{Re}((\Delta_M v) \text{sign } \bar{v}) = \text{Re}((-\bar{\lambda}v - Vv) \text{sign } \bar{v}) \leq -(\text{Re } \bar{\lambda})|v|,$$

and, hence,

$$(\Delta_M + \text{Re } \bar{\lambda})|v| \leq 0.$$

Since $v \in L^{p'}(M)$ (with $1 < p' < +\infty$) and since $\text{Re } \bar{\lambda} = \text{Re } \lambda > 0$, by Lemma 11 we get $|v| \leq 0$. Thus $v = 0$, and the surjectivity of $\lambda + H_{p,\max}$ is proven.

We now prove the property (2). Assume that $\lambda \in \mathbb{C}$ satisfies $\gamma := \text{Re } \lambda > 0$. Since $\lambda + H_{p,\max} : \text{Dom}(H_{p,\max}) \subset L^p(M) \to L^p(M)$ is injective and surjective, the inverse $(\lambda + H_{p,\max})^{-1}$ is defined on the whole $L^p(M)$. The inequality (16) follows immediately from (8). This concludes the proof of the Lemma. □

3 Proofs of main results

3.1 Proof of Theorem 3

We first prove the property (1). By Lemma 17 it follows that $(-\infty, 0) \subset \rho(H_{p,\max})$ and

$$\|((\lambda + H_{p,\max})^{-1}) \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0.$$
Thus, by [7, Theorem X.47(a)] it follows that $H_{p,\max}$ generates a contraction semigroup on $L^p(M)$. In particular, the operator $H_{p,\max}$ is $m$-accretive; see Sec. 1.4 above.

We now prove the property (2). By property (1) of this lemma it follows that $H_{p,\max}$ is $m$-accretive; hence, $H_{p,\min} = H_{p,\max}|_{C^\infty(M)}$ is accretive. By an abstract fact (see the remark preceding Theorem X.48 in [7]), the operator $H_{p,\min}$ is closable and $H_{p,\min}$ is accretive. Let $\lambda > 0$. By the proof of property (1) in Lemma 17 it follows that $\text{Ran}(\lambda + H_{p,\min})$ dense in $L^p(M)$. Using (8) and the definition of the closure of an operator (see, for example, definition below equation (5.6) in [5, Sec. III.5.3]), it follows that $\text{Ran}(\lambda + \overline{H_{p,\min}}) = L^p(M)$. Now by [7, Theorem X.48] the operator $\overline{H_{p,\min}}$ generates a contraction semigroup on $L^p(M)$. Thus, by the remark preceding Theorem X.49 in [7], the operator $\overline{H_{p,\min}}$ is $m$-accretive. Since $\overline{H_{p,\min}} \subset H_{p,\max}$ and since $\overline{H_{p,\min}}$ and $H_{p,\max}$ are $m$-accretive, it follows that $\overline{H_{p,\min}} = H_{p,\max}$.

This concludes the proof of the Theorem. □

3.2 Proof of Theorem 4

We first prove the property (1). Let $\lambda > 0$, let $0 \leq f \in L^p(M)$ be arbitrary, and let $u := (H_{p,\max} + \lambda)^{-1}f$. Then $(H_{p,\max} + \lambda)u = f \geq 0$, and, hence, by the property (3) of Lemma 15, we have $u \geq 0$. This proves the inequality $0 \leq (H_{p,\max} + \lambda)^{-1}$.

We now prove

$$(H_{p,\max} + \lambda)^{-1} \leq (A_{p,\max} + \lambda)^{-1}.$$  \hspace{1cm} (18)

Let $0 \leq f \in L^p(M)$ be arbitrary and let $u := (H_{p,\max} + \lambda)^{-1}f$. Then $0 \leq u \in \text{Dom}(H_{p,\max})$, and, hence, using (10) with $u \geq 0$ and $f \geq 0$ we get

$$u \leq (A_{p,\max} + \lambda)^{-1}f.$$  \hspace{1cm} (19)

By (19) with $u = (H_{p,\max} + \lambda)^{-1}f$, we immediately get (18). This concludes the proof of property (1).

We now prove the property (2). Let $\gamma := \text{Re} \lambda > 0$. Let $f \in L^p(M)$ be arbitrary and let $u := (H_{p,\max} + \lambda)^{-1}f$. By (10) we have

$$|(H_{p,\max} + \lambda)^{-1}f| \leq (A_{p,\max} + \gamma)^{-1}|f|,$$

and property (2) is proven.
This concludes the proof of the Theorem.

4 Proof of Lemma 11

We begin by introducing some additional notations and definitions.

4.1 Sobolev spaces $\tilde{W}^{-2,p}(M)$ and $\tilde{W}^{2,p}(M)$

Let $1 < p < +\infty$ and let $\lambda > 0$. Define

$$\tilde{W}^{-2,p}(M) := (\Delta_M + \lambda) L^p(M)$$

and

$$\tilde{W}^{2,p}(M) := \{ u \in L^p(M) : \Delta_M u \in L^p(M) \}.$$  

The norms in $\tilde{W}^{2,p}(M)$ and $\tilde{W}^{-2,p}(M)$ are given respectively by the formulas

$$\| v \|_{2,p} = \| (\Delta_M + \lambda) v \|_p, \quad \| (\Delta_M + \lambda) f \|_{-2,p} = \| f \|_p,$$  

where $\| \cdot \|_p$ is the norm in $L^p(M)$.

Let $1 < p < +\infty$ and $1/p + 1/p' = 1$. By $\langle \cdot, \cdot \rangle_S$ we denote the anti-duality

$$\langle \cdot, \cdot \rangle_S : \tilde{W}^{-2,p}(M) \times \tilde{W}^{2,p'}(M) \to \mathbb{C},$$  

of the spaces $\tilde{W}^{-2,p}(M)$ and $\tilde{W}^{2,p'}(M)$ obtained by extending the anti-duality of the pair $(L^p(M), L^{p'}(M))$ by continuity from $C_c^\infty(M) \times C_c^\infty(M)$.

The extension of the anti-duality (21) from $C_c^\infty(M) \times C_c^\infty(M)$ is well defined because $C_c^\infty(M)$ is dense in both spaces $\tilde{W}^{2,p'}(M)$ and $\tilde{W}^{-2,p}(M)$ in the corresponding norms (20). Indeed, density of $C_c^\infty(M)$ in $\tilde{W}^{2,p'}(M)$ means simply that $(\Delta_M + \lambda)C_c^\infty(M)$ is dense in $L^{p'}(M)$. To establish this, let us take $f \in (L^{p'}(M))^* = L^p(M)$ which annihilates $(\Delta_M + \lambda)C_c^\infty(M)$:

$$\langle f, (\Delta_M + \lambda) \phi \rangle = 0, \quad \text{for all } \phi \in C_c^\infty(M),$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$. 
This implies that $\Delta_M + \lambda) f = 0$ in the sense of distributions, i.e. $f$ is in the null-space of $A_{p,\text{max}} + \lambda$, where $A_{p,\text{max}}$ is as in Sec. 1.2. By Remark 7 and Remark 9 it follows that $f = 0$.

Similarly, density of $C_c^\infty(M)$ in $\tilde{W}^{-2,p}(M)$ means that $(A_{p,\text{max}} + \lambda)^{-1} C_c^\infty(M)$ is dense in $L^p(M)$. To prove this, consider $h \in (L^p(M))^* = L^{p'}(M)$ such that $h$ annihilates $(A_{p,\text{max}} + \lambda)^{-1} C_c^\infty(M)$:

$$\langle (A_{p,\text{max}} + \lambda)^{-1} \phi, h \rangle = 0, \quad \text{for all } \phi \in C_c^\infty(M),$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(L^p(M), L^{p'}(M))$.

This implies that

$$\langle \phi, ((A_{p,\text{max}} + \lambda)^{-1})^* h \rangle = 0, \quad \text{for all } \phi \in C_c^\infty(M).$$

(22)

We will now show that

$$((A_{p,\text{max}} + \lambda)^{-1})^* = (A_{p',\text{max}} + \lambda)^{-1}.$$

(23)

By Remark 7 and Remark 9 it follows that $(A_{p,\text{max}} + \lambda)^{-1}$ and $(A_{p',\text{max}} + \lambda)^{-1}$ are bounded linear operators on $L^p(M)$ and $L^{p'}(M)$ respectively. Thus, since $C_c^\infty(M)$ is dense in $L^p(M)$ and $L^{p'}(M)$, it suffices to show that

$$\langle (A_{p,\text{max}} + \lambda)^{-1} \phi, \psi \rangle = \langle \phi, (A_{p',\text{max}} + \lambda)^{-1} \psi \rangle, \quad \text{for all } \phi, \psi \in C_c^\infty(M).$$

By property (4) of Remark 7 and by Remark 9 we have for all $\phi, \psi \in C_c^\infty(M)$:

$$\langle (A_{p,\text{max}} + \lambda)^{-1} \phi, \psi \rangle = \langle (A_{2,\text{max}} + \lambda)^{-1} \phi, \psi \rangle = \langle \phi, (A_{2,\text{max}} + \lambda)^{-1} \psi \rangle = \langle \phi, (A_{p',\text{max}} + \lambda)^{-1} \psi \rangle.$$  

(24)

The second equality in (24) holds since $(A_{2,\text{max}} + \lambda)^{-1}$ is a bounded self-adjoint operator on $L^2(M)$ (it is well known that, for a complete Riemannian manifold $(M, g)$, the operator $A_{2,\text{max}}$ is a non-negative self-adjoint operator in $L^2(M)$; see, for example, [2, Theorem 3.5]).

Thus, from (22) and (23) we get $((A_{p,\text{max}} + \lambda)^{-1})^* h = (A_{p',\text{max}} + \lambda)^{-1} h = 0$. But this means that $h = 0$.

In the sequel, we will use the following lemma.

**Lemma 18** Assume that $(M, g)$ is a manifold of bounded geometry. Assume that $1 < p < +\infty$. Assume that $0 \leq \phi \in C_c^\infty(M)$ and $\lambda > 0$. Then there exists
a unique \( u \in L^p(M) \) such that

\[
(\Delta_M + \lambda) u = \phi,
\]

and \( u \geq 0 \).

**Proof** The existence and uniqueness of solution \( u \in L^p(M) \) to (25) follows by property (3) in Remark 7 and by Remark 9; just take

\[
u = (\lambda + A_{p,\text{max}})^{-1} \phi.
\]

To show that \( u \geq 0 \), we first note that \( \phi \in L^2(M) \cap L^p(M) \). By the property (4) of Remark 7 and by Remark 9 it follows that

\[
(\lambda + A_{p,\text{max}})^{-1} \phi = (\lambda + A_{2,\text{max}})^{-1} \phi.
\]

By the proof of [1, Theorem B.1] it follows that

\[
w := (\lambda + A_{2,\text{max}})^{-1} \phi
\]

satisfies (25) (with \( u \) replaced by \( w \)) and \( w \geq 0 \).

Now by (26) we have \( w = u \), and, hence, \( u \geq 0 \). This concludes the proof of the Lemma.

**Remark 19** Let \( 1 < p < +\infty \). If \( u \in L^p(M) \) satisfies (25) with \( \phi \in C_c^\infty(M) \), then it is well known (by using standard elliptic regularity and Sobolev imbedding theorems) that \( u \in C^\infty(M) \).

4.2 Proof of Lemma 11

We will adopt to the \( L^p \)-setting the arguments from [1, Appendix B] that were used in the \( L^2 \)-setting. Let \( 1 < p < +\infty \) and let \( \lambda > 0 \).

Take a test function \( \phi \in C_c^\infty(M) \) such that \( \phi \geq 0 \). We need to prove that

\[
\int u\phi \, d\mu \geq 0.
\]

Let \( p' \) satisfy \( 1/p + 1/p' = 1 \). Let us solve the equation

\[
(\Delta_M + \lambda) \psi = \phi, \quad \psi \in L^{p'}(M).
\]
By Lemma 18 and Remark 19 it follows that $\psi \in C^\infty(M)$ and $\psi \geq 0$. So we can write

$$\int u\phi d\mu = \int u(\Delta_M \psi + \lambda \psi) d\mu. \tag{27}$$

Now the right hand side can be rewritten as

$$\int u(\Delta_M \psi + \lambda \psi) d\mu = \langle (\Delta_M + \lambda) u, \psi \rangle_S = \langle \nu, \psi \rangle_S, \tag{28}$$

where $\langle \cdot, \cdot \rangle_S$ is the anti-duality of $\tilde{W}^{-2,p}(M)$ and $\tilde{W}^{2,p'}(M)$ as in Sec. 4.1.

Next, will show that

$$\langle \nu, \psi \rangle_S = \int_M \psi \nu \tag{29}$$

(the integral in the right hand side makes sense as the integral of a positive measure (see Remark 10), though it can be infinite.) Then, we will be done because the integral is obviously non-negative.

We will establish (29) by presenting the function $\psi$ as a limit

$$\psi = \lim_{k \to \infty} \psi_k, \tag{30}$$

where $\psi_k \in C_c^\infty(M)$, $\psi_k \geq 0$, $\psi_k \leq \psi_{k+1}$, and the limit is taken in the norm $\| \cdot \|_{2,p'}$. Then the equality (29) obviously holds if we replace $\psi$ by $\psi_k$, so in the limit we obtain the equality for $\psi$.

We take $\psi_k = \chi_k \psi$, where $\chi_k \in C_c^\infty(M)$, $0 \leq \chi_k \leq 1$, $\chi_k \leq \chi_{k+1}$, and for every compact $L \subset M$ there exists $k$ such that $\chi_k|_L = 1$.

Since $\psi \in L^{p'}(M)$, we obviously have $\psi_k \to \psi$ in $L^{p'}(M)$, as $k \to +\infty$. We also want to have $\Delta_M \psi_k \to \Delta_M \psi$ in $L^{p'}(M)$. Clearly,

$$\Delta_M \psi_k = \chi_k \Delta_M \psi - 2\langle d\chi_k, d\psi \rangle_{T^*_x M} + (\Delta_M \chi_k) \psi, \tag{31}$$

where $\langle \cdot, \cdot \rangle_{T^*_x M}$ denotes pointwise scalar product of 1-forms $d\chi_k$ and $d\psi$.

Since $(\Delta_M + \lambda) \psi = \phi \in C^\infty_c(M)$ and since $\psi \in L^{p'}(M)$, it follows that $\psi \in \text{Dom}(A_{p',\text{max}})$. Hence, we have

$$\chi_k \Delta_M \psi \to \Delta_M \psi \quad \text{in } L^{p'}(M).$$
Since $1 < p < +\infty$, it follows that $1 < p' < +\infty$, so by [8, Proposition 4.1], we have $\text{Dom}(A_{p',\max}) = W^{2,p'}(M)$, where $W^{2,p'}(M)$ is as in Sec. 2.1. Thus, by Remark 6 we have $d\psi \in L^{p'}(\Lambda^1T^*M)$, where $L^{p'}(\Lambda^1T^*M)$ denotes the space of $p'$-integrable 1-forms on $M$. On the other hand, $d\chi_k \to 0$ and $\Delta_M \chi_k \to 0$ in $C^\infty(M)$.

To conclude the proof, it remains to construct $\chi_k$ in such a way that

$$\sup_{x \in M} |d\chi_k(x)| \leq C, \quad \sup_{x \in M} |\Delta_M \chi_k(x)| \leq C,$$

where $C > 0$ does not depend on $k$.

In the case of any manifold of bounded geometry $(M,g)$, the construction of $\chi_k$ satisfying all the necessary properties can be found in [8, Sec. 1.4].

References


