

ON m -ACCRETIVE SCHRÖDINGER OPERATORS IN L^1 -SPACES ON MANIFOLDS OF BOUNDED GEOMETRY

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ABSTRACT. Let (M, g) be a manifold of bounded geometry with metric g . We consider a Schrödinger-type differential expression $H = \Delta_M + V$, where Δ_M is the scalar Laplacian on M and V is a non-negative locally integrable function on M . We give a sufficient condition for H to have an m -accretive realization in the space $L^1(M)$.

1. INTRODUCTION AND THE MAIN RESULTS

Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected and oriented. By $d\mu$ we will denote the Riemannian volume element of M . In any local coordinates x^1, \dots, x^n , we have $d\mu = \sqrt{\det(g_{jk})} dx^1 dx^2 \dots dx^n$.

In what follows, $C^\infty(M)$ denotes the space of smooth functions on M , the notation $C_c^\infty(M)$ stands for the space of smooth compactly supported functions on M , the notation $\|\cdot\|_p$ denotes the usual norm in $L^p(M)$, and $\mathcal{D}'(M)$ —the distributions on M . In the sequel, $d: C^\infty(M) \rightarrow C^\infty(\Lambda^1 T^*M)$ is the standard differential and d^* is the formal adjoint of d with respect to the inner product in $L^2(M)$. By $\Delta_M := d^*d$ we denote the scalar Laplacian on M .

We consider a Schrödinger type differential expression

$$H = \Delta_M + V,$$

where $V \in L^1_{\text{loc}}(M)$ is real-valued.

Operators associated to H and Δ_M . Let $1 \leq p \leq +\infty$ and let $V \in L^1_{\text{loc}}(M)$. We define the maximal operator $H_{p,\text{max}}$ in $L^p(M)$ by the formula $H_{p,\text{max}}u = Hu$ with domain

$$\text{Dom}(H_{p,\text{max}}) = \{u \in L^p(M) : Vu \in L^1_{\text{loc}}(M), \Delta_M u + Vu \in L^p(M)\}. \quad (1.1)$$

Here, the term $\Delta_M u$ in $\Delta_M u + Vu$ is understood in distributional sense.

In general, $\text{Dom}(H_{p,\text{max}})$ does not contain $C_c^\infty(M)$, but it does if $V \in L^p_{\text{loc}}(M)$. In this case, we can define $H_{p,\text{min}} := H_{p,\text{max}}|_{C_c^\infty(M)}$. In particular, the operator $H_{1,\text{min}}$ is always defined.

In the case $V = 0$, the operator $H_{p,\text{max}}$ will be denoted by $A_{p,\text{max}}$. We define $A_{p,\text{min}} := A_{p,\text{max}}|_{C_c^\infty(M)}$.

We now make an assumption on (M, g) .

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Assumption (A1). Assume that (M, g) has bounded geometry, i.e.

- (i) assume that $r_{\text{inj}} > 0$ (here, r_{inj} denotes the injectivity radius of (M, g));
- (ii) assume that

$$|\nabla^i R| \leq C_i, \quad \text{for all } i = 0, 1, 2, \dots,$$

where $C_i \geq 0$ are constants, and ∇^i denotes the i -th covariant derivative of the Riemann curvature tensor R of M .

Remark 1.1. The condition $r_{\text{inj}} > 0$ implies the completeness of (M, g) ; see, for instance, [10, Sec. A.1.1]. For more on manifolds (M, g) satisfying Assumption (A1), see [10, Sec. A.1.1] and [4].

In the sequel, by \bar{A} we denote the closure of a closable operator A .

We now state the main results.

Theorem 1.2. *Assume that (M, g) is a connected C^∞ Riemannian manifold without boundary. Assume that the Assumption (A1) holds. Assume that $0 \leq V \in L^1_{\text{loc}}(M)$. Then the following properties hold:*

- (i) *the operator $H_{1, \text{max}}$ generates a contraction semigroup on $L^1(M)$. In particular, $H_{1, \text{max}}$ is an m -accretive operator.*
- (ii) *the set $C_c^\infty(M)$ is a core for $H_{1, \text{max}}$ (i.e. $\overline{H_{1, \text{min}}} = H_{1, \text{max}}$).*

Theorem 1.3. *Under the same hypotheses as in Theorem 1.2, the following operator equality holds:*

$$H_{1, \text{max}} = A_{1, \text{max}} + V, \tag{1.2}$$

where V is understood as the maximal multiplication operator in $L^1(M)$.

In the next theorem we will use the following notation.

Positivity. Suppose that B and C are bounded linear operators on $L^p(M)$. In what follows, the notation $B \leq C$ means that for all $0 \leq f \in L^p(M)$ we have $(C - B)f \geq 0$.

Theorem 1.4. *Under the same hypotheses as in Theorem 1.2, the following properties hold:*

- (i) $0 \leq (\lambda + H_{1, \text{max}})^{-1}$, for all $\lambda > 0$;
- (ii) $(\lambda + H_{1, \text{max}})^{-1} \leq (\lambda + A_{1, \text{max}})^{-1}$, for all $\lambda > 0$.

Remark 1.5. T. Kato [7, Part A] considered the differential expression $-\Delta + V$ in spaces $L^p(\mathbb{R}^n)$, where $1 \leq p \leq \infty$, the notation Δ denotes the standard Laplacian on \mathbb{R}^n with standard metric and measure and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Assuming $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, Kato [7, Part A] proved the m -accretivity of the operator $H_{p, \text{max}}$ corresponding to $-\Delta + V$. In his proof, Kato used certain properties (specific to the \mathbb{R}^n setting) of $(-\Delta_{2, \text{max}} + \gamma)^{-1}$, where $\gamma > 0$ and $-\Delta_{2, \text{max}}$ is the self-adjoint closure of $-\Delta|_{C_c^\infty(\mathbb{R}^n)}$ in $L^2(\mathbb{R}^n)$, which enabled him to handle the cases $p = 1$ and $p = \infty$. Theorems 1.2 and 1.4 extend the corresponding results of Kato in the case $p = 1$. In the case of operators $H_{p, \text{max}}$ on manifolds of bounded geometry, where $1 < p < \infty$, Theorems 1.2 and 1.4 were proven in [8] using theory of uniformly elliptic differential operators. However, the case $p = 1$ is more delicate and requires a different approach.

2. PRELIMINARY LEMMAS

In what follows, we will use a version of Kato's inequality. For the proof of a more general version of this inequality, see [1, Theorem 5.7].

Lemma 2.1. *Assume that (M, g) is an arbitrary Riemannian manifold. Assume that $u \in L^1_{\text{loc}}(M)$ and $\Delta_M u \in L^1_{\text{loc}}(M)$. Then the following distributional inequality holds:*

$$\Delta_M |u| \leq \text{Re}((\Delta_M u) \text{sign } \bar{u}), \quad (2.1)$$

where

$$\text{sign } u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2. For the original version of Kato's inequality, see Kato [5, Lemma A].

Lemma 2.3. *Let (M, g) be a Riemannian manifold. Assume that $0 \leq V \in L^1_{\text{loc}}(M)$, $u \in \text{Dom}(H_{1, \max})$ and $\lambda \in \mathbb{C}$. Let $f := (H_{1, \max} + \lambda)u$. Then the following distributional inequality holds:*

$$(\text{Re } \lambda + \Delta_M + V)|u| \leq |f|. \quad (2.2)$$

Proof Since $u \in \text{Dom}(H_{1, \max})$ it follows that $Vu \in L^1_{\text{loc}}(M)$ and $H_{1, \max}u \in L^1(M) \subset L^1_{\text{loc}}(M)$. Thus $u \in L^1_{\text{loc}}(M)$ and $\Delta_M u \in L^1_{\text{loc}}(M)$. By Kato's inequality (2.1) we have

$$(\text{Re } \lambda + \Delta_M + V)|u| \leq \text{Re}[(\lambda + \Delta_M + V)u \text{sign } \bar{u}] = \text{Re}(f \text{sign } \bar{u}) \leq |f|,$$

and the lemma is proven. □

In the sequel, we will use a sequence of cut-off functions.

Cut-off functions. Let (M, g) be a manifold of bounded geometry. Then, there exists a sequence of functions $\{\chi_k\}$ in $C_c^\infty(M)$ such that

- (i) $0 \leq \chi_k \leq 1$, for all $k = 1, 2, \dots$;
- (ii) $\chi_k \leq \chi_{k+1}$, for all $k = 1, 2, \dots$;
- (iii) for every compact set $G \subset M$, there exists k such that $\chi_k|_G = 1$;
- (iv) for all $k = 1, 2, \dots$, the following inequalities hold:

$$\sup_{x \in M} |d\chi_k(x)| \leq \tilde{C}, \quad \sup_{x \in M} |\Delta_M \chi_k(x)| \leq \tilde{C}, \quad (2.3)$$

where the constant $\tilde{C} > 0$ does not depend on k , and $|d\chi_k(x)|$ denotes the length of the cotangent vector $d\chi_k(x) \in T_x^*M$.

For the construction of χ_k satisfying the above properties, see [10, Sec. 1.4].

In what follows, we will always assume, unless specified otherwise, that (M, g) is a manifold of bounded geometry.

Lemma 2.4. *Assume that $0 \leq V \in L^1_{\text{loc}}(M)$. Assume that $\lambda \in \mathbb{C}$ and $\gamma := \text{Re } \lambda > 0$. Then the following properties hold:*

(i) for all $u \in \text{Dom}(H_{1,\max})$, we have

$$\gamma \|u\|_1 \leq \|(\lambda + H_{1,\max})u\|_1; \quad (2.4)$$

(ii) the operator $\lambda + H_{1,\max}: \text{Dom}(H_{1,\max}) \subset L^1(M) \rightarrow L^1(M)$ is injective.

Proof We first prove (i). Let $u \in \text{Dom}(H_{1,\max})$ and $f := (\lambda + H_{1,\max})u$. By the definition of $\text{Dom}(H_{1,\max})$, we have $f \in L^1(M)$ and $Vu \in L^1_{\text{loc}}(M)$. Since $V \geq 0$, from (2.2) we get the following distributional inequality:

$$(\gamma + \Delta_M)|u| \leq |f|. \quad (2.5)$$

Thus, for all $0 \leq \psi \in C_c^\infty(M)$, we have

$$\gamma \int_M |u| \psi \, d\mu \leq \int_M |f| \psi \, d\mu - \int_M |u| (\Delta_M \psi) \, d\mu. \quad (2.6)$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined above. Clearly, the functions χ_k satisfy the following properties as $k \rightarrow +\infty$:

$$\chi_k \rightarrow 1 \quad \text{and} \quad \Delta_M \chi_k \rightarrow 0, \quad \text{a.e.} \quad (2.7)$$

Since $f \in L^1(M)$ and $u \in L^1(M)$, using the property (i) of $\{\chi_k\}$, the rightmost inequality in (2.3), the properties (2.7) and dominated convergence theorem, we have

$$\chi_k |u| \rightarrow |u|, \quad \chi_k |f| \rightarrow |f| \quad \text{and} \quad |u| (\Delta_M \chi_k) \rightarrow 0, \quad \text{in } L^1(M). \quad (2.8)$$

Substituting $\psi = \chi_k$ in (2.6), we get

$$\gamma \int_M |u| \chi_k \, d\mu \leq \int_M |f| \chi_k \, d\mu - \int_M |u| (\Delta_M \chi_k) \, d\mu. \quad (2.9)$$

Taking limit as $k \rightarrow +\infty$ in (2.9) and using (2.8), we obtain

$$\gamma \int_M |u| \, d\mu \leq \int_M |f| \, d\mu, \quad (2.10)$$

and (2.4) is proven.

We now prove (ii). Assume that $u \in \text{Dom}(H_{1,\max})$ and $(\lambda + H_{1,\max})u = 0$. Using (2.4), we get $\|u\|_1 = 0$, and hence $u = 0$. This shows that $\lambda + H_{1,\max}$ is injective. \square

Distributional inequality. Let $\lambda > 0$, and consider the following distributional inequality:

$$(\Delta_M + \lambda)u = \nu \geq 0, \quad u \in L^\infty(M), \quad (2.11)$$

where the inequality $\nu \geq 0$ means that ν is a positive distribution, i.e. $\langle \nu, \phi \rangle \geq 0$ for any $0 \leq \phi \in C_c^\infty(M)$.

Lemma 2.5. *Assume that (M, g) is a manifold of bounded geometry. Assume that $u \in L^\infty(M)$ satisfies (2.11). Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).*

For the proof of Lemma 2.5 see Sec. 6 below.

In the sequel, we will adopt certain arguments of Kato [7, Part A] to our setting.

Lemma 2.6. *Assume that $0 \leq V \in L^1_{\text{loc}}(M)$. Then the following properties hold:*

- (i) the operator $H_{1,\max}$ is closed;
- (ii) the operator $\lambda + H_{1,\max}$, where $\operatorname{Re} \lambda > 0$, has a closed range.

Proof We first prove the property (i). Let $u_k \in \operatorname{Dom}(H_{1,\max})$ be a sequence such that, as $k \rightarrow +\infty$,

$$u_k \rightarrow u, \quad f_k := H_{1,\max}u_k = \Delta_M u_k + V u_k \rightarrow f \quad \text{in } L^1(M). \quad (2.12)$$

We need to show that $u \in \operatorname{Dom}(H_{1,\max})$ and $H_{1,\max}u = f$.

By passing to subsequences, we may assume that the convergence in (2.12) is also pointwise almost everywhere.

The distributional inequality (2.2) holds if we replace u by $u_k - u_l$, f by $f_k - f_l$ and λ by 0. With these replacements, we apply a test function $0 \leq \phi \in C_c^\infty(M)$ to (2.2) and get

$$0 \leq \langle V|u_k - u_l|, \phi \rangle \leq \langle |f_k - f_l|, \phi \rangle - \langle \Delta_M |u_k - u_l|, \phi \rangle, \quad (2.13)$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(\mathcal{D}'(M), C_c^\infty(M))$.

Using integration by parts in the second term on the right hand side of the second inequality in (2.13), we get

$$0 \leq \langle V|u_k - u_l|, \phi \rangle \leq \langle |f_k - f_l|, \phi \rangle - \langle |u_k - u_l|, \Delta_M \phi \rangle. \quad (2.14)$$

Letting $k, l \rightarrow +\infty$, the right hand side of the second inequality in (2.14) tends to 0 by (2.12). Thus $V u_k \phi$ is a Cauchy sequence in $L^1(M)$, and its limit must be equal to $V u \phi$. Since $\phi \in C_c^\infty(M)$ may have an arbitrarily large support, it follows that $V u \in L^1_{\text{loc}}(M)$. Thus $V u_k \rightarrow V u$ in $L^1_{\text{loc}}(M)$ and hence in $\mathcal{D}'(M)$. Since $u_k \rightarrow u$ in $L^1(M)$ (and, hence in $L^1_{\text{loc}}(M)$), we get $\Delta_M u_k \rightarrow \Delta_M u$ in $\mathcal{D}'(M)$. Thus, $f_k = \Delta_M u_k + V u_k \rightarrow \Delta_M u + V u$ in $\mathcal{D}'(M)$. Since $f_k \rightarrow f$ in $L^1(M) \subset \mathcal{D}'(M)$, we obtain $\Delta_M u + V u = f \in L^1(M)$. This shows that $u \in \operatorname{Dom}(H_{1,\max})$ and $H_{1,\max}u = f$. This proves that $H_{1,\max}$ is closed.

We now prove the property (ii). Since $H_{1,\max}$ is closed, it immediately follows from (2.4) that $\lambda + H_{1,\max}$ has a closed range for $\operatorname{Re} \lambda > 0$. \square

Lemma 2.7. *Assume that $0 \leq V \in L^1_{\text{loc}}(M)$. Let $\lambda \in \mathbb{C}$ and $\gamma := \operatorname{Re} \lambda > 0$. Then the following properties hold:*

- (i) the operator $\lambda + H_{1,\max}: \operatorname{Dom}(H_{1,\max}) \subset L^1(M) \rightarrow L^1(M)$ is surjective;
- (ii) the operator $(\lambda + H_{1,\max})^{-1}: L^1(M) \rightarrow L^1(M)$ is a bounded linear operator with the operator norm

$$\|(\lambda + H_{1,\max})^{-1}\|_{L^1(M) \rightarrow L^1(M)} \leq 1/\gamma. \quad (2.15)$$

Proof We first prove the property (i). Since $\lambda + H_{1,\max}$ has a closed range by Lemma 2.6, it is enough to show that $(\lambda + H_{1,\min})C_c^\infty(M)$ is dense in $L^1(M)$. Let $v \in (L^1(M))^* = L^\infty(M)$, be a continuous linear functional annihilating $(\lambda + H_{1,\min})C_c^\infty(M)$:

$$\langle (\lambda + H_{1,\min})\phi, v \rangle = 0, \quad \text{for all } \phi \in C_c^\infty(M), \quad (2.16)$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(L^1(M), L^\infty(M))$.

From (2.16) we get the following distributional equality:

$$(\bar{\lambda} + \Delta_M + V)v = 0.$$

Since by hypothesis $V \in L^1_{\text{loc}}(M)$ and since $v \in L^\infty(M)$, by Hölder's inequality we have $Vv \in L^1_{\text{loc}}(M)$. Since $\Delta_M v = -Vv - \bar{\lambda}v$, we get $\Delta_M v \in L^1_{\text{loc}}(M)$. By Kato's inequality and since $V \geq 0$, we have

$$\Delta_M |v| \leq \text{Re}((\Delta_M v) \text{sign } \bar{v}) = \text{Re}((-\bar{\lambda}v - Vv) \text{sign } \bar{v}) \leq -(\text{Re } \bar{\lambda})|v|,$$

and, hence,

$$(\Delta_M + \text{Re } \bar{\lambda})|v| \leq 0.$$

Since $v \in L^\infty(M)$ and since $\text{Re } \bar{\lambda} = \text{Re } \lambda > 0$, by Lemma 2.5 we get $|v| \leq 0$. Thus $v = 0$, and the surjectivity of $\lambda + H_{1,\text{max}}$ is proven.

We now prove the property (ii). Assume that $\lambda \in \mathbb{C}$ satisfies $\gamma := \text{Re } \lambda > 0$. Since $\lambda + H_{1,\text{max}} : \text{Dom}(H_{1,\text{max}}) \subset L^1(M) \rightarrow L^1(M)$ is injective and surjective, the inverse $(\lambda + H_{1,\text{max}})^{-1}$ is defined on the whole $L^1(M)$. The inequality (2.15) follows immediately from (2.4). This concludes the proof of the lemma. \square

3. PROOF OF THEOREM 1.2

We first prove the property (i). By Lemma 2.7 it follows that $(-\infty, 0) \subset \rho(H_{1,\text{max}})$ and

$$\|(\lambda + H_{1,\text{max}})^{-1}\|_{L^1(M) \rightarrow L^1(M)} \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0.$$

Thus, by [9, Theorem X.47(a)] it follows that $H_{1,\text{max}}$ generates a contraction semigroup on $L^1(M)$. In particular, by the remark preceding Theorem X.49 in [9] the operator $H_{1,\text{max}}$ is m -accretive.

We now prove the property (ii). By property (i) of this theorem, the operator $H_{1,\text{max}}$ is m -accretive; hence, $H_{1,\text{min}} = H_{1,\text{max}}|_{C_c^\infty(M)}$ is accretive. By an abstract fact, (see the remark preceding Theorem X.48 in [9]), the operator $H_{1,\text{min}}$ is closable and $\overline{H_{1,\text{min}}}$ is accretive. Let $\lambda > 0$. By the proof of property (i) in Lemma 2.7 it follows that $\text{Ran}(\lambda + H_{1,\text{min}})$ dense in $L^1(M)$. Using (2.4) and the definition of the closure of an operator, it follows that $\text{Ran}(\lambda + \overline{H_{1,\text{min}}}) = L^1(M)$. Now by [9, Theorem X.48] the operator $\overline{H_{1,\text{min}}}$ generates a contraction semigroup on $L^1(M)$. Thus, by the remark preceding Theorem X.49 in [9], the operator $\overline{H_{1,\text{min}}}$ is m -accretive. Since $\overline{H_{1,\text{min}}} \subset H_{1,\text{max}}$ and since $\overline{H_{1,\text{min}}}$ and $H_{1,\text{max}}$ are m -accretive, it follows that $\overline{H_{1,\text{min}}} = H_{1,\text{max}}$. This concludes the proof of the theorem. \square

4. PROOF OF THEOREM 1.3

We begin with the following lemma.

Lemma 4.1. *Assume that $u \in \text{Dom}(H_{1,\text{max}})$. Assume that $\lambda \in \mathbb{C}$ with $\gamma := \text{Re } \lambda \geq 0$. Then*

$$\|(\lambda + \Delta_M)u\|_1 \leq 2\|(\lambda + H_{1,\text{max}})u\|_1, \quad \text{and} \quad \|Vu\|_1 \leq \|(\lambda + H_{1,\text{max}})u\|_1. \quad (4.1)$$

Proof Let $u \in \text{Dom}(H_{1,\text{max}})$ and $f := (\lambda + H_{1,\text{max}})u$. By the definition of $\text{Dom}(H_{1,\text{max}})$, we have $f \in L^1(M)$ and $Vu \in L^1_{\text{loc}}(M)$. By (2.2) we have the following distributional inequality:

$$(\gamma + \Delta_M + V)|u| \leq |f|. \quad (4.2)$$

Since, by assumption, $\gamma \geq 0$, we have for all $0 \leq \psi \in C_c^\infty(M)$:

$$\int_M V|u|\psi \, d\mu \leq \int_M |f|\psi \, d\mu - \int_M |u|(\Delta_M \psi) \, d\mu. \quad (4.3)$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined before Lemma 2.4. Substituting $\psi = \chi_k$ into (4.3), we get

$$\int_M V|u|\chi_k \, d\mu \leq \int_M |f|\chi_k \, d\mu - \int_M |u|(\Delta_M \chi_k) \, d\mu. \quad (4.4)$$

Since $V \geq 0$ and since $Vu \in L_{\text{loc}}^1(M)$, it follows that $V|u|\chi_k$ are non-negative integrable functions. By Fatou's lemma, (2.8) and (4.4), we have

$$\begin{aligned} \int_M V|u| \, d\mu &= \int_M (\liminf_{k \rightarrow +\infty} V|u|\chi_k) \, d\mu \leq \liminf_{k \rightarrow +\infty} \int_M V|u|\chi_k \, d\mu \leq \\ &\leq \liminf_{k \rightarrow +\infty} \left(\int_M \chi_k |f| \, d\mu - \int_M |u|(\Delta_M \chi_k) \, d\mu \right) = \int_M |f| \, d\mu. \end{aligned}$$

This shows that

$$\|Vu\|_1 \leq \|f\|_1 = \|(\lambda + H_{1,\max})u\|_1. \quad (4.5)$$

We now prove the remaining inequality in (4.1). Let $u \in \text{Dom}(H_{1,\max})$ be arbitrary. By (4.5), it follows that $Vu \in L^1(M)$. Since $(\lambda + \Delta_M)u = -Vu + (\lambda + H_{1,\max})u$, from (4.5) and triangle inequality, we obtain

$$\|(\lambda + \Delta_M)u\|_1 \leq 2\|(\lambda + H_{1,\max})u\|_1.$$

This concludes the proof of the lemma. \square

Proof of Theorem 1.3 By definition of $H_{1,\max}$ it follows that $\text{Dom}(A_{1,\max}) \cap \text{Dom}(V) \subset \text{Dom}(H_{1,\max})$. By Lemma 4.1 it follows that $\text{Dom}(H_{1,\max}) \subset \text{Dom}(V)$ and $\text{Dom}(H_{1,\max}) \subset \text{Dom}(A_{1,\max})$. Thus, $\text{Dom}(H_{1,\max}) = \text{Dom}(A_{1,\max}) \cap \text{Dom}(V)$. Now by definitions of $H_{1,\max}$, $A_{1,\max}$ and the multiplication operator V , it follows that $H_{1,\max} = A_{1,\max} + V$. This concludes the proof of the Theorem. \square

5. PROOF OF THEOREM 1.4

Throughout this section we assume that (M, g) is a manifold of bounded geometry. We begin with the following lemma.

Lemma 5.1. *Assume that $0 \leq v \in L^1(M)$ satisfies the following distributional inequality:*

$$(\Delta_M + \lambda)v \leq 0, \quad \text{for some } \lambda > 0. \quad (5.1)$$

Then $v = 0$ almost everywhere on M .

Proof Let $\lambda > 0$ be as in the hypothesis. By (5.1), for all $0 \leq \psi \in C_c^\infty(M)$, we have

$$\lambda \int_M v\psi \, d\mu \leq - \int_M v(\Delta_M \psi) \, d\mu. \quad (5.2)$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined before Lemma 2.4. Substituting $\psi = \chi_k$ in (5.2), we get

$$\lambda \int_M v \chi_k d\mu \leq - \int_M v(\Delta_M \chi_k) d\mu. \quad (5.3)$$

Since $v \in L^1(M)$, using the properties of χ_k , as in the proof of (2.8), we have

$$v \chi_k \rightarrow v \quad \text{and} \quad v(\Delta_M \chi_k) \rightarrow 0, \quad \text{in } L^1(M). \quad (5.4)$$

Taking the limit as $k \rightarrow \infty$ in (5.3) and using the hypothesis $v \geq 0$, we obtain

$$\lambda \|v\|_1 \leq 0.$$

Since $\lambda > 0$, we get $\|v\|_1 = 0$. Hence, $v = 0$ almost everywhere, and the lemma is proven. \square

Lemma 5.2. *Assume that $u \in \text{Dom}(H_{1,\max})$ satisfies $(\lambda + H_{1,\max})u \geq 0$, where $\lambda > 0$. Then $u \geq 0$ almost everywhere on M .*

Proof Let $\lambda > 0$ be as in the hypothesis, and assume that $u \in \text{Dom}(H_{1,\max})$ satisfies

$$f := (H_{1,\max} + \lambda)u \geq 0.$$

We claim that u is real. Indeed, since $(H_{1,\max} + \lambda)\bar{u} = f$, we have $(H_{1,\max} + \lambda)(u - \bar{u}) = 0$. By property (ii) of Lemma 2.4 we have $u = \bar{u}$. Since $f \geq 0$ and $\lambda > 0$, by (2.2) we have

$$(\lambda + \Delta_M + V)|u| \leq f. \quad (5.5)$$

Subtracting $f = (\lambda + H_{1,\max})u$ from both sides of (5.5) we get

$$(\lambda + \Delta_M + V)v \leq 0, \quad \text{where } v := |u| - u \geq 0. \quad (5.6)$$

Since $V \geq 0$, from (5.6) we get the following distributional inequality:

$$(\lambda + \Delta_M)v \leq 0, \quad \text{where } v = |u| - u \geq 0.$$

By Lemma 5.1 we get $v = 0$. Thus, $u = |u| \geq 0$. This concludes the proof. \square

Proof of Theorem 1.4 We first prove the property (i). Let $\lambda > 0$, let $0 \leq f \in L^1(M)$ be arbitrary, and let $u := (H_{1,\max} + \lambda)^{-1}f$. Then $(H_{1,\max} + \lambda)u = f \geq 0$, and, hence, by Lemma 5.2 we have $u \geq 0$. This proves the inequality $0 \leq (H_{1,\max} + \lambda)^{-1}$.

We now prove the property (ii). Let $\lambda > 0$ and let $0 \leq f \in L^1(M)$ be arbitrary. We will show that

$$(H_{1,\max} + \lambda)^{-1}f \leq (A_{1,\max} + \lambda)^{-1}f. \quad (5.7)$$

Define $u := (H_{1,\max} + \lambda)^{-1}f$. By property (i) of this Theorem we have $0 \leq u \in \text{Dom}(H_{1,\max})$, and, hence, by Theorem 1.3 we get $u \in \text{Dom}(A_{1,\max})$. Thus $(\Delta_M + \lambda)u \in L^1(M)$, and, hence, by (4.2) (with $u \geq 0$ and $f \geq 0$) we have the following inequality of functions:

$$(\Delta_M + \lambda)u \leq f, \quad \text{a.e. on } M. \quad (5.8)$$

By property (i) of this Theorem (with $V = 0$) it follows that $(A_{1,\max} + \lambda)^{-1} \geq 0$ as an operator $L^1(M) \rightarrow L^1(M)$. Thus, from (5.8) we get

$$u \leq (A_{1,\max} + \lambda)^{-1}f.$$

But $u = (H_{1,\max} + \lambda)^{-1}f$, and (5.7) is proven. This concludes the proof of property (ii). \square

6. PROOF OF LEMMA 2.5

In the sequel, we will use the following terms and notations. Unless specified otherwise, (M, g) is an arbitrary Riemannian manifold (not necessarily complete).

Sobolev space $W^{1,2}(M)$. By $W^{1,2}(M)$ we will denote the completion of the space $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_{W^{1,2}}$ defined by the scalar product

$$(u, v)_{W^{1,2}} := (u, v)_{L^2(M)} + (du, dv)_{L^2(\Lambda^1 T^*M)}, \quad u, v \in C_c^\infty(M).$$

Remark 6.1. If (M, g) is a complete Riemannian manifold, then by [4, Proposition 1.4] it follows that $W^{1,2}(M) = \{u \in L^2(M) : du \in L^2(\Lambda^1 T^*M)\}$.

In what follows, we will closely follow E. B. Davies [2] and [3, Sec. 1.3, 1.4, and 5.2].

Semigroups $T_p(t)$. Let $A_{2,\min}$ and $A_{2,\max}$ be as in Sec. 1. It is well known that, for a complete Riemannian manifold (M, g) , the operator $A_{2,\min}$ is essentially self-adjoint in $L^2(M)$, and $A_{2,\max} = \overline{A_{2,\min}}$; see, for example, [4, Theorem 3.5]. Moreover, by [6, Sec. VI.2.3], it follows that $A_{2,\max}$ (as the Friedrichs extension of $A_{2,\min}$) is the self-adjoint operator associated to the closure \bar{h} in $L^2(M)$ of the quadratic form

$$h(u) := \int_M |du|^2 d\mu, \quad u \in C_c^\infty(M).$$

Thus, for a complete Riemannian manifold (M, g) , the operator $A_{2,\max}$ generates a strongly continuous contraction semigroup $e^{-tA_{2,\max}}$, $t \geq 0$, on $L^2(M)$; see, for instance, [9, Sec. X.8, Example 1]. It is well known that the semigroup $e^{-tA_{2,\max}}$ is positivity preserving; see, for instance, the proof of [11, Theorem 3.6]. Moreover, for every $0 \leq f \in \text{Dom}(\bar{h}) = W^{1,2}(M)$ we have $g := \min\{f, 1\} \in \text{Dom}(\bar{h})$, and

$$\int_M |dg|^2 d\mu \leq \int_M |df|^2 d\mu.$$

Hence, the semigroup $e^{-tA_{2,\max}}$ satisfies the conditions of [3, Theorem 1.3.2] and [3, Theorem 1.3.3]. Thus, by [3, Theorem 1.4.1] it follows that the semigroup $e^{-tA_{2,\max}}$ can be extended from $L^1(M) \cap L^\infty(M)$ to a contraction semigroup $T_p(t)$, $t \geq 0$, on $L^p(M)$ for all $1 \leq p \leq +\infty$. Moreover, by [3, Theorem 1.4.1], the semigroup $T_p(t)$ is strongly continuous for $1 \leq p < +\infty$. By A_p we will denote the generator of $T_p(t)$. The operator A_p is an extension of $\Delta_M|_{C_c^\infty(M)}$ in the corresponding space $L^p(M)$; see [2, Sec. 1]. By [9, Theorem X.47(a)] it follows that $(-\infty, 0) \subset \rho(A_p)$, where $\rho(A_p)$ denotes the resolvent set of A_p , and

$$\|(A_p + \lambda)^{-1}\| \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0, \tag{6.1}$$

where $\|\cdot\|$ denotes the operator norm of the bounded linear operator $(A_p + \lambda)^{-1}: L^p(M) \rightarrow L^p(M)$. Since the semigroup $T_\infty(t)$ on $L^\infty(M)$ is not generally strongly continuous, its generator

A_∞ can be defined by

$$(A_\infty + \lambda)^{-1} = ((A_1 + \lambda)^{-1})^*, \quad \text{for all } \lambda > 0,$$

but A_∞ is not necessarily densely defined; see the remark above the formulation of Theorem 1.4.2 in [3].

Semigroup $S(t)$. As in [2, Sec. 1], we denote by $S(t)$ the positivity preserving semigroup on $L^1(M) + L^\infty(M)$ which coincides with $T_p(t)$ on $L^p(M)$ for all $1 \leq p \leq \infty$. For an arbitrary Riemannian manifold (M, g) , it is well known (see [2, Proposition 1.1]) that there exists a strictly positive C^∞ kernel K on $(0, \infty) \times M \times M$ such that

$$(S(t)f)(x) = \int_M K(t, x, y)f(y) d\mu(y), \quad \text{for all } f \in L^1(M) + L^\infty(M) \quad \text{and all } t > 0.$$

As in [2, Sec. 1], for $\lambda > 0$, by R_λ we will denote the positivity preserving operator on $L^1(M) + L^\infty(M)$ which coincides with $(A_p + \lambda)^{-1}$ on $L^p(M)$ for all $1 \leq p \leq +\infty$. By [2, eqn. (1.2)] we have

$$R_\lambda f = \int_0^{+\infty} e^{-\lambda t} S(t)f dt, \quad \text{for all } f \in L^p(M), \lambda > 0, \quad (6.2)$$

where the equation is interpreted in the strong sense for $1 \leq p < \infty$ and in the weak* sense for $p = \infty$.

We begin with the following lemma.

Lemma 6.2. *Assume that (M, g) is a Riemannian manifold (not necessarily complete). Assume that $0 \leq f \in L^\infty(M)$. Assume that $0 \leq h \in L^\infty(M)$ satisfies the following distributional inequality:*

$$(\lambda + \Delta_M)h \geq f, \quad \text{for some } \lambda > 0.$$

Let R_λ be as in (6.2) above. Then $h \geq R_\lambda f$ almost everywhere on M .

Remark 6.3. Lemma 6.2 is essentially the same as [3, Lemma 5.2.4] (or [2, Lemma 2.3]). The only difference is in that [3, Lemma 5.2.4] assumes that $0 \leq h$ is a continuous function on M , and concludes that $h \geq R_\lambda f$ everywhere. The proof of Lemma 6.2, which we give below, is the same as the proof of [3, Lemma 5.2.4].

Proof of Lemma 6.2 Let $\lambda > 0$ be as in the hypothesis. Let U_k be an increasing sequence of relatively compact open subsets of M with smooth boundaries and union equal to M . Let K_k be the self-adjoint operators on $L^2(U_k)$ given by $K_k = \Delta_M$ with Dirichlet boundary conditions. By the proof of [3, Lemma 5.2.4], we have $K_k \downarrow A_2$ in the sense of quadratic forms, where A_2 is as in (6.1). Thus, by abstract Theorem 1.2.3 in [3], we have

$$(K_k + \lambda)^{-1} \uparrow (A_2 + \lambda)^{-1}, \quad \text{as } k \rightarrow \infty,$$

in the strong operator topology.

Let χ_{U_k} denote the characteristic function of the set U_k . Define

$$g_k := (K_k + \lambda)^{-1}(f\chi_{U_k}).$$

By the definition of g_k we have

$$(\lambda + \Delta_M)g_k = f \quad \text{on } U_k, \quad \text{and} \quad g_k = 0 \quad \text{on } \partial U_k. \quad (6.3)$$

By hypotheses and by (6.3) we get

$$(\lambda + \Delta_M)(h - g_k) \geq 0 \quad \text{on } U_k, \quad \text{with} \quad (h - g_k) \geq 0 \quad \text{on } \partial U_k.$$

The maximum principle implies that $h \geq g_k$ almost everywhere on U_k .

If $j \leq k$ we obtain

$$h \geq (K_k + \lambda)^{-1}(f\chi_{U_k}) \geq (K_k + \lambda)^{-1}(f\chi_{U_j}). \quad (6.4)$$

Letting $k \rightarrow \infty$ in (6.4), we get

$$h \geq (A_2 + \lambda)^{-1}(f\chi_{U_j}) = R_\lambda(f\chi_{U_j}).$$

Finally, letting $j \rightarrow \infty$, we obtain

$$h \geq R_\lambda f, \quad \text{a.e. on } M,$$

and the lemma is proven. \square

Proof of Lemma 2.5 Let $\lambda > 0$ and $v \in L^\infty(M)$ be as in the hypothesis. By normalization, may assume that $\|v\|_\infty = \lambda^{-1}$. Define $h := \lambda^{-1} + v$. Then $h \in L^\infty(M)$ and $h \geq 0$.

By hypothesis we know that

$$\langle (\lambda + \Delta_M)v, \phi \rangle \geq 0, \quad \text{for all } 0 \leq \phi \in C_c^\infty(M).$$

Thus, for all $0 \leq \phi \in C_c^\infty(M)$ we have

$$\langle (\lambda + \Delta_M)h, \phi \rangle = \langle (\lambda + \Delta_M)\lambda^{-1}, \phi \rangle + \langle (\lambda + \Delta_M)v, \phi \rangle = \langle 1, \phi \rangle + \langle (\lambda + \Delta_M)v, \phi \rangle \geq \langle 1, \phi \rangle.$$

Hence, we get the following distributional inequality:

$$(\lambda + \Delta_M)h \geq 1.$$

Define $f := 1$. Since (M, g) has bounded geometry, by [3, Theorem 5.2.6] it follows that $R_\lambda 1 = \lambda^{-1}$.

By Lemma 6.2 with $f = 1$, it follows that $h \geq \lambda^{-1}$ almost everywhere, i.e.

$$\lambda^{-1} + v \geq \lambda^{-1} \quad \text{a.e. on } M.$$

Therefore, $v \geq 0$ a.e. on M , and the lemma is proven. \square

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