MAC 2313

VECTORS, CONIC SECTIONS, DOT PRODUCT

I. Vectors. In this section, you will use triangle and/or parallelogram law for adding vectors together with basic properties of addition and scalar multiplication. **Do not write** vectors in component form: treat them as arrows starting at a point and ending at a point. In some exercises, we will use $|PQ|$ to denote the length of a line segment PQ.

Exercise 1. In a parallelogram $ABCD$, let P and Q be the midpoints of sides CD and BC respectively. Express the vector $\overrightarrow{AB}$ in terms of $\overrightarrow{AP}$ and $\overrightarrow{AQ}$.

Answer: $\overrightarrow{AB} = \frac{4}{3}\overrightarrow{AQ} - \frac{2}{3}\overrightarrow{AP}$.

Exercise 2. Let ABC be an arbitrary triangle. On the sides AB, BC and CA of the triangle we construct parallelograms $ABDE$, $BCFG$ and $CAHI$. Show that

$$\overrightarrow{DG} + \overrightarrow{FI} + \overrightarrow{HE} = \vec{0}.$$

Exercise 3. Let ABC be an arbitrary triangle. The segment AA_1, where A_1 is the midpoint of BC, is called a *median* of ABC onto the side BC. Similarly, BB_1 and CC_1, where B_1 and C_1 are the midpoints of CA and AB respectively, are the remaining two medians of ABC. It is a well known fact in geometry that the three medians of ABC intersect at a common point T, called the *centroid* of ABC. Show that

$$\overrightarrow{AT} + \overrightarrow{BT} + \overrightarrow{CT} = \vec{0}.$$

Hint: Use the following geometric fact: $|AT| \div |TA_1| = 2 \div 1$, i.e. $|AT| = \frac{2}{3}|AA_1|$ and $|TA_1| = \frac{1}{3}|AA_1|$. The same fact holds for the remaining medians. Additionally, vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ may play a role in your proof.

Exercise 4. Let $A_1B_1C_1$ and $A_2B_2C_2$ be any two triangles in the plane with centroids T_1 and T_2 respectively. Show that

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{T_1T_2}.$$

Hint: Let O be a point in the plane. Express $\overrightarrow{A_1A_2}$, $\overrightarrow{B_1B_2}$, $\overrightarrow{C_1C_2}$ in terms of $\overrightarrow{OA_j}$, $\overrightarrow{OB_j}$, $\overrightarrow{OC_j}$ (with $j = 1, 2$) respectively. Having done this, try to bring $\overrightarrow{OT_j}$, $\overrightarrow{A_jT_j}$, $\overrightarrow{B_jT_j}$ and $\overrightarrow{C_jT_j}$ (with $j = 1, 2$) into the picture. At some point in your solution, you may want to refer to the result of Exercise 3.

Exercise 5. Prove the following statement: Triangles ABC and PQR have a common centroid if and only if $\overrightarrow{AP}+\overrightarrow{BQ}+\overrightarrow{CR}=\vec{0}$.

Hint: Feel free to cite the result of Exercise 4.

Example 1. Let ABC be an arbitrary triangle. Let M be the point on the side BC such that $|BM|\div|MC|=3\div 1$, and let N be the point on the segment AM such that $|AN|\div|NM|=2\div 3$. Let P be the intersection point of the line through B and N with the side AC. Determine the ratio $|AP|\div|PC|$.

Solution:

Note that

$$\overrightarrow{AM}=\frac{1}{4}\overrightarrow{AB}+\frac{3}{4}\overrightarrow{AC} \text{ and } \overrightarrow{AN}=\frac{2}{5}\overrightarrow{AM}=\frac{1}{10}\overrightarrow{AB}+\frac{3}{10}\overrightarrow{AC}.$$

Therefore,

$$\overrightarrow{BN}=-\overrightarrow{AB}+\overrightarrow{AN}=-\frac{9}{10}\overrightarrow{AB}+\frac{3}{10}\overrightarrow{AC}.$$

We now express $\overrightarrow{BP}$ in two ways. Since P is on the side AC, it follows that $\overrightarrow{AP}=\alpha\overrightarrow{AC}$, for some real number α. Hence,

$$\overrightarrow{BP}=\overrightarrow{BA}+\overrightarrow{AP}=-\overrightarrow{AB}+\alpha\overrightarrow{AC}.$$

Additionally, we have $\overrightarrow{BP}=\beta\overrightarrow{BN}=-\frac{9}{10}\beta\overrightarrow{AB}+\frac{3}{10}\beta\overrightarrow{AC}$, for some real number β.

Thus,

$$-\overrightarrow{AB}+\alpha\overrightarrow{AC}=-\frac{9}{10}\beta\overrightarrow{AB}+\frac{3}{10}\beta\overrightarrow{AC}.$$

Finally, we get

$$\left(\frac{9}{10}\beta-1\right)\overrightarrow{AB}=\left(\frac{3}{10}\beta-\alpha\right)\overrightarrow{AC}.$$

Suppose that $(\frac{9}{10}\beta-1)\neq 0$. Dividing both sides by $\frac{9}{10}\beta-1$, we get

$$\overrightarrow{AB}=\left(\frac{\frac{3}{10}\beta-\alpha}{\frac{9}{10}\beta-1}\right)\overrightarrow{AC}$$

From the last equality, we see that $\frac{3}{10}\beta-\alpha$ cannot equal 0 since $\overrightarrow{AB}\neq\vec{0}$. So $\overrightarrow{AB}$ is a non-zero multiple of $\overrightarrow{AC}$, and, hence, $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$. But this cannot be true since AB and AC are sides of a triangle. Hence, our assumption $(\frac{9}{10}\beta-1)\neq 0$ leads to a contradiction.

Similarly, if we assume $\frac{3}{10}\beta-\alpha\neq 0$, we get another contradiction.

Hence, we must have $\frac{9}{10}\beta - 1 = 0$ **and** $\frac{3}{10}\beta - \alpha = 0$. Thus, $\beta = \frac{10}{9}$ and $\alpha = \frac{1}{3}$, so that $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AC}$. This means that $|AP| \div |PC| = 1 \div 2$. (The two-case argument can be shortened by using the concept of *linear independence of vectors* that you will see in Linear Algebra.)

Exercise 6. Let ABC be an arbitrary triangle. Let P be the point on the side BC such that $|BP| \div |PC| = 3 \div 1$, and let Q be the point on the segment AP such that $|AQ| \div |QP| = 2 \div 3$. Let R be the intersection point of the line through C and Q with the side AB. Determine the ratio $|AR| \div |RB|$.

Hint: Use the method of Example 1. The final answer is $|AR| \div |RB| = 1 \div 6$.

II. Conic Sections.

Exercise 7. An equation of the circle centered at (p, q) with radius r is $(x-p)^2 + (y-q)^2 = r^2$. Consider an arbitrary straight line $y = kx + n$. Show that the line is tangent to the circle if and only if

$$r^2(k^2 + 1) = (q - kp - n)^2.$$

Hint: Start by plugging $y = kx + n$ into the equation of the circle. You will get a quadratic equation in x. You are interested in the case when there is exactly one solution to the quadratic equation. Recall that this happens if and only if the discriminant of the equation is zero.

Exercise 8. The ellipse centered at the origin with (horizontal) major semi-axis a and minor semi-axis b has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Consider an arbitrary straight line $y = kx + n$. Show that the line is tangent to the ellipse if and only if

$$a^2k^2 + b^2 = n^2$$

Hint: Do the same as in Exercise 7.

Exercise 9. This exercise is about quadratic equations in general and is useful in the example below. Consider a quadratic equation

$$Ax^2 + Bx + C = 0,$$

and let x_1 and x_2 denote the two solutions of the equation. Using quadratic formula for the solutions, show that

$$x_1 + x_2 = \frac{-B}{A} \qquad \text{and} \qquad x_1 \cdot x_2 = \frac{C}{A}.$$

Example 2. Consider the ellipse $\frac{x^2}{12} + \frac{y^2}{5} = 1$ and the line $x = 4$. Find all points P on the line $x = 4$ with the following property: the two tangent lines from P to the ellipse intersect at the $90°$ angle.

Solution: Since P is on $x = 4$, it follows that P has coordinates $(4, y_0)$, where y_0 is a real number. Let $y = kx + n$ be a tangent from $P(4, y_0)$ to the ellipse. By Exercise 8, the following two conditions must hold:

$$y_0 = 4k + n \qquad \text{and} \qquad 12k^2 + 5 = n^2.$$

Plugging $n = y_0 - 4k$ into $12k^2 + 5 = n^2$, we get

$$4k^2 - 8y_0k + y_0^2 - 5 = 0.$$

This is a quadratic equation in k, and its solutions are slopes of tangent lines from $P(4, y_0)$ to the ellipse. Let's denote the solutions by k_1 and k_2. We want the tangents to be perpendicular; in other words, we want $k_1 \cdot k_2 = -1$. But by Exercise 9 we have

$$k_1 \cdot k_2 = \frac{y_0^2 - 5}{4},$$

and this means that

$$\frac{y_0^2 - 5}{4} = -1.$$

Solving the last equation, we obtain $y_0 = \pm 1$, and, finally, we get the points P with coordinates $(4, 1)$ and $(4, -1)$.

Exercise 10. Consider the ellipse $\frac{x^2}{8} + \frac{y^2}{5} = 1$ and the line $y = 3$. Find all points P on the line $y = 3$ with the following property: the two tangent lines from P to the ellipse intersect at the $90°$ angle.

Hint: Use the method of Example 2. The answer is points $(2, 3)$ and $(-2, 3)$.

Exercise 11. Consider a parabola $y^2 = 4px$ with focus F. Let A be any point on the parabola, let t be the tangent to the parabola at A, and let B be the intersection point of the line t with the directrix of the parabola. Prove that the angle between the line segments FA and FB has $90°$ measure.

Hint: Let A have coordinates (x_0, y_0). Then, you see that $y_0^2 = 4px_0$. Next, use calculus (implicit differentiation) to write the equation of the tangent line t to the parabola at (x_0, y_0). Find the intersection point, say B, of t with the directrix. Show that the slopes m_{FA} and m_{FB} of the segments FA and FB satisfy the equality $m_{FA} \cdot m_{FB} = -1$.

III. Dot Product.

Exercise 12. Let $\vec{a}$ and $\vec{b}$ be two vectors such that

$$\vec{a} \perp (2\vec{a} + \vec{b}) \qquad \text{and} \qquad (2\vec{a} - \vec{b}) \perp (3\vec{a} + \vec{b}).$$

Find the angle θ between $\vec{a}$ and $\vec{b}$.

Hint: Write the given conditions in the "dot product" form. Next, get a link between $\|\vec{a}\|$ and $\|\vec{b}\|$. Then, use the formula for the angle between two vectors. You should get $\theta = 135°$.

Exercise 13. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors such that

$$\vec{c} = 2\vec{a} - \vec{b}, \|\vec{a}\| = 1, \|\vec{b}\| = 2, \text{and } \|\vec{c}\| = 3.$$

Find the value of $(2\vec{a} - \vec{c}) \cdot \vec{b} + 2\vec{a} \cdot \vec{c}$.

Hint: One way to do it is to observe that $2\vec{a} - \vec{c} = \vec{b}$. Thus,

$$(2\vec{a} - \vec{c}) \cdot \vec{b} + 2\vec{a} \cdot \vec{c} = \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{c}.$$

To get $\vec{a} \cdot \vec{c}$, start with $\vec{c} = 2\vec{a} - \vec{b}$, and "dot" both sides of the equality with $\vec{a}$, $\vec{b}$, and $\vec{c}$ separately, so that you get three new equalities. From those equalities, you can deduce that $\vec{a} \cdot \vec{c} = \frac{\|\vec{c}\|^2 - \|\vec{b}\|^2 + 4\|\vec{a}\|^2}{6}$. Finally, you will get $(2\vec{a} - \vec{c}) \cdot \vec{b} + 2\vec{a} \cdot \vec{c} = 7$.

Exercise 14. Let $\vec{a}$ and $\vec{b}$ be two vectors such that

$$\|\vec{a} - \vec{b}\| = 5, \|2\vec{a} + \vec{b}\| = 5, \text{ and } \|\vec{a} + 2\vec{b}\| = \sqrt{10}.$$

Find the value of $\vec{a} \cdot \vec{b}$.

Hint: You can bring $\vec{a} \cdot \vec{b}$ into the picture by squaring each inequality. You'll get lots of terms, but you can use elimination to find $\vec{a} \cdot \vec{b} = -5$.

Exercise 15. Let ABC be a triangle such that the medians AA_1 and BB_1 are perpendicular. Show that

$$\|\overrightarrow{BC}\|^2 + \|\overrightarrow{CA}\|^2 = 5\|\overrightarrow{BA}\|^2.$$

Hint: Express $\overrightarrow{AA_1}$ and $\overrightarrow{BB_1}$ in terms of $\overrightarrow{BC}$ and $\|\overrightarrow{CA}\|$. Then, use (i) the perpendicularity condition for medians and (ii) $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ to finish the proof.

Exercise 16. Let ABC be a triangle, and let O be a point on the side AB such that $|OA| = |OB| = |OC|$. Show that the angle of the triangle at C has $90°$ measure.

Hint: Use vectors to prove that $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$.