MAC 2313
LINES AND PLANES

Exercise 1. You are given vectors $\vec{a} = \langle 1, -2, 2\rangle$ and $\vec{b} = \langle 2, 0, -3\rangle$. For what value of t does the vector $\vec{a} + t\vec{b}$ lie in the plane determined by the vectors $\vec{p} = \langle 1, 1, 1\rangle$ and $\vec{q} = \langle 1, 2, 3\rangle$?

Hint: Try to use triple scalar product. The final answer is $t = 7$.

Exercise 2. Use properties of the cross product and triple scalar product to simplify the following expressions:

$$\text{(i)}\ \ (\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{a}) \qquad \text{(ii)}\ \ (\vec{a} \times \vec{b}) \cdot \vec{a}$$

Exercise 3. You are given the points $A(1, 5, -1)$, $B(3, 2, -2)$ and $C(1, 3, a)$, where a is a real number. Let l be the line determined by A and the vector $\vec{s} = \langle 1, -1, 3\rangle$. Find the value of a so that the plane determined by A, B and C contains the line l.

Answer: $a = -15$.

Exercise 4. Consider the line

$$l: \quad \frac{x-4}{3} = \frac{y+3}{-4} = \frac{z-3}{2}.$$

Find an equation of the line that passes through the point $(3, 2, 0)$ and is perpendicular to l.

Hint: First, find an equation of the plane containing the point $(3, 2, 0)$ and is perpendicular to l. Use this plane to get the second point on the line perpendicular to l. The final answer is $\frac{x-3}{4} = \frac{y-2}{-9} = \frac{z-0}{5}$.

Exercise 5. Let l_1 be defined as the line of intersection of the planes $y = 3$ and $2x + y - z = 6$. Additionally, let the line l_2 be defined as

$$l_2: \quad \frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{1}.$$

Show that the lines l_1 and l_2 belong to the same plane.

Hint: You need to show that l_1 and l_2 intersect at some point or that l_1 and l_2 are parallel. In other words, you need to show that l_1 and l_2 are **not skew lines**.

Exercise 6. You are given the planes

$$\pi_1 : \ 2x - y + z = 7, \qquad \pi_2 : \ x - y = 4.$$

Let m be the line defined as the intersection of the planes π_1 and π_2. You are also given the line

$$l : \quad \frac{x-2}{0} = \frac{y+1}{1} = \frac{z+1}{4}.$$

Let T_1 and T_2 be the points of intersection of l with the planes π_1 and π_2 respectively. Find the projections of T_1 and T_2 onto m.

Hint: First, you need to find T_1 and T_2. You should get $T_1(2, 0, 3)$ and $T_2(2, -2, 5)$. Then, you need to find the line m, and you will get $m : \frac{x-4}{1} = \frac{y-0}{1} = \frac{z+1}{-1}$. The projection of T_1 onto m is the intersection of m and the plane through T_1 perpendicular to m. The same holds for T_2. The final answer is as follows: the projections of T_1 and T_2 onto m are $(2, -2, 1)$ and $(4, 0, -1)$ respectively.

Exercise 7. Consider the plane $\pi : x - 3y + 2z + 5 = 0$ and the sphere of radius 12 centered at $(5, -14, 9)$.

(i) Find the point P on π with the smallest distance from the sphere.

(ii) How far is P from the center of the sphere?

(iii) How far is P from the sphere?

Answer: (i) $P(0, 1, -1)$; (ii) $5\sqrt{14}$; (iii) $5\sqrt{14} - 12$.

Exercise 8. Let S be the sphere with the following properties: the radius of S is 3, the sphere S touches the plane $6x + 3y + 6z = 9$ at the point $(1, -1, 1)$, and S is located in the half-space $6x + 3y + 6z \geq 9$.

(i) Write an equation of the sphere S.

(ii) Find all intersection points of the sphere S with the plane $y = 3$.

Answer: (i) $(x - 3)^2 + y^2 + (z - 3)^2 = 9$; (ii) point $(3, 3, 3)$.