MAC 2313

LIMITS AND CONTINUITY

I. LIMITS. In this section, we will keep in mind two limit definitions:

(i) $\epsilon - \delta$ definition from the book, and

(ii) "sequence" definition:

Suppose that $f(x, y)$ is defined in a neighborhood of (x_0, y_0), except possibly at (x_0, y_0). We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for every sequence (x_n, y_n) of points (in the domain of f) such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to L$, as $n \to \infty$.

It is sometimes convenient to use (ii) to show that the limit of f at (x_0, y_0) does not exist. You can do this, for instance, by finding two sequences (x_n, y_n) and (a_n, b_n) of points (in the domain of f) such that $(x_n, y_n) \to (x_0, y_0)$ and $(a_n, b_n) \to (x_0, y_0)$, but $f(x_n, y_n)$ and $f(a_n, b_n)$ converge to different values. Here is an example.

Example 1. Does the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} \tag{1}$$

exist?

Solution: Consider sequences $(\frac{1}{n}, \frac{1}{n})$ and $(\frac{1}{n}, \frac{2}{n})$. Both sequences clearly converge to $(0, 0)$. However,

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{2}{n^2}}{\frac{2}{n^2}} \to 1 \quad \text{and} \quad f\left(\frac{1}{n}, \frac{2}{n}\right) = \frac{\frac{4}{n^2}}{\frac{5}{n^2}} \to \frac{4}{5}.$$

Hence, the limit (1) does not exist.

Another way to see this is to consider the limits of f as $(x, y) \to (0, 0)$ along two straight lines (or curves) and show that the results are different. For instance, finding the limits along $y = x$ and then along $y = 2x$, you get the answers 1 and $\frac{4}{5}$ respectively. Different answers mean that the limit (1) does not exist.

Exercise 1. Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x - y)^2}$$

Exercise 2. Find the following limit:

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2-xy+y^2}.$$

Outline: We will use the "squeeze theorem," which works the same way as in Calculus 1.

Our goal is to make the following estimate (which works for all $x \neq 0$ and $y \neq 0$):

$$0 \leq \left|\frac{x+y}{x^2-xy+y^2}\right| \leq \frac{1}{|y|}+\frac{1}{|x|}. \tag{2}$$

Clearly, the rightmost side of (2) goes to 0 as $(x,y) \to (+\infty,+\infty)$, and the leftmost side of (2) is 0. Hence, by "squeeze theorem" the middle term must go to 0.

Let's make the estimate (2). The first inequality in (2) is easy: absolute value must be greater than or equal to zero. To prove the second inequality, we begin by showing that for all numbers x and y we have

$$x^2-xy+y^2 \geq xy.$$

Then, explain why it follows that

$$|x^2-xy+y^2| \geq |xy|.$$

Thus, for all x and y (with $x \neq 0$ and $y \neq 0$) we get

$$0 \leq \left|\frac{x+y}{x^2-xy+y^2}\right| \leq \left|\frac{x+y}{xy}\right| = \left|\frac{1}{x}+\frac{1}{y}\right| \leq \frac{1}{|x|}+\frac{1}{|y|}.$$

Explain why these inequalities are true. (For instance, notice that the last inequality follows from the celebrated triangle inequality for numbers: $|u+v| \leq |u|+|v|$).

Exercise 3. Show that

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{x^2+y^2}{x^4+y^4} = 0.$$

Hint: Using the kind of reasoning as in Exercise 2, show that for all $x \neq 0$ and $y \neq 0$ we have:

$$0 < \frac{x^2+y^2}{x^4+y^4} \leq \frac{1}{x^2}+\frac{1}{y^2}.$$

Then use the "squeeze theorem."

Exercise 4. Show that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = 1.$$

Outline: First, since we are considering the limit as $(x, y) \to (0, 0)$, it is enough to consider just those points that are sufficiently close to $(0, 0)$, say points $0 < x^2 + y^2 < 1$.

Our goal will be to make the following estimate for $0 < x^2 + y^2 < 1$:

$$(x^2 + y^2)^{\frac{(x^2+y^2)^2}{4}} \le (x^2 + y^2)^{x^2y^2} \le 1. \tag{3}$$

Now, to explain (3), we first show that for any x and y we have:

$$x^2y^2 \le \frac{1}{4}(x^2 + y^2)^2,$$

and, then, to finish the explanation, we use the fact that we are considering only $0 < x^2 + y^2 < 1$.

Having done this, we try to use the squeeze theorem. For this, we need to show that

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{\frac{(x^2+y^2)^2}{4}} = 1.$$

This does not look like an easy task. However, we can make it easier by passing to polar coordinates. Indeed, this will be very nice as our function depends only on the radius $r = \sqrt{x^2 + y^2}$:

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{\frac{(x^2+y^2)^2}{4}} = \lim_{r\to 0+} (r^2)^{\frac{r^4}{4}}.$$

Now use the "0^0" case of L'Hopital's rule, and you will get 1.

II. CONTINUITY

Example 2. Briefly explain why the function

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

is continuous at all points $(x, y) \ne (0, 0)$. Why is f discontinuous at $(0, 0)$?

Solution: As a quotient of two polynomial functions, f is continuous at all points (x, y) where the denominator is not equal to 0, i.e. for all $(x, y) \ne (0, 0)$.

Recall from Example 1 that the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$$

does not exist. Hence, f is not continuous at $(0, 0)$.

Exercise 5. Briefly explain why the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous at all points $(x, y) \neq (0, 0)$. Why is f discontinuous at $(0, 0)$?

Exercise 6. (i) Show that the function

$$f(x,y) = \begin{cases} \frac{x^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0). \end{cases}$$

is continuous at all points $(x, y) \neq (0, 0)$.

(ii) Redefine the function f at $(0, 0)$ so that the new function is continuous at $(0, 0)$.

Outline: As in previous exercises, explain why f must be continuous for all $(x, y) \neq (0, 0)$.

Then, consider the point $(0, 0)$. Clearly, f is defined at $(0, 0)$ with the value $f(0, 0) = 2$. Unlike in Example 2 above, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2}$$

exists and is equal to 0. Explain why this is true. For instance, you may pass to polar coordinates or use "squeeze theorem."

This, however, does not mean that f is continuous at $(0, 0)$. In fact, from the definition of f we know that $f(0, 0) = 2$, and we have just found out the value of the limit is 0. Hence, f is not continuous at $(0, 0)$.

So, how can we redefine f at $(0, 0)$ so that the new function is continuous at $(0, 0)$?

The kind of discontinuity described in this example is known as a "removable discontinuity."

Exercise 7. (i) Show that the following function is defined for all $(x, y) \neq (0, 1)$:

$$f(x,y) = \frac{x(y-1)(x^2-y^2+2y-1)}{x^2+y^2-2y+1}.$$

(ii) Show that

$$\lim_{(x,y)\to(0,1)} \frac{x(y-1)(x^2-y^2+2y-1)}{x^2+y^2-2y+1} = 0$$

(iii) Extend the definition of f so that the new function is continuous at $(0, 1)$.

Hint: Show that f can be written as

$$f(x,y) = \frac{x(y-1)(x^2-(y-1)^2)}{x^2+(y-1)^2}.$$

This will answer (i).

To answer (ii), make a polar-type change of coordinates $x = r\cos\theta$ and $y - 1 = r\sin\theta$. Explain why $(x, y) \to (0, 1)$ means that $r \to 0+$. Compute the limit in new coordinates. Now you can answer (iii).

Exercise 8. (i) Show that the function

$$f(x,y) = \frac{x+y}{x^3+y^3}$$

is defined and continuous at all points (x, y) such that $x + y \neq 0$.

(ii) Consider the points (x_0, y_0) such that $(x_0, y_0) \neq (0, 0)$ and $x_0 + y_0 = 0$. Show that at these points,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{x+y}{x^3+y^3} = \frac{1}{x_0^2 - x_0y_0 + y_0^2}.$$

Explain why the quantity on the right hand side is finite.

(iii) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^3+y^3} = +\infty.$$

(iv) Explain why discontinuities in (ii) are removable, and why the one in (iii) is not removable.

Example 3. Using ϵ-δ definition show that $f(x, y) = x + y$ is continuous at all points (x_0, y_0) in $\mathbb{R}^2$.

Solution: Consider a point (x_0, y_0) in $\mathbb{R}^2$. Fix an arbitrary $\epsilon > 0$. Our goal is to find $\delta > 0$ such that for all points (x, y) with

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta, \tag{4}$$

we have

$$|f(x,y) - f(x_0,y_0)| = |(x+y) - (x_0+y_0)| < \epsilon.$$

To find a suitable δ, we make the following estimates:

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &= |(x+y) - (x_0+y_0)| = |(x-x_0) + (y-y_0)| \\ &\leq |x-x_0| + |y-y_0| \\ &\leq (\sqrt{2})\sqrt{(x-x_0)^2+(y-y_0)^2}. \qquad (5) \end{aligned}$$

The last two inequalities in (5) follow from the triangle inequality

$$|A+B| \leq |A|+|B|$$

and the inequality

$$|A|+|B| \leq \sqrt{2(A^2+B^2)}.$$

Let us pick $\delta = \frac{\epsilon}{\sqrt{2}}$. Looking at (5) we see that for all (x,y) satisfying (4), we have

$$|f(x,y)-f(x_0,y_0)| \leq (\sqrt{2})\sqrt{(x-x_0)^2+(y-y_0)^2} < \sqrt{2}\delta = \sqrt{2}\cdot\frac{\epsilon}{\sqrt{2}} = \epsilon.$$

Hence, f is continuous at an arbitrary point (x_0, y_0) in $\mathbb{R}^2$.

Exercise 9. Using ϵ-δ definition, show that $f(x,y) = \sqrt{1+x^2+y^2}$ is continuous at the point $(0,0)$.