MAC 2313 **DIFFERENTIABILITY**

1.) Consider the function

$$f(x,y) = \begin{cases} \frac{5x^2y}{x^3+y^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(i) Using the definition, find the partials $f_x(0,0)$ and $f_y(0,0)$.

Outline: By definition of $f_x(0,0)$, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

since, by definition of f, we have $f(h,0) = \frac{5(h)^2\cdot 0}{h^3+0^3} = 0$ and $f(0,0) = 0$.
Follow the same outline for $f_y(0,0)$.

(ii) Show that $f(x,y)$ is not continuous at $(0,0)$.

Hint: Show that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by the usual technique.

(iii) Using (ii) conclude that $f(x,y)$ is not differentiable at the point $(0,0)$. (Consult the appropriate theorem from the book.)

2.) Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(i) Find the partials $f_x(x,y)$ and $f_y(x,y)$ at points (x,y) such that $(x,y) \neq (0,0)$.

Hint: Do not use the limit definition here because the expression $\frac{xy(x^2-y^2)}{x^2+y^2}$ looks fine for $(x,y) \neq (0,0)$. Instead, use the usual rules for finding partials. The final answer is

$$f_x(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \quad f_y(x,y) = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}.$$

(ii) Find the partials $f_x(0,0)$ and $f_y(0,0)$.

Hint: Here, you will use the limit definition as in part (i) of Exercise 1 above. The final answer is $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

(iii) Show that the functions $f_x(x,y)$ and $f_y(x,y)$ obtained in part (i) of this problem are continuous at the point $(x,y) = (0,0)$.

Hint: You will show that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

and

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = f_y(0,0).$$

(iv) Using part (iii) conclude that $f(x,y)$ is differentiable at the point $(x,y) = (0,0)$. (Consult the appropriate theorem from the book.)

(v) Find the second partials $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Hint: Remember that the second partial f_{xy} is the derivative with respect to y of the function $f_x(x,y)$. Hence, you can calculate $f_{xy}(0,0)$ by using the formula

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}.$$

The quantity $f_x(0,h)$ can be calculated from part (i) of this exercise, while $f_x(0,0)$ was obtained in part (ii). With small adjustments of the outlined procedure, you can find $f_{yx}(0,0)$. The final answer is $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

(vi) Observe that this example demonstrates that the partials f_{xy} and f_{yx} **need not be equal** at every point.