

# Divergence theorems in path space II: degenerate diffusions

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## Abstract

Let  $x$  denote an elliptic diffusion process defined on a smooth compact manifold  $M$ . In a previous work, we introduced a class of vector fields on the path space of  $x$  and studied the admissibility of this class of vector fields with respect to the law of  $x$ . In the present note, we extend this study to the case of degenerate diffusions.

## Résumé

Laisser  $x$  indiquer une diffusion elliptique définie sur une variété compacte lisse  $M$ . Dans un travail précédent, nous avons introduit une classe de champs de vecteur sur l'espace de chemin de  $x$  et avons étudié l'admissibilité de cette classe de champs de vecteur par rapport à la loi de  $x$ . Dans la note présente, nous étendons cette étude au cas de diffusions dégénérées.

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Let  $M$  denote a closed compact  $d$ -dimensional  $C^\infty$  manifold and  $X_1, \dots, X_n$  smooth vector fields defined on  $M$ . Consider the following Stratonovich stochastic differential equation<sup>2</sup> (SDE) with fixed initial point  $o \in M$

$$dx_t = \sum_{i=1}^n X_i(x_t) \circ dw_i, \quad t \in [0, T]. \quad (1)$$

Let  $C_o(M)$  denote the space of continuous paths from  $[0, T]$  into  $M$  originating at  $o$ .

**Definition.** A vector field  $Z$  on the path space  $C_o(M)$  is admissible (with respect to the law of  $x$ ) if there exists an  $L^1$  random variable  $Div(Z)$  such that the equality

$$E[(Z\Phi)(x)] = E[\Phi(x)Div(Z)] \quad (2)$$

holds for a dense class of smooth functions  $\Phi$  on  $C_o(M)$ .

In [1], we developed a method for establishing the admissibility of a class of vector fields  $Z$  on  $C_o(M)$  of the form

$$Z_t = \sum_{i=1}^n X_i(x_t)h^i(t), \quad t \in [0, T] \quad (3)$$

where  $h^i : [0, T] \mapsto \mathbf{R}$  are adapted processes,  $i = 1, \dots, n$ . It was assumed in [1] that the SDE (1) is *elliptic*, i.e. the vector fields  $X_1, \dots, X_n$  span  $TM$  at all points of  $M$ . The purpose of this note is to study the admissibility of vector fields of the form (3) in the *degenerate* case, i.e. when the ellipticity condition fails. This problem has also been treated by Elworthy, Le Jan & Li using an approach based on filtering (cf. [4, Section 4.1]).

For each  $x \in M$ , define  $E_x$  to be the subspace of  $T_xM$  spanned by the vectors  $X_1(x), \dots, X_n(x)$ . Assume these vector spaces have the same dimension for all  $x \in M$ , and define  $E$  to be the sub-bundle of  $TM$ ,  $E = \cup_{x \in M} E_x$ . Following Elworthy, Le Jan & Li ([4]), we Riemannianize  $E$  by defining  $\langle \cdot, \cdot \rangle$  to be the inner product on each  $E_x$  induced from the Euclidean space  $\mathbf{R}^n$  by the map  $X(x) \in L(\mathbf{R}^n, E_x)$ , where  $X(x)(h_1, \dots, h_n) \equiv X_i(x)h_i$ . Here, and in the sequel, we assume that whenever an index in a product is repeated, that index is summed on. Further following [4], we define a connection  $\nabla$  on  $E$ , compatible with this metric, by  $\nabla_V Z = X(x)d_V(X^*Z)$ ,  $Z \in \Gamma(E)$ ,  $V \in T_xM$ , where  $d$  represents the usual derivative of the function  $x \in M \mapsto X(x)^*Z(x) \in \mathbf{R}^n$ .

We define a collection of 1-forms  $\omega^{jk}$ ,  $1 \leq j, k \leq n$  on  $M$  by

$$\omega^{jk}(V) = \langle \nabla_{X_j} X_k, V \rangle - \langle \nabla_V X_j, X_k \rangle - \langle T(X_j, V), X_k \rangle, \quad V \in TM,$$

where  $T$  is the torsion tensor of the connection  $\nabla$ .

<sup>2</sup>The assumption of no drift in equation (1) is purely for notational convenience.

**Theorem 1** *Suppose that the sub-bundle  $E$  satisfies the integrability condition*

$$\text{span}\{[X_i, X_j](x), 1 \leq i, j \leq n\} \subseteq E_x, \quad \forall x \in M. \quad (4)$$

Let  $r = (r^1, \dots, r^n)$  be a path in the  $n$ -dimensional Cameron-Martin space and suppose  $h^1, \dots, h^n$  are real-valued processes with initial value 0 satisfying the system of SDE's

$$dh^k = \omega^{jk}(\circ dx)h^j + \dot{r}^k dt, \quad 1 \leq k \leq n. \quad (5)$$

Then the vector field  $Z$  on  $C_o(M)$  defined by (3) is admissible.

*Sketch of proof.* Let  $g : C_0(\mathbf{R}^n) \mapsto C_o(M)$  denote the Itô map  $w \mapsto x$  defined by equation (1). Following the approach in [1], we lift  $Z$  to the Wiener space via  $g$ , i.e. we construct a vector field  $r$  on  $C_0(\mathbf{R}^n)$  such that the following diagram commutes

$$\begin{array}{ccc} T(C_0(\mathbf{R}^n)) & \xrightarrow{dg} & TC_o(M) \\ r \uparrow & & \uparrow Z \\ C_0(\mathbf{R}^n) & \xrightarrow[g]{} & C_o(M) \end{array}$$

Of course, since the map  $g$  is non-differentiable in the classical sense,  $dg$  must be interpreted in the extended sense of the Malliavin calculus. The tangent space  $T(C_0(\mathbf{R}^n))$  is defined as the space of processes of the form  $\int_0^\cdot h_s ds + \int_0^\cdot A_s ds$ , where  $h$  and  $A$  are continuous adapted processes with values in  $\mathbf{R}^n$  and the space of  $n \times n$  skew-symmetric matrices  $so(n)$ , respectively (this notion of tangent space was inspired by Driver's work [3], see also [5]).

The starting point of the proof is equation (3.6) in [1], which states that  $r$  is a lift of  $Z$  if and only if the following SDE is satisfied

$$X_i(x_t) \circ dh^i = [X_j, X_i](x_t)h^j \circ dw_i + X_i(x_t)dr^i. \quad (6)$$

Now the metric  $\langle \cdot, \cdot \rangle$  has the property  $V = \langle V, X_i(x) \rangle X_i(x)$ , for all  $V \in E_x$ . In view of condition (4), we can use this property to solve equation (6) for  $dh$  and obtain

$$dh^k = \langle [X_j, X_i](x_t), X_k(x_t) \rangle h^j \circ dw_i + dr^k.$$

The idea is to now decompose the diffusion coefficient in this equation into a *tensorial* term in  $X_i$  and a term that is *skew-symmetric* in the indices  $i$  and  $k$ . To this end, we define

$$a_{ik}^j(t) = \langle \nabla_{X_j} X_i(x_t), X_k(x_t) \rangle - \langle \nabla_{X_j} X_k(x_t), X_i(x_t) \rangle, \quad 1 \leq i, j, k \leq n.$$

Note that these terms are skew-symmetric in  $i$  and  $k$ . We then have

$$\langle [X_j, X_i](x_t), X_k(x_t) \rangle = a_{ik}^j(t) + \omega_{jk}(X_i(x_t))$$

Proceeding as in [1], we write the process  $h$  in equation (5) in the form

$$dh^k = \langle [X_j, X_i](x_t), X_k(x_t) \rangle h^j \circ dw_i + d\tilde{r}^k$$

where

$$d\tilde{r}^k = \tilde{r}^k dt - a_{ik}^j(t) h^j \circ dw_i.$$

Thus  $\tilde{r}$  is a lift of  $Z$ . Let  $\Phi$  be a test (i.e. smooth cylindrical) function on  $C_o(M)$ . By definition of the lift, we have

$$E[(Z\Phi)(x)] = E[\tilde{r}(\Phi \circ g)(w)] = E[\Phi(x) \text{Div}(\tilde{r})]$$

where  $\text{Div}$  denotes the divergence operator in the classical Wiener space. The form chosen for  $\tilde{r}$  ensures that  $\text{Div}(\tilde{r})$  exists (cf. Theorems 2.3 and 2.4 in [1]), and the theorem follows.

In general, it is of interest to know if a given vector field on  $C_o(M)$  admits a lift to the Wiener space. We now show that if Hörmander's condition holds, then the integrability condition (4) is *necessary* for the existence of lifts, for almost all vector fields  $Z$  on  $C_o(M)$  of the form (3).

**Theorem 2** *Suppose condition (4) fails at some point  $m \in M$ . Define a (proper) subspace  $V$  of  $\mathbf{R}^n$  by*

$$V \equiv \left\{ (c_1, \dots, c_n) \in \mathbf{R}^n / \text{span} \{ c_j [X_j, X_k](m), 1 \leq k \leq n \} \subseteq E_m \right\}.$$

*Let  $h = (h^1, \dots, h^n)$  denote a continuous adapted process in  $\mathbf{R}^n$  such that  $P(h(t_0) \notin V) > 0$  for some  $t_0 \in (0, T)$ . Suppose  $X_1, \dots, X_n$  satisfy Hörmander's condition everywhere on  $M$ . Then the vector field  $Z$  on  $C_o(M)$  defined by (3) admits no lift to  $C_0(\mathbf{R}^n)$  via the Itô map.*

*Sketch of proof.* By hypothesis, there exists  $1 \leq k \leq n$  such that  $P([X_j, X_k](m) h_{t_0}^j \notin E_m) > 0$ . This implies the existence of a neighborhood  $N$  of  $m$  on which this condition holds. By a result of Léandre [7, Theorem II.1),  $P(x_{t_0} \in N) > 0$ . In particular, there exists a positive stopping time  $\tau$  such that with positive probability

$$[X_j, X_k](x_t) h_t^j \notin E_{x_t}, \forall t \in [t_0, t_0 + \tau]. \quad (7)$$

Now suppose there exists a lift of  $Z$  to  $C_0(\mathbf{R}^n)$ . Then equation (6) implies

$$[X_j, X_k](x_t) h_t^j \circ dw_k \in E_{x_t}. \quad (8)$$

Together, (7) and (8) imply there exists a non-vanishing continuous adapted process  $a = (a_1, \dots, a_n)$  such that with positive probability  $a_k(t) \circ dw_k = 0, \forall t \in [t_0, t_0 + \tau)$ . However, using the Itô rules  $dw_i dw_j = \delta_{ij} dt, dw_i dt = 0$ , we see this is impossible. This proves that no such lift exists, as claimed.

The following result, which provides a natural setting for Theorem 2, is easy to verify.

**Proposition.** *Suppose the SDE (1) is degenerate and the vector fields  $X_1, \dots, X_n$  satisfy Hörmander's condition everywhere on  $M$ . Then the set of points at which condition (4) fails is dense in  $M$ .*

*Remark.* The lifting method was originally used by Malliavin [8] to study the hypoellipticity of the differential operator  $\sum_{i=1}^n X_i^2$ . In this context, it suffices to construct the lift of vector fields on  $M$  under the endpoint map  $g_t : w \mapsto x_t$ , for fixed  $t > 0$ . Now, it is well-known that when the diffusion (1) is degenerate, Hörmander's condition on  $X_1, \dots, X_n$  implies the existence of lifts for *all* smooth vector fields on  $M$  under  $g_t$  (see e.g. [8], [6], [2]). By contrast, Theorem 2 and the Proposition imply that the set of liftable vector fields on the *path space* of the diffusion is very sparse. The problem of lifting vector fields at the path space level thus has a strikingly different character to that encountered in earlier work on the endpoint problem.

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