CHAPTER 1

Brief Introduction to Vectors and Matrices

In this chapter, we will discuss some needed concepts found in introductory course in linear algebra. We will introduce matrix, vector, vector-valued function, and linear independency of a group of vectors and vector-valued functions.

1. Vectors and Matrices

A **matrix** is a group of numbers(elements) that are arranged in rows and columns. In general, an $m \times n$ matrix is a rectangular array of mn numbers (or elements) arranged in m rows and n columns. If $m = n$ the matrix is called a square matrix. For example a 2×2 matrix is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and an 3×3 matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Generally, we use bold phase letter, like $\boldsymbol{A}$, to denote a matrix, and lower case letters with subscripts, like $\boldsymbol{a_{ij}}$, to denote element of a matrix. Here $\boldsymbol{a_{ij}}$ would be the element at $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column. So $\boldsymbol{a_{11}}$ is an element at $\boldsymbol{1^{st}}$ row and column. Sometime we use the abbreviation $\boldsymbol{A = (a_{ij})}$ for a matrix with elements $\boldsymbol{a_{ij}}$.

1.1. Special matrices.

0 denotes the zero matrix whose elements are all zeroes. So $\mathbf{2 \times 2}$ and $\mathbf{3 \times 3}$ zero matrices are

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Another special matrix is the identity matrix, denoted by $\boldsymbol{I}$, a identity matrix is an matrix whose main diagonal elements are 1, and all other elements are 0. So $\mathbf{2} \times \mathbf{2}$ and $\mathbf{3} \times \mathbf{3}$ zero matrices are

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

A vector is a matrix with one row or one column. In this chapter, a vector is always a matrix with one column as

$$\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$$

for a two-dimensional vector and

$$\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix}$$

for a three dimensional vector. Here the element has only one index that denotes the row position (Sometimes we use different variable to denote number in different position such as using

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

for a 2-dimensional vector). We use bold lower case, such as $\boldsymbol{v}$, to denote a vector.

1.2. Operations on Matrices. Arrange number in rectangular fashion, as a matrix, itself is not something terribly interesting. The most important advantage from that kind arrangement is that we can define matrix addition, multiplication, and scalar multiplication.

DEFINITION 1.1.

(i) **Equality:** *Two matrix* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *and* $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ *are equal if corresponding elements are equal, i.e.* $\boldsymbol{a_{ij}} = \boldsymbol{b_{ij}}$*.*
(ii) **Addition:** *If* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *and* $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ *and the sum of* $\boldsymbol{A}$*and* $\boldsymbol{B}$ *is* $\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{c_{ij}}) = \boldsymbol{a_{ij}} + \boldsymbol{b_{ij}}$*.*
(iii) **Scalar Product:** *If* $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *is matrix and* $\boldsymbol{k}$ *is number(scalar), the* $\boldsymbol{kA} = (\boldsymbol{ka_{ij}})$ *is product of* $\boldsymbol{k}$ *and* $\boldsymbol{A}$*.*

From the above definition, we see that, to multiply a matrix by a number $\boldsymbol{k}$, we simply multiply each of its entries by $\boldsymbol{k}$; to add two matrices we just add their corresponding entries; $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\mathbf{1})\boldsymbol{B}$.

EXAMPLE 1.1. *Let*

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ -\mathbf{1} & \mathbf{4} \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix},$$

find (a) $\boldsymbol{A}+\boldsymbol{B}$, *(b)* $\mathbf{3}\boldsymbol{A}$, *(c)* $4\boldsymbol{A}-\boldsymbol{B}$.

Solution
(a)

$$\boldsymbol{A}+\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{2+0} & \mathbf{3+5} \\ \mathbf{-1+3} & \mathbf{4+(-4)} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{8} \\ \mathbf{2} & \mathbf{0} \end{bmatrix}$$

(b)

$$\mathbf{3}\boldsymbol{A} = \mathbf{3}\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix} = \begin{bmatrix} \mathbf{6} & \mathbf{9} \\ \mathbf{-3} & \mathbf{12} \end{bmatrix}$$

(c)

$$4\boldsymbol{A}-\boldsymbol{B} = \begin{bmatrix} \mathbf{8} & \mathbf{12} \\ \mathbf{-4} & \mathbf{16} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix} = \begin{bmatrix} \mathbf{8} & \mathbf{7} \\ \mathbf{-7} & \mathbf{20} \end{bmatrix}$$

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The following fact lists all properties of matrix addition and scalar product.

THEOREM 1.1. *Let* $\boldsymbol{A}$, $\boldsymbol{bB}$, *and* $\boldsymbol{C}$ *be matrices. Let* $\boldsymbol{a}, \boldsymbol{b}$ *be scalars (numbers). We have*

(1) $\boldsymbol{A}+\mathbf{0}=\mathbf{0}+\boldsymbol{A}=\boldsymbol{A}$, $\boldsymbol{A}-\boldsymbol{A}=\mathbf{0}$;
(2) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ *(commutativity);*
(3) $\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$, $(\boldsymbol{ab})\boldsymbol{A}=\boldsymbol{a}(\boldsymbol{bA})$ *(associativity);*
(4) $\boldsymbol{a}(\boldsymbol{A}+\boldsymbol{B})=\boldsymbol{aA}+\boldsymbol{aB}$, $(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{A}=\boldsymbol{aA}+\boldsymbol{bA}$ *(distributivity)*

When we have a row vector and a column vector with the same number of elements, we can define the dot product as

DEFINITION 1.2. **Dot Product:**

- **in 2-dimension:** *Let* $\boldsymbol{x} = \begin{bmatrix} \mathbf{x_1} & \boldsymbol{x_2} \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} \mathbf{y_1} \\ \boldsymbol{y_2} \end{bmatrix}$, *the dot product of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is,*

$$\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x_1y_1}+\boldsymbol{x_2y_2}$$

- **in 3-dimension:** *Let* $\boldsymbol{x} = \begin{bmatrix} \mathbf{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} \mathbf{y}_1 \\ \boldsymbol{y}_2 \\ \boldsymbol{y}_3 \end{bmatrix}$,

  the dot product of $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is,*

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}_1\boldsymbol{y}_1 + \boldsymbol{x}_2\boldsymbol{y}_2 + \boldsymbol{x}_3\boldsymbol{y}_3$$

DEFINITION 1.3. **Matrix product**

Let $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ *and* $\boldsymbol{B} = (\boldsymbol{b_{ij}})$, *if the number of columns of* $\boldsymbol{A}$ *is the same as number of rows of* $\boldsymbol{B}$, *then the product of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is given by* $\boldsymbol{AB} = (\boldsymbol{c_{ij}})$ *where* $\mathbf{c_{ij}}$ *is dot product of* $\boldsymbol{i^{th}}$ *row of* $\boldsymbol{A}$ *with* $\boldsymbol{j^{th}}$ *column of* $\boldsymbol{B}$.

EXAMPLE 1.2. *Let*

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix},$$

find $\boldsymbol{AB}$

Solution

$$\boldsymbol{AB} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{2\cdot(0)+3\cdot(3)} & \mathbf{2\cdot 5+3\cdot(-4)} \\ \mathbf{(-1)\cdot(0)+4\cdot 3} & \mathbf{-1\cdot 5+4\cdot(-4)} \end{bmatrix} = \begin{bmatrix} \mathbf{9} & \mathbf{-2} \\ \mathbf{12} & \mathbf{-21} \end{bmatrix}$$

Notice, the first element of $\boldsymbol{AB}$ is $\mathbf{2\cdot(0)+3\cdot(3)}$ which is the dot product of first row of $\boldsymbol{A}$, $\begin{bmatrix} \mathbf{2} & \mathbf{3} \end{bmatrix}$ and first column of $\boldsymbol{B}$, $\begin{bmatrix} \mathbf{0} \\ \mathbf{3} \end{bmatrix}$ ⊣

The following fact gives properties of matrix product,

THEOREM 1.2. *Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *be three matrices and* $\boldsymbol{r}$ *be a scalar, we have*

- $\boldsymbol{A(BC) = (AB)C}$, $\boldsymbol{r(AB) = A(rB)}$ *(associativity)*
- $\boldsymbol{A(B+C) = AB + AC}$ *(distributivity)*

Notice, in general $\boldsymbol{AB \neq BA}$, that is for most of the times, $\boldsymbol{AB}$ is not equal to $\boldsymbol{BA}$.

Using the matrix notation and matrix product, we can write the following system of equations

$$\begin{cases} \boldsymbol{ax_1 + bx_2} &= \boldsymbol{y_1} \\ \boldsymbol{cx_1 + dx_2} &= \boldsymbol{y_2} \end{cases}$$

as $\boldsymbol{Ax} = \boldsymbol{y}$ with

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix},$$

$\boldsymbol{x} = \begin{bmatrix} \mathbf{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} \mathbf{y_1} \\ \boldsymbol{y_2} \end{bmatrix}$.

Definition 1.4. *A square (ex. $\mathbf{2} \times \mathbf{2}$ or $\mathbf{3} \times \mathbf{3}$) matrix $\boldsymbol{A}$ is invertible if there is a matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.*

Theorem 1.3. *Let*

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$$

be a $\mathbf{2} \times \mathbf{2}$ matrix, if $\boldsymbol{A}$ is invertible, we have

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\boldsymbol{ad} - \boldsymbol{bc}} \begin{bmatrix} \mathbf{d} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$$

So if $\boldsymbol{A}$ is invertible, to solve $\boldsymbol{Ax} = \boldsymbol{y}$, we need to simply multiply both sides with $\boldsymbol{A}^{-1}$, that is

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}.$$

Example 1.3. *Solve the system of equation*

$$\begin{cases} \mathbf{3}\boldsymbol{x_1} - \mathbf{4}\boldsymbol{x_2} & = \mathbf{2} \\ -\mathbf{2}\boldsymbol{x_1} + \mathbf{5}\boldsymbol{x_2} & = \mathbf{7} \end{cases}$$

Solution The equation can be rewrite $\boldsymbol{Ax} = \boldsymbol{y}$ with

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & -\mathbf{4} \\ -\mathbf{2} & \mathbf{5} \end{bmatrix},$$

$\boldsymbol{x} = \begin{bmatrix} \mathbf{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} \mathbf{2} \\ \mathbf{7} \end{bmatrix}$. So in matrix form the system of equation is

$$\begin{bmatrix} \mathbf{3} & -\mathbf{4} \\ -\mathbf{2} & \mathbf{5} \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \boldsymbol{x_2} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{7} \end{bmatrix}.$$

Now the inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\mathbf{3(5) - (-2)(-4)}} \begin{bmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix},$$

so the solution is

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{7}} \begin{bmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \mathbf{2} \\ \mathbf{7} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{38}}{\mathbf{7}} \\ \frac{\mathbf{25}}{\mathbf{7}} \end{bmatrix}$$

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EXAMPLE 1.4. *Solve the system of equation*

$$\begin{cases} 3x_1 - 4x_2 + 5x_3 & = & 2 \\ -2x_1 + 5x_2 & = & 7 \\ x_1 - 5x_2 + 8x_3 & = & -1 \end{cases}$$

Solution The equation can be rewrite $\boldsymbol{Ax} = \boldsymbol{y}$ with

$$\boldsymbol{A} = \begin{bmatrix} 3 & -4 & 5 \\ -2 & 5 & 0 \\ 1 & -5 & 8 \end{bmatrix},$$

$\boldsymbol{x} = \begin{bmatrix} \mathrm{x}_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$. So in matrix form the system of equation, $\boldsymbol{Ax} = \boldsymbol{y}$, is

$$\begin{bmatrix} 3 & -4 & 5 \\ -2 & 5 & 0 \\ 1 & -5 & 8 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}.$$

It is a little harder to compute the inverse of a 3×3 matrix, we will use Mathcad to solve the equation. Here is how to do it,

- Type `A:[Ctrl][M]` at a blank area to bring up the matrix definition screen, put 3 in the both input boxes and click OK, you will get a 3×3 matrix place holder like

  $$\boldsymbol{A} := \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

  Fill the entries of $\boldsymbol{A}$ in the corresponding position, using [Tab] key to navigate among the place holders(or just click each one).
- Type `b:[Ctrl][M]` in another blank area, the matrix definition screen is up again. This time put 3 in the number of row box, and 1 in the number of column box and click OK. You will get b:= $\begin{bmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{bmatrix}$ put the values of $\boldsymbol{y}$ in the corresponding position.
- Type `A^-1 *b=` you will get the solution, which is,

  $$\begin{bmatrix} \frac{154}{81} \\ \frac{175}{81} \\ \frac{80}{81} \end{bmatrix}$$

- Notice, by default, Mathcad will display the results as decimal, you can double click on the result vector to change it to fraction, after you double click the result you will have a popup menu such as

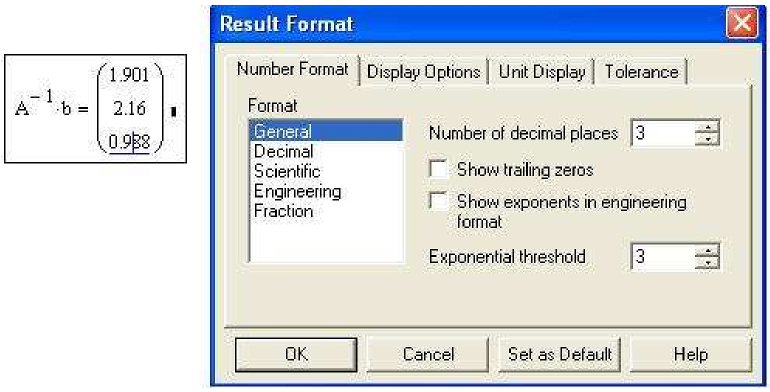

FIGURE 1. Format Result

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The next example shows how we can determine the unknown constants typically found in the initial value problems of system of differential equations.

EXAMPLE 1.5. *Let* $\boldsymbol{x_1(t) = C_1e^t + C_2e^{\ 2t}}$ *and* $\boldsymbol{x_2 = 2C_1e^t - C_2e^{\ 2t}}$. *If* $\boldsymbol{x_1(0) = 2,\ x_2(0) = 3,}$ *find* $\boldsymbol{C_1}$ *and* $\boldsymbol{C_2}$

Solution From $\boldsymbol{x_1(t) = C_1e^t + C_2e^{\ 2t}}$, set $\boldsymbol{t = 0}$ we have $\boldsymbol{x_1(0) = C_1e^0 + C_2e^{\ 2(0)} = C_1 + C_2}$.

Similarly, $\boldsymbol{x_2(t) = 2C_1e^t - C_2e^{\ 2t}}$, gives $\boldsymbol{x_2(0) = 2C_1e^(0) - C_2e^{\ 2(0)} = 2C_1 - C_2}$.

Together with $\boldsymbol{x_1(0) = 2,\ x_2(0) = 3}$ we have the following system of equations,

$$\begin{cases} \boldsymbol{C_1 + C_2} & \boldsymbol{= 1} \\ \boldsymbol{2C_1 - C_2} & \boldsymbol{= 3} \end{cases}$$

Rewrite the equation in matrix form

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} \mathrm{C}_1 \\ \boldsymbol{C_2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix},$$

and using Mathcad we find the solution is

$$\begin{bmatrix} \mathrm{C}_1 \\ \boldsymbol{C_2} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{4}}{\mathbf{3}} \\ -\frac{\mathbf{1}}{\mathbf{3}} \end{bmatrix}$$

So $x_1(t) = \frac{4}{3}e^t - \frac{1}{3}e^{\ \ 2t}$ and $x_2 = \frac{8}{3}e^t + \frac{1}{3}e^{\ \ 2t}$. They are solution of the following system of differential equations,

$$\begin{cases} x_1'(t) &= -x_1(t) + x_2(t) \\ x_2'(t) &= 2x_1 \end{cases}$$

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1.3. Eigenvalues and Eigenvectors. If $A = \begin{bmatrix} \mathrm{a} & b \\ c & d \end{bmatrix}$ the determined of A is defined as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. For a 3×3 matrix

$$\begin{bmatrix} \mathrm{a}_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we can compute the matrix as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In Mathcad , type the vertical bar — to bring up the absolute evaluator |▪|, put the matrix in the place holder and press = to compute the determinant. The following screen shot shows an example,

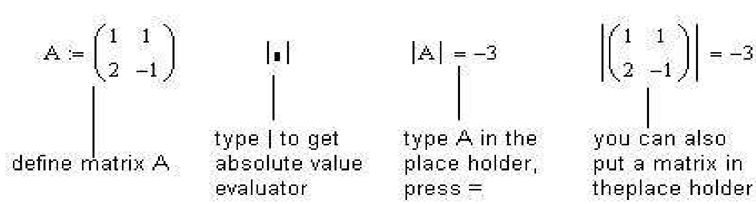


FIGURE 2. Compute determinant in Mathcad

The concepts of eigenvalue and eigenvector play an important role in find solutions to system of differential equations.

DEFINITION 1.5. *We say* λ *is an eigenvalue of a matrix* A *(*2×2 *or* 3×3*) if the determinant*

$$|A - \lambda I| = 0.$$

An nonzero vector v *is an eigenvector associated with* λ *if*

$$Av = \lambda v.$$

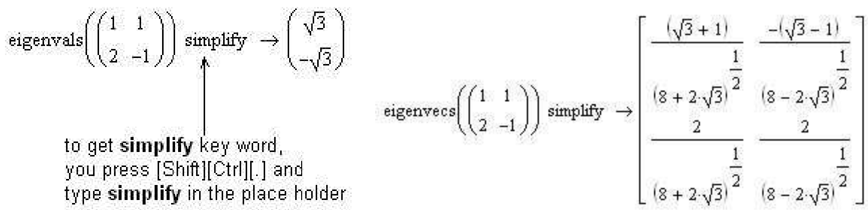
REMARK 1.1.

- *The above definition of eigenvector and eigenvalue is valid for any square matrix with* $\boldsymbol{n}$ *rows and columns.*
- $\boldsymbol{p(\lambda) = |A - \lambda I|}$ *is a polynomial of degree* $\boldsymbol{n}$ *for* $\boldsymbol{n \times n}$ *matrix* $\boldsymbol{A}$**,** *which is called the characteristic polynomial of* $\boldsymbol{A}$**.**
- *If we view* $\boldsymbol{A}$ *as an transform that maps a vector* $\boldsymbol{x}$ *to* $\boldsymbol{Ax}$*, an eigenvector* $\boldsymbol{v}$ *defines a straight line passing origin that is invariant under* $\boldsymbol{A}$**.**
- *If* $\boldsymbol{v}$ *is an eigenvector then for and number* $\boldsymbol{s \neq 0}$*,* $\boldsymbol{sv}$ *is also an eigenvector. This is especially useful when using Mathcad to get eigenvectors, the result of Mathcad might look "bad", you might need to remove the common factor of the component of the vector to make it "better."*

Computing eigenvalues and eigenvectors of a given matrix is quite tedious, Mathcad provides two functions **eigenvals()** and **eigenvecs()** to compute eigenvalues and eigenvectors of a matrix.

> In Mathcad , eigenvecs(M) Returns a matrix containing the eigenvectors. The nth column of the matrix returned is an eigenvector corresponding to the nth eigenvalue returned by eigenvals.

The results of these functions by default is in decimal, you can change it by using **simplify** key word as shown in the following diagram.



(a) Find eigenvalue (b) Find eigenvector

FIGURE 3. Compute eigenvalue and eigenvector in Mathcad

Notice, in the diagram, the eigenvalues are listed as vector $\begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$ and the eigenvectors are listed in a matrix

$$\begin{bmatrix} \frac{1+\sqrt{3}}{(8+2\sqrt{3})^{\frac{1}{2}}} & \frac{-(\sqrt{3}-1)}{(8-2\sqrt{3})^{\frac{1}{2}}} \\ \frac{2}{(8+2\sqrt{3})^{\frac{1}{2}}} & \frac{2}{(8-2\sqrt{3})^{\frac{1}{2}}} \end{bmatrix},$$

each column represents a eigenvector. Since multiplying an eigenvector by a nonzero constant you still get an eigenvector, so we can simplify the eigenvectors as $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{1+\sqrt{3}} \\ \mathbf{2} \end{bmatrix}$, and $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{1-\sqrt{3}} \\ \mathbf{2} \end{bmatrix}$

Here is how to use Mathcad ,

- Define the matrix by type `A:[Ctrl][M]` and specify the row and column number, fill the entries.
- type `eigenvals(`, you will get $\boldsymbol{eigenvals(\blacksquare)}$ and in the place holder type $\boldsymbol{A}$.
- Click at end of the $\boldsymbol{eigenvals(A)}$ and press [Shift][Ctrl][.], you will get $\boldsymbol{eigenvals(A)}\blacksquare \rightarrow \mathbf{.}$ In the place holder type in key word **simple**. And click any area outside the box to get result.
- Using the same procedure for find eigenvector using eigenvecs() function.

2. Vector-valued functions

A vector-valued function over [a, b] is a function whose value is a vector or matrix. For example the following functions are vector-valued functions,

EXAMPLE 2.1. (1) $\boldsymbol{v(t)} = \begin{bmatrix} \mathbf{t} \\ \boldsymbol{t^2} \end{bmatrix}$

(2) $\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{t^2} \\ \boldsymbol{e^t} \end{bmatrix}$

(3) $\boldsymbol{A(t)} = \begin{bmatrix} \mathbf{1} & \boldsymbol{t^3 - 4t + 5} \\ \mathbf{0} & \mathbf{sin}\boldsymbol{(t)} \end{bmatrix}$

2.1. Arithmetics of vector-valued function.

- To add two vector-valued function is to add their corresponding components.
- To multiply a vector-valued function by a scalar function to to multiply each entry by the scalar function.
- To multiply a vector(matrix) valued function to another vector-valued function is same as multiply a matrix with a vector.

The following example illustrate how to add/subtract two vector-valued functions and how to multiply a vector-valued function by a scalar function and how to apply a vector-valued function that is matrix to a vector value function.

EXAMPLE 2.2. *Suppose* $\boldsymbol{v}(\boldsymbol{t}) = \begin{bmatrix} \mathrm{t} \\ t^2 \\ t^3 - 2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ t^2 \\ e^t \end{bmatrix}$, *and*

$$\boldsymbol{A}(\boldsymbol{t}) = \begin{bmatrix} 1 & t^3 - 4t + 5 & 1 \\ 0 & 2 & \sin(t) \\ 2 & 0 & 1 \end{bmatrix}.$$

(a) *Find* $\boldsymbol{v}(\boldsymbol{t}) + \boldsymbol{x}(\boldsymbol{t})$;
(b) *Let* $\boldsymbol{f}(\boldsymbol{t}) = e^t$, *find* $\boldsymbol{f}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t})$;
(c) *Find* $\boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t})$

Solution

(a)

$$\boldsymbol{v}(\boldsymbol{t}) + \boldsymbol{x}(\boldsymbol{t}) = \begin{bmatrix} \mathrm{t} \\ t^2 \\ t^3 - 2 \end{bmatrix} + \begin{bmatrix} 1 \\ t^2 \\ e^t \end{bmatrix} == \begin{bmatrix} \mathrm{t} + 1 \\ 2t^2 \\ t^3 - 2 + e^t \end{bmatrix};$$

(b)

$$\boldsymbol{f}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) = e^t \begin{bmatrix} 1 \\ t^2 \\ e^t \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{t}} \\ t^2 e^t \\ e^{2t} \end{bmatrix};$$

(c)

$$\boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) == \begin{bmatrix} 1 & t^3 - 4t + 5 & 1 \\ 0 & 2 & \sin(t) \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t^2 \\ e^t \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \mathrm{t}^2(\mathrm{t}^3 - 4\mathrm{t} + 5) + \mathrm{e}^{\mathrm{t}} \\ 2t^2 + e^t \sin(t) \\ 2 + \sin(t) \end{bmatrix}$$

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2.2. derivative and integrations of vector-valued functions.

- A vector-valued function is **continuous** if each of its entries are continuous.
- A vector-valued function is **differentiable** if each of its entries are differentiable.
- If $\boldsymbol{v}(\boldsymbol{t})$ is an vector-valued function, then the derivative $\frac{d\boldsymbol{V}(t)}{dt} = \boldsymbol{v}'(\boldsymbol{t})$ of $\boldsymbol{v}(\boldsymbol{t})$ is a vector-valued function whose entries are the derivative of corresponding entries of $\boldsymbol{v}(\boldsymbol{t})$**.** That is to find derivative of a vector-valued function we just need to find derivative of each of its component.

- The antiderivative $\int \boldsymbol{v(t)}\,\boldsymbol{dt}$ of an vector-valued function $\boldsymbol{v(t)}$ is a vector-valued function whose entries are the antiderivative of corresponding entries of $\boldsymbol{v(t)}$.

EXAMPLE 2.3. *Find derivative of* $\boldsymbol{x(t)} = \begin{bmatrix} \mathbf{3t^2 - 5} \\ \mathbf{sin}(\boldsymbol{t}) \end{bmatrix}$

Solution

$$\boldsymbol{x'(t)} = \frac{\boldsymbol{dx(t)}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}} \begin{bmatrix} \mathbf{3t^2 - 5} \\ \mathbf{sin}(\boldsymbol{t}) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{3t^2 - 5}) \\ \frac{\boldsymbol{d}}{\boldsymbol{dt}}(\mathbf{sin}(\boldsymbol{t})) \end{bmatrix} = \begin{bmatrix} \mathbf{6t} \\ \mathbf{cos}(\boldsymbol{t}) \end{bmatrix}$$

⊣

EXAMPLE 2.4. *Find antiderivative of* $\boldsymbol{x(t)} = \begin{bmatrix} \mathbf{3t^2 - 5} \\ \mathbf{sin}(\boldsymbol{t}) \end{bmatrix}$

Solution

$$\int \boldsymbol{x(t)}\,\boldsymbol{dt} = \int \begin{bmatrix} \mathbf{3t^2 - 5} \\ \mathbf{sin}(\boldsymbol{t}) \end{bmatrix} \boldsymbol{dt} = \begin{bmatrix} \int(\mathbf{3t^2 - 5})\,\mathrm{dt} \\ \int \mathbf{sin}(\boldsymbol{t})\,\boldsymbol{dt} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{t^3 - 5t + C_1} \\ -\mathbf{cos}(\boldsymbol{t}) + \boldsymbol{C_2} \end{bmatrix} = \begin{bmatrix} \mathbf{t^3 - 5t} \\ -\mathbf{cos}(\boldsymbol{t}) \end{bmatrix} + \begin{bmatrix} \mathbf{C_1} \\ \boldsymbol{C_2} \end{bmatrix}$$

⊣

THEOREM 2.1. *Suppose* $\boldsymbol{v(t)}$, $\boldsymbol{x(t)}$, $\boldsymbol{A(t)}$ *are differentiable vector-valued functions (*$\boldsymbol{A(t)}$ *is matrix), and* $\boldsymbol{f(t)}$ *is differentiable scalar function. We have,*

(1) **Sum and Difference rule:**
  - $[\boldsymbol{v(t)} \pm \boldsymbol{x(t)}]' = \boldsymbol{v'(t)} \pm \boldsymbol{x'(t)}$,
  - $\int \boldsymbol{v(t)} \pm \boldsymbol{x(t)}\,\boldsymbol{dt} = \int \boldsymbol{v(t)}\,\boldsymbol{dt} \pm \int \boldsymbol{x(t)}\,\boldsymbol{dt}$.

(2) **Product rule:**
  - $[\boldsymbol{f(t)v(t)}]' = \boldsymbol{f'(t)v(t)} + \boldsymbol{f(t)v'(t)}$,
  - $[\boldsymbol{A(t)x(t)}]' = \boldsymbol{A'(t)x(t)} + \boldsymbol{A(t)x'(t)}$,

Using Mathcad to find derivative or antiderivative of a vector-valued function using Mathcad , you need to find derivative or antiderivative component wise as shown in the following screen shot,

$$x(t) := \begin{bmatrix} t^2 \\ t\cdot e^t \\ \left(\sin(t) + e^{2t}\right) \end{bmatrix} \quad dx(t) := \begin{bmatrix} \frac{d}{dt}t^2 \\ \frac{d}{dt}t\cdot e^t \\ \frac{d}{dt}\left(\sin(t) + e^{2t}\right) \end{bmatrix} \quad ix(t) := \begin{bmatrix} \int t^2\,dt \\ \int t\cdot e^t\,dt \\ \int \left(\sin(t) + e^{2t}\right) dt \end{bmatrix}$$

$$dx(t)\ \text{simplify} \rightarrow \begin{pmatrix} 2\cdot t \\ \exp(t) + t\cdot\exp(t) \\ \cos(t) + 2\cdot\exp(2\cdot t) \end{pmatrix} \quad ix(t) \rightarrow \begin{pmatrix} \frac{1}{3}\cdot t^3 \\ t\cdot\exp(t) - \exp(t) \\ -\cos(t) + \frac{1}{2}\cdot\exp(2\cdot t) \end{pmatrix}$$

FIGURE 4. Differentiate and integrate vector-valued function

Notice:

- Press [Shift][/]to get the derivative operator and press [Ctrl][I] to get the antiderivative operator.
- To get $\boldsymbol{dx(t) simplify} \rightarrow$ you type $\boldsymbol{dx(t)}$ and press [Shift][Ctrl][.] and type the key word **simplify** in the place holder before $\rightarrow$ **.**
- To execute symbolically ($\rightarrow$ operator), just press [Ctrl][.]

3. Linearly independency

3.1. Linearly independency of vectors. Let $\boldsymbol{x_1, x_2, \cdots, x_n}$ be $\boldsymbol{n}$ vectors, $\boldsymbol{C_1, C_2, \cdots, C_n}$ are $\boldsymbol{n}$ scalars(numbers), the expression

$$\boldsymbol{C_1 x_1 + C_2 x_2 + \cdots + C_n x_n}$$

is called a **linear combination** of vectors $\boldsymbol{x_1, x_2, \cdots, x_n}$.

Definition 3.1. *$\boldsymbol{n}$ vectors $\boldsymbol{x_1, x_2, \cdots, x_n}$ is linearly independent if*

$$\boldsymbol{C_1 x_1 + C_2 x_2 + \cdots + C_n x_n = 0}$$

leads to $\boldsymbol{C_1 = 0, C_2 = 0, \cdots, C_n = 0}$.

A set of vectors are linearly dependent if they are not linearly independent.

- If $\mathbf{0}$ is one of $\boldsymbol{x_1, x_2, \cdots, x_n}$, then they linearly dependent.
- Two nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent if and only if $\boldsymbol{x = sy}$ for some $\boldsymbol{s \neq 0}$.
- $\boldsymbol{n}$ nonzero vectors are linearly independent if one can be represented as linear combination of the others.
- Any three or more 2-dimensional vectors (vectors with two entries) are linear dependent.
- Any four or more 3-dimensional(vectors with three entries) vectors are linear dependent.

To determine if a given set of vectors are linearly independent, create a matrix so that the row of the matrix are given vectors. Using Mathcad function **rref(▪)** to find the **reduced echelon form** of the matrix, if the result contains one or more rows that are **entirely zero** the vectors are linearly dependent, otherwise the vectors are linearly independent.

EXAMPLE 3.1. *For* $\boldsymbol{x_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\boldsymbol{x_2} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$, *and* $\boldsymbol{x_3} = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$, *we can form a matrix,*

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -4 \\ 4 & 8 & 0 \end{bmatrix},$$

apply **rref***(type rref and in the place holder type* $\boldsymbol{A}$*, then press =),*

$$\boldsymbol{rref(A)} = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

.

We see that the vectors are linearly dependent as the last row is entirely zero.

3.2. Linearly independency of functions. We can also define linearly independency for a group of functions over an given interval [a, b]. Let $\boldsymbol{f_1, f_2, \cdots, f_n}$ be $\boldsymbol{n}$ functions defined over [a, b], $\boldsymbol{C_1, C_2, \cdots, C_n}$ are $\boldsymbol{n}$ scalars(numbers), the expression

$$\boldsymbol{C_1 f_1 + C_2 f_2 + \cdots + C_n f_n}$$

is called a **linear combination** of functions $\boldsymbol{f_1, f_2, \cdots, f_n}$.

DEFINITION 3.2. $\boldsymbol{n}$ *functions* $\boldsymbol{f_1, f_2, \cdots, f_n}$ *is linearly independent over [a, b]if*

(1) $\boldsymbol{C_1 f_1 + C_2 f_2 + \cdots + C_n f_n = 0}$ *for all* $\boldsymbol{a \le t \le b}$

leads to $\boldsymbol{C_1 = 0, C_2 = 0, \cdots, C_n = 0}$.

A set of function are linearly dependent if they are not linearly independent.

- If $\boldsymbol{0}$ function is one of $\boldsymbol{f_1, f_2, \cdots, f_n}$, then they linearly dependent.
- Two nonzero functions $\boldsymbol{f(t)}$ and $\boldsymbol{g(t)}$ are linearly dependent over [a, b] if and only if $\boldsymbol{f(t) = sg(t)}$ for a constant $\boldsymbol{s \neq 0}$ and all $\boldsymbol{a \le t \le b}$, for example, $\boldsymbol{f(t) = t}$ and $\boldsymbol{g(t) = 4t}$ are linearly dependent but $\boldsymbol{f(t) = t}$ and $\boldsymbol{g(t) = 4t^2}$ are not, even $\boldsymbol{f(0) = 4g(0)}$ and $\boldsymbol{f(1) = 4g(1)}$.
- There are exists infinite many functions that are linearly independent. For example the set $\boldsymbol{\{1, t, t^2, t^3, \cdots, t^n, \cdots\}}$ is a linearly independent set.

Here are some sets of linearly independent functions that we encounter in solving a system of differential equations, assume $\boldsymbol{k_1, k_2, \cdots, k_n}$ are different numbers,

- $\{t^{k_1}, t^{k_2}, \cdots, t^{k_n}\}$.
- $\{e^{k_1 t}, e^{k_1 t}, \cdot, e^{k_1 t}\}$.
- $\{\sin(k_1 t), \sin(k_2 t), \cdot, \sin(k_n t)\}$.
- $\{\cos(k_1 t), \cos(k_2 t), \cdot, \cos(k_n t)\}$.
- The mixing of above sets.
- For each above set, when multiplying each element by an common nonzero factor, we get another linearly independent set.

The following screen shot displays a heuristic Mathcad function that tries to determine if a given set of functions are linearly independent.

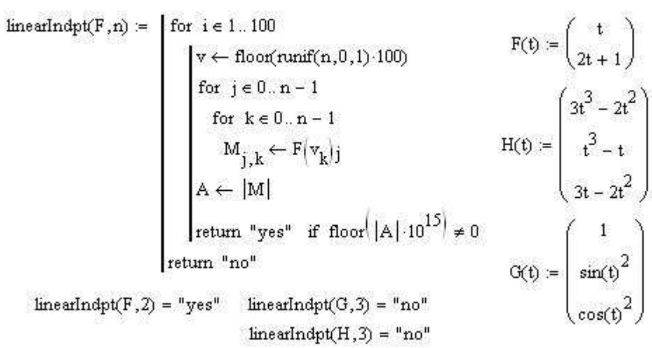

FIGURE 5. Calculus tool bar

One warning, the result of the program is not very reliable, the user should check the result manually to confirm the result.

To manually check if an set of functions are linearly independent on [a, b], one need to show that the only solutions are $\boldsymbol{C_1 = 0, C_2 = 0, \cdots, C_n = 0.}$ if equation (1) holds for all $\boldsymbol{t}$ in [a, b], which requires strong algebraic skill.

One method is to choose $\boldsymbol{n}$ different numbers $\{\boldsymbol{t_1,\ t_2,\ \cdots,\ t_n}\}$ from [a, b] and using the functions to create an matrix, the compute the determinant of the matrix $\boldsymbol{A = (f_i(t_j))}$, if the determinant is not zero, the functions are linearly independent, but if the determinant is zero, it is inconclusive(most likely are linearly dependent).

EXAMPLE 3.2. *Determine if* $\boldsymbol{f_1(t) = t^2 - 2t + 3,\ f_2(t) = 2t^2 - 5t - 6,}$ *and* $\boldsymbol{f_3(t) = 5t^2 - 11t + 4}$ *are linearly independent.*

Solution Choose $t_1 = -1$, $t_2 = 0$, and $t_3 = 1$,

$$\begin{bmatrix} f_1(t_1) & f_1(t_2) & f_1(t_3) \\ f_2(t_1) & f_2(t_2) & f_2(t_3) \\ f_3(t_1) & f_3(t_2) & f_3(t_3) \end{bmatrix} = \begin{bmatrix} 7 & 3 & 2 \\ 1 & -6 & -9 \\ 20 & 4 & -2 \end{bmatrix}$$

Compute the determinant,

$$\left| \begin{bmatrix} 7 & 3 & 2 \\ 1 & -6 & -9 \\ 20 & 4 & \\ -2 & & \end{bmatrix} \right| = 50,$$

so the functions are linearly independent. ⊣

Project

At beginning you should enter: Project title, your name, ss#, and due date in the following format

Project One: Define and Graph Functions

John Doe
SS# 000-00-0000

Due: Mon. Nov. 23rd, 2003

You should format the text region so that the color of text is different than math expression. You can choose color for text from **Format–>Style** select normal and click **modify**, then change the settings for font. You can do this for headings etc.

(1) **Independent of functions as vectors**
Goal: Familiar your self with the concept of linearly independency.
- Use the Mathcad code provided at at the website **www.unf.edu/∼mzhan/linear** to check if given set of functions are linearly independent or not.
  $\{\sin(x), \sin(2x), \sin(3x)\}$
  $\{t^2, 2t^2 - 2t + 4, 3t, 6\}$
  $\{e^t, te^t, t^3e^t\}$
  $\{e^{2t}, e^{\ t}, e^{\ 3t}\}$
- Using algebraic arguments or reasoning to verify the conclusion of the Mathcad code.

(2) **Condition Number** In solving $\boldsymbol{Ax} = \boldsymbol{b}$, one number is very important, it is called the condition number, which can be defined as $\boldsymbol{C(A)} = |\frac{s}{l}$, where $\boldsymbol{\lambda_s}$ is the eigenvalue with smallest absolute value and $\boldsymbol{lambda_l}$ is the eigenvalue with largest absolute value, if $\boldsymbol{C(A)}$ is too large or too small, a little change in $\boldsymbol{b}$ will result in a large in the solution $\boldsymbol{x}$. We say the system $\boldsymbol{Ax} = \boldsymbol{b}$ is not stable. Now if $\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

- Find all eigenvalues, all eigenvectors, and $\boldsymbol{C(A)}$.
- Find solution of $\boldsymbol{Ax} = \boldsymbol{b}$ if $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- Change $\boldsymbol{b}$ a little to $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1.1 \end{bmatrix}$ we get different solution, which component of the new solution change most? The change of the third component if $\mathbf{10\%}$ what is the percentage change of the most changed component?

Note:

- Our definition of condition number is not accurate, the true definition is $\boldsymbol{C(A)} = \frac{1}{\|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\|}$ where $\|\cdot\|$ is a given norm (metric).
- Mathcad provides three functions $\boldsymbol{cond1(A)}$, $\boldsymbol{cond2(A)}$ and $\boldsymbol{condi(A)}$ in compute condition number for $\boldsymbol{A}$ in different metric.