

CHAPTER 1

Explicitly Solvable First Order Differential Equations

In this chapter we will introduce the first order differential equations and several type of it that can be solved explicitly using integration methods covered in standard Calculus course.

From Calculus, we know that the derivative $\frac{df(t)}{dt}$ of a function $f(t)$ is the rate at which the quantity $x = f(t)$ is changing with respect to the **independent variable** t . An equation relating an unknown function and one or more of its derivatives is called an differential equation. Particulary, an equation of the form $x' = f(t, x)$ is called a first order ordinary differential equation(ODE), more precisely a semi linear first order ordinary differential equation. In this chapter will demonstrate how to find explicit solution(s) to a given ODE . In general one can't find explicit solution to a given ODE , but for special types of $f(t, x)$, we will have luck to do that.

Some clearance of notations is needed here.

In $x' = f(t, x)$, t is the independent variable and $x(t)$ is the unknown function, x' is $\frac{dx}{dt}$, the first derivative of $x(t)$, and $f(t, x)$ is a given function of two variables t and x . So to find an explicit solution is to find an formula for $x(t)$ so that when substitute the formula in $f(t, x)$, we will get an expression in t only that matches $x'(t)$. For example, for example, suppose we have $f(t, x) = x + t$, our ODE reads, $x' = x + t$. If we set $x(t) = 2e^t - t - 1$ then $f(t, x) = x + t = 2e^t - t - 1 + t = 2e^t - 1$ and $x'(t) = 2e^t - 1$ so $x' = f(t, x)$ and $x(t) = 2e^t - t - 1$ is an solution.

On the other hand if we write $y' = f(x, y)$, then in this ODE , x is the independent variable, $y(x)$ is the unknown function to be found, and y' is $\frac{dy}{dx}$, the first derivative of $y(x)$. So for $y' = y + x$, we would get an solution $y(x) = e^x - x - 1$.

Sometimes the independent variable does not appear in the equation such as $y' = f(y)$ or $x' = f(x)$ in those case, one could assume the independent variable is either x (for $y' = f(y)$) or t (for $x' = f(x)$) and write the solution accordingly. Hence, for $y' = y$ one solution would be $y(x) = e^x$ and for $x' = x$, one would write as solution as $x(t) = e^t$.

NOTE 0.1. *It is very important for students to know what variable represents an unknown function and what variable represents the independent variable that the unknown function is defined with respect to.*

1. Finding antiderivative using Mathcad

You can use Mathcad to find both definite and indefinite integral. To access those two operators, you can bring up the popup menu, (See



FIGURE 1. Calculus tool bar

chapter 1 for how to bring up the Calculus popup menu.) Click on the \int_a^b symbol to get the operator for definite integral and on the \int symbol to get the operator for the indefinite operator. Then just fill in the place holders accordingly. For definite integral, hit = to get the value and for indefinite integral, you need either

- Select the entire expression and from the **Symbolics** menu following the menu item **evaluate** to the **symbolically**(Or [Shift][F9]).

or

- Hold down the [Ctrl] key and press the [.] (period) key.

You can also access the integration operators by hot keys [Shift][F7] for definite integral and [Ctrl][I] for indefinite integral.

The integration operators appear on the workplace as shown in the following diagram,

Now lets see some examples. The first example is for finding definite integral.

EXAMPLE 1.1. *Find the definite integral $\int_0^3(x^5 - e^{3x} - \sin(3x)) dx$.*

Solution At any blank space in the workplace of Mathcad , type [Shift][F7], that is hold the [Shift] key and press [F7] key to place the definite integration operator

$$\int_{\blacksquare}^{\blacksquare} \blacksquare d\blacksquare$$

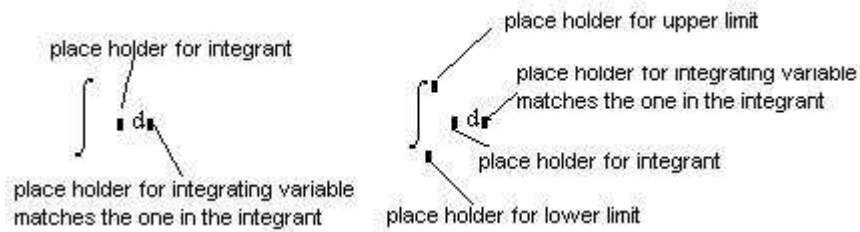


FIGURE 2. Integration Operators

. Click the upper limit place holder, type 3 and click the lower limit place holder type 0. Now click the integrant place holder type

$$x^5 - e^{3x} - \sin(3x)$$

notice you need it the space bar **twice** before $-\sin(3x)$. Of course you need hit space bar once before entering $-e^{3x}$ to exit the power mode. Finally in the integration variable place holder type x (see Figure 2 for the position of each place holder.) Now hit [=] key if in automatic mode or [=F9] otherwise to find the value -2.5×10^3 .

The second example is for find indefinite integral.

EXAMPLE 1.2. Find the definite integral $\int(\sqrt{x+2}-e^{3x}-\sin(3x)) dx$.

Solution At any blank space in the workplace of Mathcad, type [Ctrl][I], that is hold the [Ctrl] key and press [I] key to place the indefinite integration operator

$$\int \cdot d\cdot$$

Click the integrant place holder and type

$$\sqrt{x+2} - e^{3x} - \sin(3x)$$

Notice you need to hit the space bar **twice** before $-\sin(3x)$. Of course you also need hit space bar **twice** too before entering $-e^{3x}$ to exit the square root operator.

In general, you will need to hit the space bar as many times an need to escape the power(subscript, division, etc.) mode before you enter next term.

The backslash \ brings up the square root operator. Finally in the integration variable place holder type x (see Figure 2 for the position of each place holder.) Now hold down [Ctrl] to type [.] key and **click on**

any region outside the bounding box, if not in the auto execution mode hit [F9] too, to find the antiderivative

$$\int \sqrt{x+2} - e^{3x} - \sin(3x) dx \rightarrow \frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{1}{3}\exp^{3x} + \frac{1}{3}\cos(3x)$$

. Here exp represent exponential function. From Calculus, we know that

$$\int \sqrt{x+2} - e^{3x} - \sin(3x) dx = \frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{1}{3}\exp^{3x} + \frac{1}{3}\cos(3x) + C.$$

Mathcad omits the C for believing you know it! Remember C is important when we try to find a particular solution for a given, so called, "initial" condition.

EXAMPLE 1.3. Find a function $x(t)$ such that $x'(t) = 2t$ and $x(0) = 3$.

Solution If we take antiderivative on both sides, we have

$$\int x'(t) dt = \int 2t dt$$

which leads to

$$(1) \quad x(t) = t^2 + C.$$

Evaluate this $x(t)$ at $t = 0$ we have

$$x(0) = 0^2 + C = C$$

So the requirement $x(0) = 3$ implies $C = 3$. Hence $x(t) = t^2 + 3$.

Notice if we drop the C from (1), we would not be able to find $x(t)$ that satisfies $x(0) = 3$! \dashv

The problem of finding a particular solution with initial condition is called initial value problem, which we will discuss in the later section. \dashv

Practice

1. Find the following definite integral using Mathcad .

(a) $\int_{-1}^2 x^3 - x \cos(x^2 + 1) dx$

(b) $\int_1^4 \frac{x^3+4x-5}{x+1} dx$

(c) $\int_{-5}^0 e^{3x} - \frac{x}{x^2+3} dx$

(d) $\int_{-1}^{10} x^2 - x \sec(x^2 + 1) dx$

2. Find the following indefinite integral using Mathcad .

(a) $\int x^5 - x\sqrt{(x^2 + 1)} dx$

(b) $\int \frac{x^3+4x-5}{x^2+1} dx$

- (c) $\int \frac{x-3}{(x^2+3)(x+5)} dx$
 (d) $\int x \tan(x^2 + 1) dx$

2. Separable equations

If we let x be the independent variable and y be the dependent variable, a separable ODE is given by,

$$y' = f(x)g(y)$$

where both f and g are given one variable functions and $y' = \frac{dy}{dx}$ is the derivative of $y(x)$ with respect to x and $y(x)$ is the unknown function we want to find out.

When $g(y) \equiv 1$, that is $g(y)$ is a constant function whose value is always 1, the separable ODE becomes

$$y' = f(x)$$

, which is sometimes called integrable ODE . The general solution to the integrable ODE is the antiderivative of $f(x)$, i.e.

$$y(x) = \int f(x) dx$$

So if we can find the antiderivative of $f(x)$ we can find the explicit general solution.

EXAMPLE 2.1. Find solution $y' = x^2 + 3$

Solution The solution is,

$$y(x) = \int x^2 + 3 dx = \frac{1}{3}x^3 + 3x + C$$

. Here we used the power rule of antiderivative,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

To find the solution in Mathcad , you just click any blank area in the workplace and type [Ctrl][I] to get the indefinite operator

$$\int \blacksquare d\blacksquare$$

and type $x^2 + 3$ in the place holder for integrant and x in the integrating variable place holder. Then press [Ctrl][.] and **click on any region outside the bounding box** to get the result. Sometime you might need to hit [F9] too if the auto execution mode is turn off. \dashv

When $g(y)$ is not a constant function, the general solution to

$$y' = f(x)g(y)$$

is given by the equation

$$(2) \quad \int \frac{dy}{g(y)} = \int f(x) dx,$$

which is obtained by dividing both sides of the equation by $g(y)$ and then taking antiderivative to both sides.

So one has to find two antiderivatives $\int \frac{dy}{g(y)}$ and $\int f(x) dx$. Sometimes we can solution for $y(x)$ explicit from the (2), sometimes we can't find explicit formula for $y(x)$, in that case we say $y(x)$ is an implicitly defined function.

Notice, in the process of finding the general solution to the separable ODE, we divide both sides of the equation by $g(y)$, that implicitly requires that $g(y) \neq 0$. On the other hand if $g(c) = 0$ then $y(x) = c$ is an solution to the separable equation

$$y' = f(x)g(y)$$

This kind solutions are called **stationary or steady-state solutions**. Sometimes they are also called **singular solution**.

EXAMPLE 2.2. *In the study of Epidemics, the spread of disease in a population can be modelled by $I' = (b(t)S(t) - r(t))I$ where $S(t)$ is number of people that is susceptible to the disease but not infected yet. $I(t)$ is number of people actually infected. b, r are proportional function. Now suppose $S(t) = e^{-3t}$, $b = t^2$, and $r = 4$, find all solutions.*

Solution First notice that $I(t) \equiv 0$ is the steady-state solution. Then, to

$$I' = \frac{dI}{dt} = (b(t)S(t) - r(t))I,$$

divide its both sides by I , multiplying its both sides by dt and taking antiderivative we have

$$\int \frac{dI}{I} = \int b(t)S(t) - r(t) dt$$

The left side is easy to find,

$$(3) \quad \int \frac{dI}{I} = \ln |I| + C$$

Plug in the given functions, we get the right hand side

$$\int b(t)S(t) - r(t) dt = \int t^2 e^{-3t} - 4 dt$$

Now open Mathcad window, click any blank area, type [Ctrl][I] to get the antiderivative operator

$$\int \blacksquare d\blacksquare$$

type $t^2 * e^{-3t} - 4$ in the integrand place holder (again using space bar to exit the power mode) and t in the other place holder, then type [Ctrl][.], click any area outside the box. You will get

$$(4) \quad \int t^2 e^{-3t} - 4 dt = -\frac{1}{3}t^2 e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t} - 4t + C$$

Put (3) and (4) together, we have

$$\ln |I(t)| = -\frac{1}{3}t^2 e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t} - 4t + C$$

Solve for $I(t)$, we finally arrive at

$$\begin{aligned} I(t) &= e^{-\frac{1}{3}t^2 e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t} - 4t + C} \\ &= C e^{-\frac{1}{3}t^2 e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t} - 4t} \end{aligned}$$

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The next example shows that sometimes we can find explicit formula for the solution of an separable ODE .

EXAMPLE 2.3. Find general solution of $y' = \frac{e^x + 2x}{\sin(y) + 3y^2 + 3}$

Solution Write y' as $\frac{dy}{dx}$, multiply both sides of the equation by $(\sin(y) + 3y^2 + 3)dy$ and take antiderivative, we have

$$\int \sin(y) + 3y^2 + 3 dy = \int e^x + 2x dx$$

Now using Mathcad or compute by hand, we find

$$\int \sin(y) + 3y^2 + 3 dy = -\cos(y) + y^3 + 3y + C$$

and

$$\int e^x + 2x dx = e^x + x^2 + C$$

So we have an implicitly defined solution $y(x)$ by the equation

$$-\cos(y) + y^3 + 3y = e^x + x^2 + C$$

Here, we can't find an explicit formula for $y(t)$.

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Practice

Project**3. Linear equations and Bernoulli equations****3.1. Linear Equations.** An ODE of form

$$y' + a(x)y = b(x)$$

is called a linear equation. Any equation that can be converted into this form can also be called a linear equation. Such as $x^2y' + 4x^2e^xy = \sin(x)$, which can be written as $y' + e^xy = \frac{\sin(x)}{x^2}$, is a linear equation.

The general solution of the linear equation is

$$y(x) = e^{\int a(x) dx} \left(\int b(x)e^{-\int a(x) dx} dx + C \right)$$

Notice that C is inside the parenthesis. It is common mistake of students forgetting multiplying C by $e^{\int a(x) dx}$.

EXAMPLE 3.1. *According to Newton's law of cooling, the time rate of change of the temperature $T(t)$ of a air conditioned house with external temperature $A(t)$ is proportional to the difference $A - T$. That is,*

$$\frac{dT}{dt} = k(A - T),$$

where k is positive constant. Now, suppose $A(t) = 80 - 10 \cos(\omega t)$ find the general solution.

Solution Since we want to use Mathcad to find the antiderivative for us, the first thing is to find how to get the Greek letter ω . To get ω , first bring the math tool bar from the **View** menu



FIGURE 3. Math Toolbar

Click on the



FIGURE 4. Greek Letter Button

which will bring up



FIGURE 5. Greek Letter Bar

and select the ω letter.

Now if we rewrite and put in $A(t)$

$$\frac{dT}{t} = k(A - T),$$

we get

$$\frac{dT}{t} = k(80 - 10 \cos(\omega t)) - kT,$$

and is a linear equation with $a(t) = k$ and $b(t) = k(80 - 10 \cos(\omega t))$, it easy to see that $\int a(t) dt = \int k dt = kt$. Apply the solution formula we have

$$T(t) = e^{-kt} \left(\int k e^{kt} (80 - 10 \cos(\omega t)) dt + C \right).$$

After bring up the indefinite operator

$$\int \blacksquare d\blacksquare$$

type $k * e^{k * t} (80 - 10 \cos(\omega t))$ as integrant and t as integration variable, press [Ctrl][.], click any place outside the box, we get,

$$\int k e^{kt} (80 - 10 \cos(\omega t)) dt = 80 e^{kt} - 10k \left(\frac{k}{k^2 + \omega^2} e^{kt} \cos(\omega t) + \frac{\omega}{k^2 + \omega^2} e^{-kt} \sin(\omega t) \right)$$

So the solution is

$$T(t) = e^{-kt} \left(80 e^{kt} - 10k \left(\frac{k}{k^2 + \omega^2} e^{kt} \cos(\omega t) + \frac{\omega}{k^2 + \omega^2} e^{kt} \sin(\omega t) \right) + C \right)$$

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EXAMPLE 3.2. If in Example 3.1, we set $k = 0.2$ and $\omega = \frac{\pi}{4}$, we would have general solution

$$T(t) = e^{-0.2t} \left(80e^{0.2t} - 5 \left(\frac{3.2}{0.64 + \pi^2} e^{0.2t} \cos(\omega t) + \frac{4\pi}{0.64 + \pi^2} e^{0.2t} \sin(\omega t) \right) + C \right)$$

The following graph displayed curves of several solutions with different initial values $T(0)$.

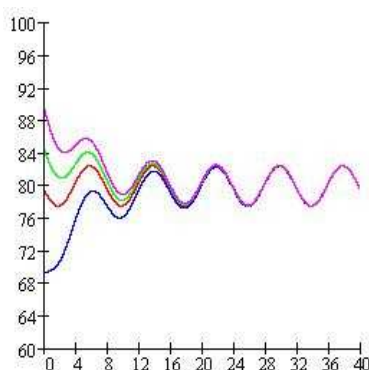


FIGURE 6. Greek Letter Bar

The picture above exhibits the situation when the air condition fails at $t = 0$, so after a out 14 hours, the room temperature is the same as the outside temperature!

Practice

- (1) Find general solution to the following problem

- (a) $y' + 4xy = x^2$
- (b) $e^x y' + e^x y = \sin(x)$
- (c) $xy' = 2y + x^3 \cos(x)$
- (d) $y' = 2xy + 3x^2 e^{x^2}$
- (e) $(x^2 + 4)y' + 3xy = x$
- (f) $(x^2 + 1)y' + 3x^3 y = 6xe^{-\frac{3}{2}x^2}$

- (2) **Application:** The equation $y' = ky$ models wide range of natural phenomena—any involving a quantity whose time rate of change is proportional to its current size.

Continuously Compounded Interest Rate: Let $P(t)$ denote principle at time t , which will earn interest with rate $r(t)$ and compounded continuously, we have $\frac{dP}{dt} = rP$. Now

suppose $r(t) = \cos^2(t)$, find the general solution and graph the solution for several different value for the constant C in the same coordinate. Explain the long time behavior of the solutions.

Drug Elimination: Let $A(t)$ be the amount of certain drug in the bloodstream, measured by the excess over the natural level of the drug. Then in many situations $A(t)$ will decline at a rate proportional to the current excess amount. That is

$$\frac{dA}{dt} = -\lambda A,$$

where $\lambda > 0$ is called the elimination parameter of the drug. Suppose for one type of drug, $\lambda(t) = \frac{1}{t^2+4}$ find the general solution. Discuss its longtime behavior by graphing several solutions in the same coordinate.

3.2. Bernoulli Equations. A Bernoulli equation is an ODE of the form

$$(5) \quad \frac{dy}{dx} + a(x)y = b(x)y^n.$$

You can see that when $n = 1$, we get linear equation. So linear equation is a special case of Bernoulli equation. On the other hand if we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

Dividing both side of (5) by y^n , we have

$$y^{-n} \frac{dy}{dx} + a(x)y^{1-n} = b(x)$$

So

$$(6) \quad \frac{1}{1-n} \frac{dv}{dx} + a(x)v = b(x).$$

That is using the transformation $v = y^{1-n}$, the Bernoulli equation for y becomes a linear equation for v .

Therefore to find the general solution for Bernoulli equation (5) we first find the general solution for the linear equation (6) of v , then using $y = v^{\frac{1}{1-n}}$ to find y .

EXAMPLE 3.3. *In modelling a population with its births(per unit time) proportional to current population level and the deaths(per unit time) is proportional to the square of the current population, we have*

$$\frac{dx}{dt} = ax - bx^2$$

, which is called logistic differential equation. It is a Bernoulli equation with $n = 2$ (Notice it is also a separable equation if a, b are constants.)

Solution Let $v = x^{1-2} = x^{-1}$, then $xv = 1$ apply product rule of differentiation, we have $x'v + v'x = 0$, plug in $x' = ax - bx^2$, we have

$$(ax - bx^2)v + v'x = axv - bx^2v + v'x = a - bx + v'x = 0 \text{ since } xv = 1$$

So

$$(7) \quad a - bx + v'x = 0$$

Divide (7) by x and notice $\frac{1}{x}$ is v , we have

$$v' + av - b = 0,$$

or

$$v' + av = b.$$

Apply the solution formula for linear equation we have

$$v = e^{-\int a dt} \left(\int be^{\int a dt} dt + C \right)$$

So

$$x = \frac{1}{v} = \frac{e^{\int a dt}}{\left(\int be^{\int a dt} dt + C \right)}$$

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EXAMPLE 3.4. *In the logistic model,*

$$\frac{dx}{dt} = ax - bx^2$$

, suppose $a > 0, b > 0$ are constants, we will discuss behavior of the solutions.

Solution From Example 3.3,

$$x = \frac{e^{\int a dt}}{\left(\int b e^{\int a dt} dt + C \right)}$$

So

$$x = \frac{e^{at}}{\left(\int b e^{at} dt + C \right)} = \frac{e^{at}}{\left(\frac{b}{a} e^{at} + C \right)},$$

is the general solution.

When $C = 0$ we have $x = \frac{a}{b}$, a steady-state solution. Also, $x = 0$ is another steady-state solution, which can't be represented in the general solution.

We can rewrite $x(t)$ as

$$x(t) = \frac{a}{b + aC e^{-at}},$$

since $\lim_{t \rightarrow \infty} e^{-at} = 0$ So we have $\lim_{t \rightarrow \infty} x(t) = \frac{a}{b}$. That is, as time goes on the population will gradually settle itself at the level of $\frac{a}{b}$. The following picture shows that the solution curves approach the line $y = \frac{a}{b} = 10$, with $a = 0.2$ and $b = 0.002$.

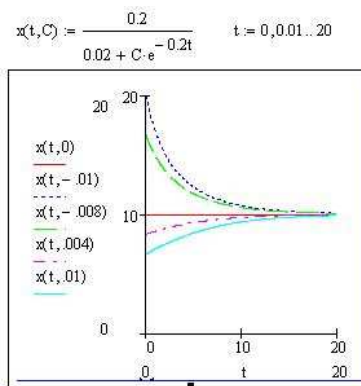


FIGURE 7. Population $a = 0.2$ and $b = 0.002$

Here is how to get the graph in Mathcad :

- we first defined a function $x(t, C) := \frac{0.2}{0.02 + C e^{-0.2t}}$ with two arguments t and C by typing $x(t, C) : 0.2 / 0.02 + C * e^{-0.2t}$,
- then we defined a range variable $t := 0, 0.1 \dots 20$ by typing $t : 0, 0.1 ; 20$,
- finally, type @ at a blank area to get the xy-plot; in the function place holder, we put $x(t, 0), x(t, -0.01), x(t, -0.008), x(t, 0.004), x(t, 0.01)$ and type t in the variable place holder, and click at outside of the box to get the graph.

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4. Initial value problem and existence theorem

The problem of finding a particular solution $x(t)$ of an ODE $x' = f(t, x)$ that satisfying condition $x(t_0) = x_0$ is called an initial value problem. The problem can be written

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

Since solving an ODE requires a lots of hard work, it is wise to know if a given equation has any solution before march on the journey of find a solution. For the linear first order ODE $x' + a(t)x = b(t)$ we following existence-uniqueness theorem.

THEOREM 4.1. *If the function $a(t)$ and $b(t)$ are continuous on the open interval I containing the point t_0 , then the initial value problem $x' + a(t)x = b(t)$, $x(t_0) = x_0$ has a unique solution $x(t)$ on I , given by*

$$(8) \quad x(t) = e^{\int a(t) dt} \left(\int b(t) e^{-\int a(t) dt} dt + C \right)$$

with an appropriate value of C .

So solving an initial value problem for linear first order ODE is easy. One just apply the formula (8) and using the equation $x(t_0) = x_0$ to find C .

EXAMPLE 4.1. *Solve the initial value problem $x' + 2tx = 4t^3$, $x(0) = 1$.*

Solution First we apply the formula (8)

$$x(t) = e^{\int a(t) dt} \left(\int b(t) e^{-\int a(t) dt} dt + C \right),$$

with $a(t) = 2t$ and $b(t) = 4t^3$. So $\int a(t) dt = \int 2t dt = t^2$, and

$$x(t) = e^{-t^2} \left(\int 4t^3 e^{t^2} dt + C \right).$$

Using method of substitution or Mathcad , we find

$$x(t) = e^{-t^2} \left(2t^2 e^{t^2} - 2e^{t^2} + C \right)$$

Let $t = 0$ we have $x(0) = C - 2$. From initial condition $x(0) = 1$ we see that $C = 3$. So the solution is

$$x(t) = e^{-t^2} \left(2t^2 e^{t^2} - 2e^{t^2} + 3 \right)$$

+

However, for general first order ODE , the problem is not so nice, the following theorem is a little bit hard to state and to understand.

THEOREM 4.2. *Let $f(t, x)$ be defined on interval I . If the partial derivative of $f(t, x)$ with respect to y (denoted as $\frac{\partial f(t, x)}{\partial y}$, reads "partial $f(t, x)$ over partial y ") and $f(t, x)$ are continuous at an open disk centered at (t_0, x_0) , then the initial value problem*

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has an unique solution on some open interval containing the point t_0 .

REMARK 4.1. *We will not discuss this existence and uniqueness theorem for general ODE . But we would like to make some comments.*

- To find $\frac{\partial f(t, x)}{\partial t}$, you just need to treat x variable as a constant, for example, let $f(t, x) = t^2 x^3 + 3t - 4x$, we have $\frac{\partial f(t, x)}{\partial t} = 2tx^3 + 3$, here $4x$ is a constant with respect to t , so the derivative is 0, and x^3 is also a constant, so $t^2 x^3$ when taking derivative against t we have $2tx^3$. Therefore finding partial derivative is as easy as to find ordinary derivative.
- This theorem only guarantees a solution defined for t in an open interval, which might be a bounded interval. For example $x' = x^2$, we see that $x(t) = -\frac{1}{t+c}$ is the general solution, if $c = -2$ then the solution is only defined on interval $(0, 2)$ as $x(t) = -\frac{1}{t-2}$ will become undefined at $t = 2$.
- Some time we could have more than one solution that satisfies the same initial condition, for example, for $x' = x^{\frac{2}{3}}$ and $x(0) = 0$ we have two different solution, one is $x(t) = 0$ another one

is $x(t) = t^{\frac{1}{3}}$. The reason is that for $f(t, x) = x^{\frac{2}{3}}$, $\frac{\partial f(t, x)}{\partial x} = \frac{1}{\sqrt[3]{x}}$ which is undefined at $(0, 0)$. So the conditions of the theorem are violated.

5. Phase and vector field diagrams

Sometimes it is more important to know the behavior of solutions than to know precisely detail of point-by-point of some particular solutions. Here we will discuss two diagrams to get an "feel" of behavior of solutions for a ODE without actually find any solution. The first, **phase diagram**, is used for the autonomous equations and is easy to draw. The second, **vector field diagram**, can be used for any equation but harder to draw. These two diagrams make use of the facts:

- (1) The derivative $f'(x)$ represent the slope of tangent line to the graph $y = f(x)$ at the point $(x, f(x))$.
- (2) The sign of $f'(x)$ tell whether the graph is going up or down.
 - If $f'(x) > 0$ the graph is increasing.
 - If $f'(x) < 0$ the graph is going down.
- (3) The second derivative $f''(x)$ gives information about concavity:
 - If $f''(x) > 0$ the graph is concave upward.
 - If $f''(x) < 0$ the graph is concave downward.

The concepts of concavity (up or down) and monotonicity (going up or down) is illustrated in the following diagram.

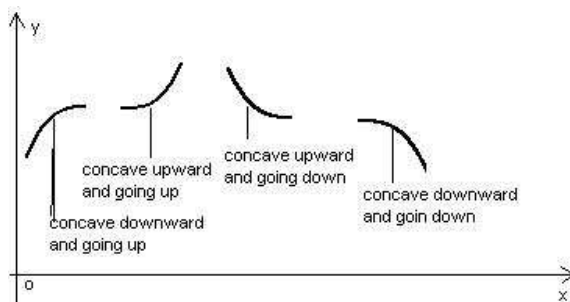


FIGURE 8. Concavity and Monotonicity

5.1. The phase diagram. First-order equations whose right-hand side does not depend on the independent variable are called autonomous. That is, autonomous equations are those of the form $x' = \frac{dx}{dt} = f(x)$ (The independent variable is t) or $y' = \frac{dy}{dx} = f(y)$ (The independent variable is x). To draw a phase diagram of $x' = f(x)$, you would

- Step one:** Set $f(x) = 0$ solve the equation. The solution is called the **critical number** of the equation $x' = f(x)$.
- Step two:** Plot the solution on a horizontal (or vertical) number line. The solutions will divide the line into intervals (segments), pick any number from each interval (segment) and determine the sign of the value of $f(x)$ at the picked value.
- Step three:** Using the sign in **Step two** to draw arrow on each segment, a right (up) arrow for positive sign and a left (down) arrow for negative sign.

EXAMPLE 5.1. Draw phase diagram for $x' = (x - 1)(x + 2)(x - 3)$.

Solution

Step one: Here $f(x) = (x - 1)(x + 2)(x - 3)$ so $f(x) = 0$ gives

$$(x - 1)(x + 2)(x - 3) = 0,$$

and it has three solutions, $x = -2, 1, 3$

Step two: Plot the three solution on the number line,



FIGURE 9. Number Line

we see that we have four intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. The following table organizes our computation,

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
x-value picked	-3	0	2	4
Value of $f(x) = (x-1)(x+2)(x-3)$	$f(-3) = (-3-1)(-3+2)(-3-3) = (-4)(-1)(-6) = -24$	$f(0) = (0-1)(0+2)(0-3) = (-1)(2)(-3) = 6$	$f(2) = (2-1)(2+2)(2-3) = (1)(4)(-1) = -4$	$f(4) = (4-1)(4+2)(4-3) = (3)(6)(1) = 18$
Sign of $f(x)$	-	+	-	+

Step three: Draw the arrows on the number line. A right arrow for positive sign and a left arrow for negative sign. \dashv



FIGURE 10. Phase Diagram

If c is a critical number of $x' = f(x)$, i.e. $f(c) = 0$,

- c is a **sink** if arrows on both side of it point toward it.
- c is a **source** if arrows on both side of it point away from it.
- c is a **saddle** if one arrow points away from and another points toward it.

In Example ??, $c = 1$ is a sink, $c = -2, 3$ are source.

From phase diagram and the sign of derivative of $x'' = \frac{df(x)}{dt} = \frac{df(x)}{dx} \frac{dx}{dt} = f(x) \frac{df(x)}{dx}$, we can get an very good picture about the behavior of the solutions.

EXAMPLE 5.2. For $x' = f(x) = (x - 1)(x + 2)(x + 3)$, we have $f(x) \frac{df(x)}{dx} = (x - 1)(x + 2)(x + 3)(3x^2 - 4x - 5)$, which has the following sign distribution,

- (a) $\frac{d^2x(t)}{dt^2} > 0$ on intervals, $(-2, \frac{2}{3} - \frac{1}{3}\sqrt{19})$, $(1, \frac{2}{3} + \frac{1}{3}\sqrt{19})$, and $(3, \infty)$.
- (b) $\frac{d^2x(t)}{dt^2} < 0$ on intervals, $(\infty, -2)$, $(\frac{2}{3} - \frac{1}{3}\sqrt{19}, 1)$, and $(\frac{2}{3} + \frac{1}{3}\sqrt{19}, 3)$.

With this information and the phase diagram we have the following picture of typical solutions for $x' = (x - 1)(x + 2)(x + 3)$,

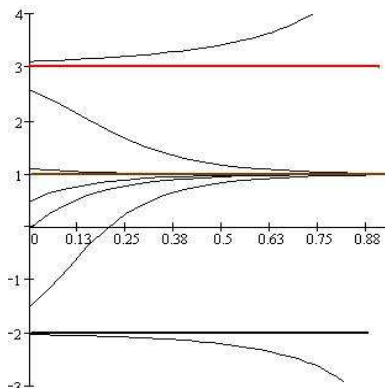
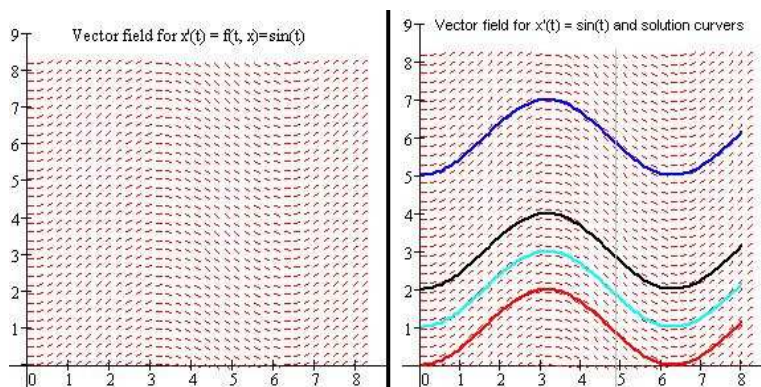


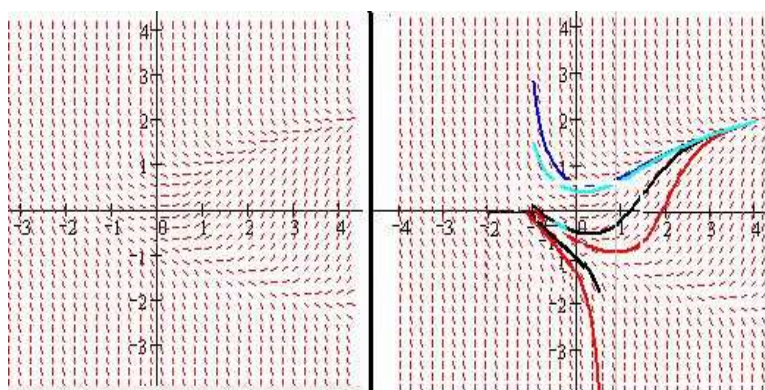
FIGURE 11. Typical solutions for $x' = (x - 1)(x + 2)(x + 3)$

5.2. Vector field diagram. We know that the derivative $x'(t_0)$ of $x(t)$ at t_0 is the slope of the tangent line to the graph of $x(t)$ at the point $(t_0, x(t_0))$, and the equation of the tangent line is $y - x(t_0) = x'(t_0)(t - t_0)$. The idea of vector field for an ODE $x' = f(t, x)$ is to draw a segment of tangent line to many chosen points (t_0, x_0) with slope $x'(t_0) = f(t_0, x_0)$. When we draw enough such line segments, patterns shall emerge and we shall have a clear idea (with some training) of the behavior of solutions to the equation. The following diagram shows the vector field of $x' = \sin(t)$. Clearly we see the wave pattern of typical solution $x(t) = -\cos(t) + C$.

Here two diagrams are shown, the right one is provided for novel eyes, which has solutions curves superimposed on the vector field. For complicated vector field only trained eyes will see the subtle patterns

FIGURE 12. Vector field for $f(t, x) = \sin(t)$

of the dynamics of solutions as shown in the next not so complicated diagram,

FIGURE 13. Vector field for $f(t, x) = t - x^2$

To graph the vector field diagram for $y' = f(x, y)$ by hand over rectangular region $[a, b] \times [c, d]$, you would do the following:

- (1) Divide interval $[a, b]$ into N equal length subintervals and $[c, d]$ into M equal length subintervals, for $[a, b]$ the length is $h = \frac{b-a}{N}$ and the subintervals are $[a, a+h], [a+h, a+2h], \dots$; for $[c, d]$ the subinterval length is $s = \frac{d-c}{M}$ and the subintervals are $[c, c+s], [c+s, c+2s], \dots$.
- (2) If we use $x_i = a + ih, i = 0, 1, \dots, N$ to denote the endings of subintervals of $[a, b]$ and $y_j = c + js, j = 0, 1, 2, \dots$ to denote the endings of subintervals of $[c, d]$, then (x_i, y_j) form the grid points, $i = 0, 1, \dots, N; j = 0, 1, \dots, M$. For each point (x_i, y_j) compute the slope $m_{ij} = f(x_i, y_j)$ and, from the

equation of tangent line $y - y_j = m_{ij}(x - x_i)$ we get another point $((x_i + \frac{h}{2}, y_j + m_{ij}\frac{h}{2}))$.

- (3) Draw line segment from (x_i, y_j) to $((x_i + h, y_j + m_{ij}h)$ for $i = 0, 1, \dots, N; j = 0, 1, \dots, M$.

EXAMPLE 5.3. Suppose we want to graph vector field for $y' = x$ over $[0,1] \times [0,1]$ and divide $[0, 1]$ into two subinterval, $h = s = 0.5$. Then we will have nine grid points $(0, 0)$, $(0, 0.5)$, $(0, 1)$, $(0.5, 0)$, $(0.5, 0.5)$, $(0.5, 1)$, $(1, 0)$, $(1, 0.5)$, and $(1, 1)$. For the grid point $(0.5, 0.5)$ we have slope $m = 0.5$ as the function $f(x, y) = x$. So we can get another point $(0.5 + \frac{h}{2}, 0.5 + m\frac{h}{2}) = (0.75, 0.625)$, connect $(0.5, 0.5)$ to $(0.75, 0.625)$ you get one line segment, continuous this we get the following diagram.

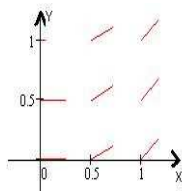


FIGURE 14. Vector field for $f(x, y) = x$

So it is obviously tedious to draw the vector field by hand.

To graph vector field in Mathcad is not an easy task. This is especially true when you want to superimpose solution curve over the graph. You need to write some code to instruct how you want Mathcad to graph the vector field for what function. you can get the code from the website www.unf.edu/~mzhan.

Project

At beginning you should enter: Project title, your name, ss#, and due date in the following format

Project One: Define and Graph Functions

John Doe
SS# 000-00-0000

Due: Mon. Nov. 23rd, 2003

You should format the text region so that the color of text is different than math expression. You can choose color for text from **Format**→**Style** select normal and click **modify**, then change the settings for

font. You can do this for headings etc. **Graph Vector field**

Goal: Familiar your self with an many different kind of vector fields, defined by $x' = f(t, x)$, as possible and identify many general features of solutions, especially the following features,

- Does the equation has constant solutions? A constant solution is shown in vector field as horizontal line segments since constant solution has slope equals zero.
- Does solutions converge to an particular solution in the lang run?
- Do solutions grow to infinite an a finite time? Her vertical bars indicate a solution might become unbounded in short time.
- Do solutions display any periodicity? We know $\sin(t)$ has period of 2π and solution curve repeats itself. Here large interval for t might be needed.

You can use the following functions

- $f(t, x) = \sin(x)$
- $f(t, x) = \cos(t)$
- $f(t, x) = x(x - 1)$
- $f(t, x) = e^t x + \sin(x)$
- $f(t, x) = x + x\sin(t)$

You should use the provide Mathcad file at the website

www.unf.edu/~mzhan/vectorfield.mcd

as a starting point and superimpose solutions to the vector fields so support your observation.