

# Stochastic approximations of perturbed Fredholm Volterra integro-differential equation arising in mathematical neurosciences

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## Abstract

This paper extends the results of synaptically generated wave propagation through a network of connected excitatory neurons to a continuous model, defined by a Fredholm Volterra integro-differential equation (FVIDE), which includes memory effects of the past in the propagation. Stochastic approximation and numerical simulations are discussed.

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## 1. Introduction

Noisy systems can be modelled in several different ways: For example, Langevin's equation describes a linear physical system to which white noise is added, and the linear theory for it has been extended to nonlinear stochastic differential equations with additive white noise, see [8]. This approach is based on additive white noise. Another approach was derived from work of Bogoliubov, see [9,10] on averaging nonlinear oscillatory systems. In this system parameters are allowed to be random processes, and methods based on averaging and ergodic theory provide useful predictions from the model. This approach is referred to as being based on parametric noise. A third approach is to derive models, such as Markov Chains, for the systems state variable as being random variables, and then use the method of probability theory; for analysis, see [11].

Nonlinear Fredholm integro differential equation arises whenever the connections of excitable neurons are distributed according to the normal probability distribution and it is appropriate to take kernel as a Gaussian, or normal distribution of influence from any site. Gaussian functions, or normal distributions, arise naturally in neuroscience model, see [5] and [6]. It is known from the central limit theorem of probability theory that most random processes, when correctly scaled have (approximately) normal distributions. If we include some

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memory effects of the past and then Fredholm integro-differential equation becomes Fredholm Volterra integro-differential equation. The purpose of this paper is to apply a random perturbation method to Fredholm Volterra integro differential equation that has a parametric noise, and demonstrate that for relatively small variance of noise the propagation sustains. We have shown this result using perturbation method in Fredholm Volterra integro differential equation described in [2].

## 2. Model

Consider the initial-value problem for the Fredholm Volterra integro differential equation of convolution type

$$\left\{ \begin{array}{l} \dot{\theta}_x(s, t) = 1 + \cos \theta(s, t) + F(t) \\ \quad + \mu \int_0^t \left( K(t-t', y(t'/\epsilon)) \int_{-\infty}^{\infty} k(s-s') \dot{\theta}_x(s', t') ds' \right) dt' \\ \quad \equiv f_x(s, t, y(t/\epsilon), \theta_x, \alpha) \\ \theta_x(s, 0) = g(s), \end{array} \right. \quad (1)$$

$-\infty \leq s \leq \infty, t \geq 0$  with a given smooth initial function  $g$  and  $\dot{\theta}_x(s', t') \equiv \frac{d}{dt'} \theta_x(s', t')$ .  $\xi$  is vector of random processes that satisfies certain natural conditions to be listed below. The kernel  $K(t, \xi)$  is defined by

$$K(t, \xi) = A(\xi) \frac{1}{\alpha} e^{-\frac{t}{\alpha}}, \quad (2)$$

where  $A(\xi) = \xi$  with  $\alpha > 0$ . In applications  $\xi = y(t/\epsilon)$  and  $\epsilon \ll 1$  is a small parameter. It measures the ratio of time scales between noise (fast) and the system response. The strength of the connection is defined by

$$k(s) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2\sigma^2}} \quad (3)$$

with  $\sigma > 0$ .  $F(t)$  is the forcing function.  $0 < \mu \ll 1$  is small, represents dimensionless parameter, reflecting the strength of connection between neurons.  $\xi$  takes values in a measurable space  $Y \subseteq R^N$ . We suppose that  $\xi$  satisfies the following conditions:

- It is a stationary Markov processes.
- It is ergodic process (in the sense of Birkhoff) having ergodic measure, say  $\rho$ , in  $Y$ .
- It satisfies strong mixing conditions.
- $k$  is a measurable function of  $Y$  with respect to  $\rho$ .

When these conditions are satisfied we can obtain a useful approximations to the solution

$$\theta_x(s, t) = \bar{\theta}_x(s, t) + \sqrt{\epsilon} \tilde{\theta}_x(s, t) + \text{error} \quad (4)$$

and

$$d\tilde{\theta}_x(s, t) = \bar{f}_{\tilde{\theta}_x}(t, s, \bar{\theta}_x, \alpha) \tilde{\theta}_x(s, t) dt + B(s, t) dw, \quad (5)$$

where derivation  $B(s, t)$  is shown in Section 4 and  $dw = \sqrt{dt}N(0, 1)$  and  $\bar{\theta}_x(s, t)$  solves corresponding averaged system is

$$\left\{ \begin{array}{l} \dot{\bar{\theta}}_x(s, t) = 1 + \cos \bar{\theta}(s, t) + F(t) \\ \quad + \mu \int_0^t \left( \bar{K}(t-t') \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \\ \quad \equiv \bar{f}_x(s, t, \bar{\theta}_x, \alpha) \\ \bar{\theta}_x(s, 0) = g(s), \end{array} \right. \quad (6)$$

$-\infty \leq s \leq \infty, t \geq 0$  with initial function  $g$  and  $\dot{\bar{\theta}}_x(s', t') \equiv \frac{d}{dt'} \bar{\theta}_x(s', t')$ .

$$\bar{K}(t) = \int_Y K(t, y) \rho(dy) \equiv \rho_1 K(t, y_1) + \rho_2 K(t, y_2). \tag{7}$$

The problem (1) is a continuous analog of a discrete Voltage Controlled Oscillator Neuron (VCON) model with perturbed kernel of transmission line in neural networks discussed in [5,6]. The angle  $\theta_\alpha(s, t)$  represents the phase at time  $t$  associated with a neuron located at the point  $s$ , and the integral appearing in (1) corresponds to the influence of the neighboring neuron along with some memory effects of the past.

It is shown by numerical experimentation that solution can be expanded using Central Limit Theorem for Markov processes (see [2,3]): In particular

$$\theta_\alpha = \bar{\theta}_\alpha + \sqrt{\epsilon} \tilde{\theta}_\alpha + \text{error}, \tag{8}$$

where

$$d\tilde{\theta}_\alpha = \bar{f}_{\tilde{\theta}_\alpha}(t, s, \bar{\theta}_\alpha, \alpha) \tilde{\theta}_\alpha dt + B(t) dw, \tag{9}$$

where,  $B(t)$  shown in Section 4 and  $dw = \sqrt{dt}N(0, 1)$ .

A solution strategy for (1) is based on the use of Gaussian rules and interpolation to a regular grid on bounded interval for some final time as we used in our earlier paper [4] and we also used random path described by 2-state transition probability matrix. In Section 3, we gave numerical method to solve the perturbed model. In Section 4, we gave experimental results. Finally, we investigate the asymptotic behavior of the model as  $\epsilon \rightarrow 0$ .

### 3. Simulating stochastic process

Noise can arise in the network in a variety of ways:

- The cells have irregularities in their metabolism and in their membrane properties.
- The chemical bath in which they reside fluctuates.
- Input signal arriving from other cells or from outside the body are irregular, etc.

We will use two state Markov process to generate this noise. Thus consider the case where  $y(t) = y(t/\epsilon)$  takes values in a countable set

$$Y = \{y_1, y_2\}, \tag{10}$$

where we choose  $y_1 = 0, y_2 = 1$ . The transition probability matrix  $P(\epsilon, y_i, y_j)$ , where  $i, j = 1, 2$  can be represented by the  $2 \times 2$  matrix  $P$  such that  $P_{i,j} = P(\epsilon, y_i, y_j)$  denotes the probability  $y(\epsilon) = y_j$  when starting with  $y(0) = y_i$ . Let

$$P = \begin{pmatrix} P_{0 \rightarrow 0} & P_{0 \rightarrow 1} \\ P_{1 \rightarrow 0} & P_{1 \rightarrow 1} \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}, \tag{12}$$

where  $0 < a, b < 1$  and  $0 < \epsilon < \frac{2}{a+b}$ . The matrix  $P$  satisfies the following properties (see [5]).

- $P$  is nonnegative (its entries are probabilities).
- $P$  is stochastic, i.e.,  $Pe = e$  (for sure one of the  $y_j$  is the outcome of the transition from  $y_i$ ).
- $P$  is irreducible, i.e., any state  $y_j$  can be eventually be reached in a finite number of steps with nonzero probabilities.
- $P$  is aperiodic (or acyclic, e.g., not a permutation). This implies that  $|\lambda_2| < 1$ .

The above conditions guarantee the existence of a unit vector  $\rho > 0$  such that

$$\rho^T \mathbf{e} = 1 \tag{13}$$

and  $\lim_{k \rightarrow \infty} P^k = \mathbf{e}\rho^T$ . Thus  $\rho$  is the unique positive left eigen vector associated to  $\lambda_1 = 1$  satisfying (13). It is natural to consider the spectral decomposition (see [3]) and thus

$$P = 1 \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix} + (1 - a - b) \begin{pmatrix} \frac{a}{a+b} & \frac{-a}{a+b} \\ \frac{-b}{a+b} & \frac{a}{a+b} \end{pmatrix} = 1P_1 + (1 - a - b)P_2$$

with  $P_i^2 = P_i, P_i P_j = 0$ . Then

$$P^k = (\lambda_1 P_1 + \lambda_2 P_2)^k = \lambda_1^k P_1^k + \lambda_2^k P_2^k = P_1 + (1 - a - b)^k P_2 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} = \mathbf{e}\rho^T.$$

Thus  $\rho_1 = \frac{b}{a+b}$  and  $\rho_2 = \frac{a}{a+b}$ . This tells us such a system, in the long run, will be in state 0 with probability  $\frac{b}{a+b}$  and in state 1 with probability  $\frac{a}{a+b}$ , irrespective of the initial state in which the system started.

#### 4. An Euler–Hermite approach

In this section we will describe numerical method for where integral term  $\int_{-\infty}^{\infty}$  is approximated by the Gauss–Hermite quadrature rule for the kernel  $k$  defined by (1). Then using (1) in (1) we have

$$\begin{cases} \dot{\theta}_x(s, t) = 1 + \cos \theta_x(s, t) + F(t) \\ \quad + \mu \int_0^t \left( K(t - t', y(t'/\epsilon)) \frac{\mu}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(s-s')^2}{\sigma^2}} \dot{\theta}_x(s', t') ds' \right) dt' \\ \theta_x(s, 0) = g(s), \end{cases} \tag{14}$$

$-\infty \leq s \leq \infty, t \geq 0$ . Using change of variables  $\frac{s-s'}{\sigma} = \eta, s' = s - \sigma\eta, ds' = -\sigma d\eta$ , (14) takes the form,

$$\begin{cases} \dot{\theta}_x(s, t) = 1 + \cos \theta_x(s, t) + F(t) \\ \quad + \frac{\mu}{\sqrt{2\pi}} \int_0^t \left( K(t - t', y(t'/\epsilon)) \int_{-\infty}^{\infty} e^{-\eta^2} \dot{\theta}_x(s - \sigma\eta, t') d\eta \right) dt' \\ \theta_x(s, 0) = g(s), \end{cases} \tag{15}$$

$-\infty \leq s \leq \infty, t \geq 0$ . To approximate (15) we restrict the range of the variable  $s$  to the symmetric interval  $[-A, A]$ , where  $A$  is large enough, and introduce the grid  $s_j = -A + (j - 1)\Delta s, j = 1, 2, 3, \dots, 2N + 1$ . Then  $s_1 = -A, s_{N+1} = 0, s_{2N+1} = A$ . Then (15) for  $s = s_j, j = 1, \dots, 2N + 1$ , and  $\bar{\mu} = \frac{\mu}{\sqrt{2\pi}}$  takes the form,

$$\begin{cases} \dot{\theta}_x(s_j, t) = 1 + \cos \theta_x(s_j, t) + F(t) \\ \quad + \bar{\mu} \int_0^t \left( K(t - t', y(t'/\epsilon)) \int_{-\infty}^{\infty} e^{-\eta^2} \dot{\theta}_x(s_j - \sigma\eta, t') d\eta \right) dt' \\ \theta_x(s_j, 0) = g(s_j), \end{cases} \tag{16}$$

$j = 1, \dots, 2N + 1$ . We discretize by the Gauss–Hermite quadrature rule

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sum_{k=1}^n w_k f(x_k) + \frac{n! \sqrt{\pi}}{2^n (2n)!} f^{(2n)}(\xi) \tag{17}$$

$-\infty \leq \xi \leq \infty$ . The rule (17) is convergent if the function  $f$  satisfies the inequality

$$|f(x)| \leq \frac{e^{x^2}}{|x|^{1+\rho}}$$

for some  $\rho > 0$ , compare [7]. Moreover, if  $f$  has bounded derivatives of any order then the errors of (17) tend to zero as  $n \rightarrow \infty$  with spectral rate. Then,

$$\begin{cases} \dot{\theta}_\alpha(s_j, t) = 1 + \cos \theta_\alpha(s_j, t) + F(t) + \bar{\mu} \int_0^t \left( K(t-t', y(t'/\epsilon)) \sum_{k=1}^{NH} w_k \dot{\theta}_\alpha(s_j - \sigma\eta_k, t') \right) dt' \\ \theta_\alpha(s_j, 0) = g(s_j), \end{cases} \tag{18}$$

$j = 1, 2, \dots, 2N + 1$ . Here  $w_1, \dots, w_{NH}$  and  $\eta_1, \eta_2, \dots, \eta_{NH}$  are the weights and abscissas of the Gauss–Hermite quadrature rules, which can be computed using eigen values of the tridiagonal matrices (see [7]). Now introducing the grid  $t_i = (i - 1)\Delta t$ ,  $\Delta t = \frac{T}{M}$ ,  $i = 1, 2, \dots, M + 1$  on the time interval  $[0, T]$ . Then for  $t = t_i$ , (18) takes the form

$$\begin{cases} \dot{\theta}_\alpha(s_j, t_i) = 1 + \cos \theta_\alpha(s_j, t_i) + F(t(i)) + \bar{\mu} \int_0^{t_i} \left( K(t_i - t', y(t'/\epsilon)) \sum_{k=1}^{NH} w_k \dot{\theta}_\alpha(s_j - \sigma\eta_k, t') \right) dt' \\ \theta_\alpha(s_j, t_1) = g(s_j), \end{cases} \tag{19}$$

$j = 1, 2, \dots, 2N + 1$ . This implies

$$\begin{cases} \dot{\theta}_\alpha(s_j, t_i) = 1 + \cos \theta_\alpha(s_j, t_i) + F(t(i)) + \bar{\mu} \sum_{l=1}^{i-1} \int_{t_l}^{t_{l+1}} \left( K(t_i - t', y(t'/\epsilon)) \sum_{k=1}^{NH} w_k \dot{\theta}_\alpha(s_j - \sigma\eta_k, t') \right) dt' \\ \theta_\alpha(s_j, t_1) = g(s_j), \end{cases} \tag{20}$$

where  $j = 1, 2, \dots, 2N + 1, i = 1, 2, \dots, M + 1$ . We will approximate  $\dot{\theta}_\alpha(s_j - \sigma\eta_k, t')$  by  $\dot{\theta}_\alpha(s_j - \sigma\eta_k, t_l)$ , if  $t' \in [t_l, t_{l+1}]$

$$\begin{cases} \dot{\theta}_\alpha(s_j, t_i) = 1 + \cos \theta_\alpha(s_j, t_i) + F(t(i)) + \bar{\mu} \sum_{l=1}^{i-1} \int_{t_l}^{t_{l+1}} \left( K(t_i - t', y(t'/\epsilon)) \sum_{k=1}^{NH} w_k \dot{\theta}_\alpha(s_j - \sigma\eta_k, t_l) \right) dt' \\ \theta_\alpha(s_j, t_1) = g(s_j), \end{cases} \tag{21}$$

$j = 1, 2, \dots, 2N + 1, i = 1, 2, \dots, M + 1$ . Now introducing mini-grid  $t'_n = t_l + (n - 1)\frac{\Delta t}{M_\epsilon}$ ,  $n = 1, 2, \dots, M_\epsilon + 1$ ,  $M_\epsilon = \frac{1}{\epsilon}$ . Then  $t'_1 = t_l, t'_{M_\epsilon+1} = t_{l+1}$ . Then (21) takes the form

$$\begin{cases} \dot{\theta}_\alpha(s_j, t_i) = 1 + \cos \theta_\alpha(s_j, t_i) + F(t(i)) + \bar{\mu} \sum_{l=1}^{i-1} \frac{\Delta t}{M_\epsilon} \sum_{n=1}^{M_\epsilon} \left( K(t_i - t'_n, y(t'_n/\epsilon)) * \sum_{k=1}^{NH} w_k \dot{\theta}_\alpha(s_j - \sigma\eta_k, t_l) \right) dt' \\ \theta_\alpha(s_j, t_1) = g(s_j), \end{cases} \tag{22}$$

$j = 1, 2, \dots, 2N + 1, i = 1, 2, \dots, M + 1$ . Discretizing  $\dot{\theta}_\alpha(s_j, t_i)$  and  $\dot{\theta}_\alpha(s_j - \sigma\eta_k, t_l)$  appearing in (22) takes the form

$$\begin{cases} \frac{\theta_\alpha(s_j, t_{l+1}) - \theta_\alpha(s_j, t_l)}{\Delta t} = 1 + \cos \theta_\alpha(s_j, t_i) + F(t(i)) + \frac{\bar{\mu}}{M_\epsilon} \sum_{l=1}^{i-1} \sum_{n=1}^{M_\epsilon} (K(t_i - t'_n, y(t'_n/\epsilon)) \\ * \sum_{k=1}^{NH} w_k (\theta_\alpha(s_j - \sigma\eta_k, t_{l+1}) - \theta_\alpha(s_j - \sigma\eta_k, t_l))) dt' \\ \theta_\alpha(s_j, t_1) = g(s_j), \end{cases} \tag{23}$$

$j = 1, 2, \dots, 2N + 1, i = 1, 2, \dots, M + 1$ . Then for  $n = 1, 2, \dots, M_\epsilon + 1$ ,

$$\frac{t'_n}{\epsilon} = M_\epsilon t'_n = M_\epsilon \left( t_l + (n - 1) \frac{\Delta t}{M_\epsilon} \right) = M_\epsilon \left( (l - 1)\Delta t + (n - 1) \frac{\Delta t}{M_\epsilon} \right) = \underbrace{\left( (l - 1)M_\epsilon + n - 1 \right)}_{\text{index}} \Delta t \tag{24}$$

so that  $y\left(\frac{t_i}{\epsilon}\right) = y((l-1)M_\epsilon + n)$  and thus (24) takes the form

$$\begin{cases} \theta_x(s_j, t_{i+1}) = \theta_x(s_j, t_i) + \Delta t(1 + \cos \theta_x(s_j, t_i) + F(t(i))) \\ \quad + \frac{\bar{\mu}\Delta t}{M_\epsilon} \sum_{l=1}^{i-1} \sum_{n=1}^{M_\epsilon} \left( K(t_i - t_n^l, y((l-1)M_\epsilon + n)) * \sum_{k=1}^{NH} w_k(\theta_x(s_j - \sigma\eta_k, t_{l+1}) - \theta_x(s_j - \sigma\eta_k, t_l)) \right) dt' \\ \theta_x(s_j, t_1) = g(s_j), \end{cases} \tag{25}$$

$j = 1, 2, \dots, 2N + 1, i = 1, 2, \dots, M + 1.$

Thus using (25) Euler–Hermite method takes the form

$$\begin{cases} \theta_x(s_j, t_{i+1}) = \theta_x(s_j, t_i) + \Delta t(1 + \cos \theta_x(s_j, t_i) + F(t(i))) + \bar{\mu} \frac{\Delta t}{M_\epsilon} \sum_{l=1}^{i-1} \sum_{n=1}^{M_\epsilon} K(t_i - t_n^l, y(n + M_\epsilon(l-1))) \\ \quad * \sum_{k=1}^{NH} w_k(\theta_x(s_j - \sigma\eta_k, t_{l+1}) - \theta_x(s_j - \sigma\eta_k, t_l)) \\ \theta_x(s_j, t_1) = g(s(j)), \quad j = 1, 2, \dots, 2N + 1, \quad i = 1, 2, \dots, M. \end{cases} \tag{26}$$

We have to approximate  $\theta_x(s_j - \sigma\eta_k, t_{l+1})$  and  $\theta_x(s_j - \sigma\eta_k, t_l)$  using the values  $\theta_x(s_j, t_l)$  and  $\theta_x(s_j, t_{l+1})$  defined at the grid points  $s_j$ . If

$$s_j - \sigma\eta_k \leq s_1$$

then for any  $t$  we use

$$\theta_x(s_j - \sigma\eta_k, t) = \theta_x(s_j, t).$$

If

$$s(1) < s_j - \eta_k \leq s_{2N+1}$$

then we use piecewise linear interpolation

$$\theta_x(s_j - \sigma\eta_k, t) = \theta(s_{j-1}, l + 1) + (\theta_x(s_j, l + 1) - \theta_x(s_j, l)) * \frac{s_j - \sigma\eta_k - s_{j-1}}{\Delta s},$$

where  $s_j - \sigma\eta_k \in (x_{q-1}, s_q]$ . This corresponds to the index  $q$  defined by

$$q = \text{ceil}\left(\frac{s_j - \sigma\eta_k - s_1}{\Delta t}\right) + 1.$$

If  $s_j - \sigma\eta_k \geq s_{2N+1} = A$ , we choose

$$\theta_x(s_j - \sigma\eta_k, t) = \theta_x(s_{2N+1}, t).$$

We also generate random sequence  $y$  using random number in MATLAB generated function `rand` for two state transition matrix (see Section 2) given by (8) with initial state  $y(1) = 0$ . Since  $t_i = (i-1)\Delta t, i = 1, 2, \dots, M + 1, M + 2, \dots, M_\epsilon M + 1$ . Then  $t_{M_\epsilon M + 1} = M_\epsilon M \Delta t = M_\epsilon M \frac{T}{M} = M_\epsilon T$ . Thus for  $i = 1, 2, \dots, M_\epsilon * M$ , if `rand` <  $1 - a$  and  $y(i) = 0$  then  $y(i + 1) = 0$ ; if `rand`  $\leq 1 - a$  and  $y(i) = 0$  then  $y(i + 1) = 1$ ; if `rand` <  $b$  and  $y(i) = 1$  then  $y(i + 1) = 0$ ; if `rand`  $\leq b$  and  $y(i) = 1$  then  $y(i + 1) = 1$ .

### 5. Stochastic approximation

In this section we derived explicit expression for Stochastic Correction Equation defined by (9) to estimate the asymptotic behavior of

$$|\theta_x - (\bar{\theta}_x + \sqrt{\epsilon}\tilde{\theta}_x)| = o(\epsilon) \tag{27}$$

as  $\epsilon \rightarrow 0$ , where  $\theta_x$  and  $\bar{\theta}_x$  are the solution of (1) and (6) respectively and  $\tilde{\theta}_x$  is the solution of the Stochastic Correction Eq. (9), for more detail see [3], is given by

$$B^2(t) = \frac{b}{a(a+b)} g(t, s, \theta_x, y_1, \alpha)^2 [2 - a - b], \tag{28}$$

where derivation of  $g(t, s, \theta_x, y_1, \alpha)$  is obtained using  $f(t, s, \bar{\theta}_x, y_1, \alpha)$ ,  $f(t, s, \bar{\theta}_x, y_2, \alpha)$  and  $\bar{f}(t, s, \bar{\theta}_x, \alpha)$ , such that  $\rho_1 g(t, s, \theta_x, y_1, \alpha) + \rho_2 g(t, s, \theta_x, y_2, \alpha) = 0$ , with  $\rho_1 + \rho_2 = 1$ , where

$$g(t, s, \theta_x, y_1, \alpha) \equiv \bar{f}(t, s, \bar{\theta}_x, \alpha) - f(t, s, \bar{\theta}_x, y_1, \alpha) \tag{29}$$

and

$$g(t, s, \theta_x, y_2, \alpha) \equiv \bar{f}(t, s, \bar{\theta}_x, \alpha) - f(t, s, \bar{\theta}_x, y_2, \alpha). \tag{30}$$

Thus using (7), (6) takes the form

$$\begin{aligned} \bar{f}(t, s, \bar{\theta}_x, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) \\ &+ \mu \int_0^t \left( (\rho_1 K(t-t', y_1) + \rho_2 K(t-t', y_2)) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt'. \end{aligned} \tag{31}$$

Thus again using (2), we obtain

$$\begin{aligned} \bar{f}(t, s, \bar{\theta}_x, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) \\ &+ \mu \int_0^t \left( \left( \rho_1 y_1 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} + \rho_2 y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \right) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt'. \end{aligned} \tag{32}$$

Then using  $y = y_1$  and  $y = y_2$ , in (1) we have

$$\begin{aligned} f(t, s, \bar{\theta}_x, y_1, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) + \mu \int_0^t \left( \bar{K}(t-t', y_1) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \\ &= 1 + \cos \bar{\theta}_x(s, t) + F(t) \end{aligned} \tag{33}$$

and

$$\begin{aligned} f(t, s, \bar{\theta}_x, y_2, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) + \mu \int_0^t \left( \bar{K}(t-t', y_2) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \\ &= 1 + \cos \bar{\theta}_x(s, t) + F(t) + \mu \int_0^t \left( \rho_1 y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt'. \end{aligned} \tag{34}$$

Thus using (33) and (31), (29) becomes,

$$\begin{aligned} g(t, s, \theta_x, y_1, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) \\ &+ \mu \int_0^t \left( \left( \rho_1 y_1 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} + \rho_2 y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \right) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' - 1 \\ &- \cos \theta_x(s, t) - F(t) - \mu \int_0^t \left( y_1 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \\ &= \mu \int_0^t \left( \rho_2 y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \end{aligned} \tag{35}$$

and thus using (34), (31), and (30) becomes,

$$\begin{aligned} g(t, s, \theta_x, y_2, \alpha) &= 1 + \cos \bar{\theta}_x(s, t) + F(t) \\ &+ \mu \int_0^t \left( \left( \rho_1 y_1 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} + \rho_2 y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \right) \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' - 1 \\ &- \cos \theta_x(s, t) - F(t) - \mu \int_0^t \left( y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \\ &= \mu \int_0^t \left( (\rho_2 - 1) y_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s-s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt'. \end{aligned} \tag{36}$$

Using (29), (28), takes the form

$$B^2(s, t) = \frac{b(2 - a - b)}{a(a + b)} \left( \mu \int_0^t \left( \rho_2 \gamma_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s - s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \right)^2. \tag{37}$$

Thus

$$B(s, t) = \sqrt{\frac{b(2 - a - b)}{a(a + b)}} \mu \int_0^t \left( \rho_2 \gamma_2 \frac{1}{\alpha} e^{\frac{-1}{\alpha}(t-t')} \int_{-\infty}^{\infty} k(s - s') \dot{\bar{\theta}}_x(s', t') ds' \right) dt' \tag{38}$$

and thus the Stochastic correction equation becomes

$$d\bar{\theta}_x = [-\sin \bar{\theta}_x(s, t)] \tilde{\theta}_x dt + B(s, t) dw. \tag{39}$$

### 6. Numerical results

We solved Eq. (1) numerically over time interval [0, 10] using

1. Euler–Hermite with smaller time scale.
2. Evaluate  $\bar{\theta}_x(s, t) + \sqrt{\epsilon} \tilde{\theta}_x(s, t)$ , where  $\bar{\theta}_x(s, t)$  and  $\tilde{\theta}_x(s, t)$  are computed using Eqs. (6) and (39) respectively.
3. Compare 2 with 1 where  $\bar{\theta}_x(s, t)$  is the solution of (6) using large time scale,  $H$  and  $B^2(s, t)$  is also computed using the solution of (38).

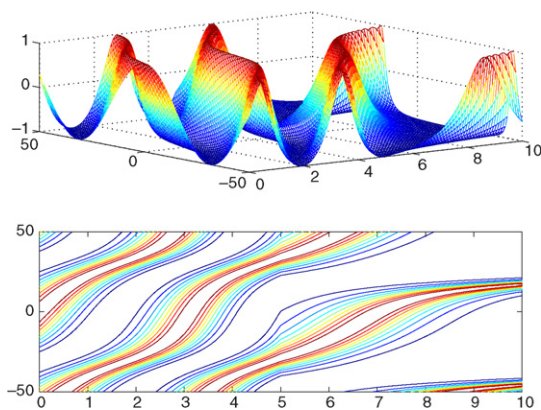


Fig. 1. Contour plot of  $\cos \theta_x(s, t)$  for the Euler–Hermite quadrature method with piecewise linear interpolation of FVIDE.

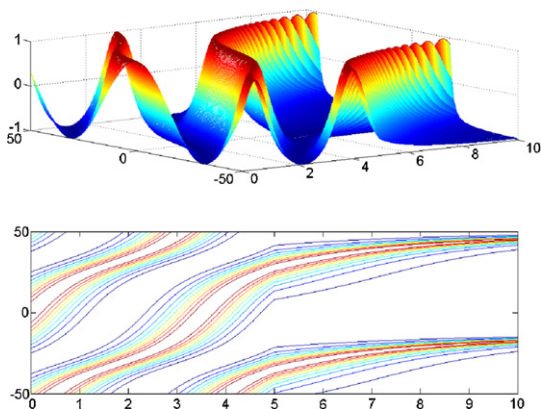


Fig. 2. Contour plot of  $\cos \theta_x(s, t)$  for Euler–Hermite quadrature method with piecewise linear interpolation of average FVIDE.

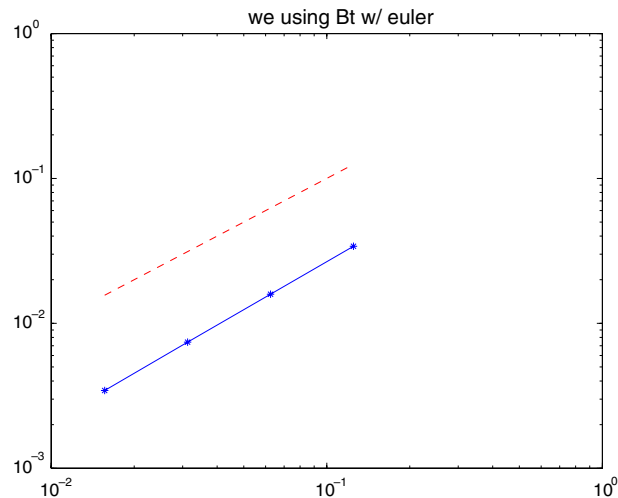


Fig. 3. Strong error plots using Euler–Hermite and dashed line is the line with slope 1.

Using  $E|\theta_x(s, t) - (\bar{\theta}_x(s, t) + \sqrt{\epsilon}\tilde{\theta}_x(s, t))|$ , where  $E$  denotes the expected value, leads the concept of strong convergence. Thus numerical experimentations suggests that  $\theta_x(s, t) = \bar{\theta}_x(s, t) + \sqrt{\epsilon}\tilde{\theta}_x(s, t)$  for  $\epsilon \rightarrow 0$  for perturbed Fredholm Volterra integro-differential equation. Thus the asymptotic result for Volterra integral equations given in, [2] can be extended to more general FVIDE (see Figs. 1–3).

## 7. Concluding remarks

In simulation we observed, the elapsed time for solving the averaged system approximately 0.001 times that for solving the perturbed system. While experimenting the error, we implicitly assumed that number of other sources of error are negligible, including the following, for more detail, (see [1]):

- Sampling error: error arising from approximating an expected value by sample mean.
- Random number bias: inherent error in the random generator.
- Rounding error: floating point roundoff errors.

For a typical computation the sampling error is likely to be the most significant of these three. In preparing the programs in these simulations we found that some experimentation is required to make the number of samples sufficiently large and the time step is sufficiently small for the predicted order of convergence to be observable. Experimentation indicates that as step size decreases, the lack of independence in the samples from a random generator typically degrades the computation before rounding errors becomes significant. Future generalization of this work can be extended in the networks of excitatory and inhibitory neurons with sparse and random connectivity.

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