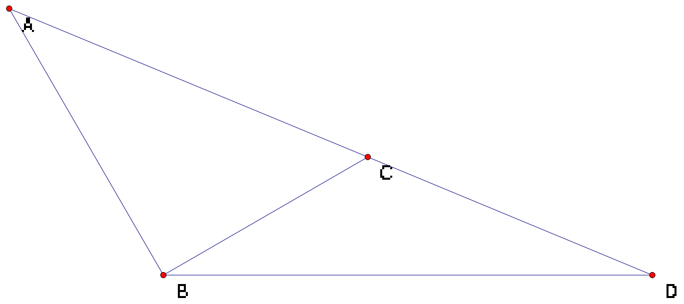
# Proof of the Cube Root Construction

We begin with the following diagram:




The diagram is constructed with the following conditions:

- $\overline{AB} = \overline{CD} = 1$
- $\angle ABC = 90°$ and $\angle CBD = 30°$

The claim is that the segment $\overline{AC}$ has length $\sqrt[3]{2}$.

**Proof:** Just to make things easier, we will use $x$ to represent the length of $\overline{AC}$. By the Pythagorean Theorem, we can say that the length of $\overline{BC}$ is $\sqrt{x^2 - 1}$. By addition, we can say the length of $\overline{AD}$ is $x + 1$.

The *Law of Sines* says that if $ABC$ is a triangle with side lengths $a$, $b$, and $c$ opposite the angles at $A$, $B$, and $C$ respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Using this on the small triangle $BCD$, we can say:

$$\frac{\sin \angle BDC}{\overline{BC}} = \frac{\sin \angle CBD}{\overline{CD}}$$

Substituting all we know, this says:

$$\frac{\sin \angle BDC}{\sqrt{x^2 - 1}} = \frac{\sin 30°}{1}$$

Square this to say $\sin^2 \angle BDC = (x^2 - 1)/4$.

Now we use the law of sines on the big triangle $ABD$:

$$\frac{\sin \angle ADB}{\overline{AB}} = \frac{\angle ABD}{\overline{AD}}$$

Substituting all we know, this says:

$$\frac{\sin \angle ADB}{1} = \frac{\sin 120°}{x + 1}$$

Square this to say $\sin^2 \angle ADB = 3/(4(x+1)^2)$.

Finally, since $\angle ADB = \angle BDC$, we can set these two equations together:

$$\frac{3}{4(x+1)^2} = \frac{x^2 - 1}{2}$$

Cross multiply and expand, and this reduces to:

$$x^4 + 2x^3 - 2x - 4 = 0$$

which factors to

$$(x^3 - 2)(x + 2) = 0$$

The only positive root of this equation is $\sqrt[3]{2}$!!