The Rules: (They are the same as last time.)

1) You may use any non-human source on this exam, including books, calculators, Maple, notes, websites, and code written for homework. You MAY NOT ask anyone about problems on this exam except for Dr. Dan. Now that you have the exam in hand, you MAY NOT share code with any other student in the class. Unless specifically restricted, you may use any built-in function in Maple.

2) Problems can be submitted as Maple worksheets or they can be submitted on paper, depending on what works best for you. If you submit worksheets, make absolutely certain that I can read your work. Also make sure that you submit all necessary work. As usual, worksheets will be submitted via Blackboard.

3) Exam is due on November 17 at 6:00. Late exams will NOT be excepted. If you don’t finish the exam, then submit all that you have done.

4) When submitting the exam, write out the following and sign it: "I did not give help on this exam to any other student, and I did not receive help on this exam from any other student."

5) Please note that I will not provide as much help on this exam as I have on the homework (meaning I may not answer your questions). I am still willing to help you as much as needed on Maple.

6) In case you didn’t get the point last time: past experience would imply that you should quit reading these rules and start this exam now. DO NOT wait until Monday before starting this.

7) Enjoy!

The Exam:

1) A formula for the $n$-th Legendre polynomial is:

$$P_n(x) = \frac{(-1)^n n!}{(2n)!} \frac{d^n}{dx^n} (1 - x^2)^n$$

a) Use this formula to find $P_6(x)$.

b) Use this formula to prove

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0$$

whenever $m \neq n$. This problem requires a major application of integration by parts.

c) Use the formula to show that all of the roots of $P_n(x)$ occur between $-1$ and $1$. (Don’t do this problem. If I figure out a good way to do it, then I’ll give you a major hint, and then
you can do the problem. Unless I give you a major hint, you won’t need to do or turn in part c.)

2) a) Prove that the quadrature formula

$$\sum_{i=1}^{n} c_i f(x_i)$$

on \([a, b]\) cannot have a degree of accuracy greater than \(2n - 1\), no matter what the \(c_i\)'s or the \(x_i\)'s are. You should do this by beginning with a fixed set of \(c_i\)'s and \(x_i\)'s and then constructing a \(2n\) degree polynomial whose integral on \([a, b]\) is guaranteed to be different from the quadrature formula.

b) Show that the Gaussian quadrature using two points is the only two-point quadrature formula on \([-1, 1]\) with a degree of accuracy equal to 3. This can also be done for the \(n\)-point Gaussian quadrature (though I think it would be difficult to do a general case, and I am not asking you to do it). Combining this with part a), we have shown that Gaussian quadrature is the best quadrature (at least as far as degree of accuracy is concerned).

3) a) Hey, ya know, we never talked about error terms for Gaussian quadrature! Silly us. So let’s do that. Find an error term for the three-point Gaussian quadrature on \([-1, 1]\). You are going to do this in the same way you did it for Simpson’s rule in the homework: calculate

$$\int_{-1}^{1} f(x) \, dx = \frac{5}{9} f \left( -\sqrt{\frac{3}{5}} \right) + \frac{8}{9} f(0) + \frac{5}{9} f \left( \sqrt{\frac{3}{5}} \right) + k f^{(6)}(\xi)$$

using \(f(x) = x^6\) to determine what \(k\) has to be.

b) Verify your error is correct by approximating the integral of \(f(x) = \cos(\pi x/2 + \pi/4)\) on \([-1, 1]\) using the three-point Gaussian quadrature, and then comparing the real error to the estimated error.

4) a) Find the formula for the closed Newton-Cotes quadrature for a function \(f(x)\) on the interval \([a, b]\) with \(n = 8\). Don’t need the error term yet: that’s part c). And give me the formula in terms of fractions instead of decimals. It looks better.

b) Give the formula for Romberg \(R_{4,4}\) for a function \(f(x)\) on the interval \([a, b]\). Note that quadrature formulas from a) and b) use the same nodes, but they have different coefficients.

c) Determine the degree of accuracy and the error term for both quadrature formulas. For the Newton-Cotes, you are allowed to use theorems if you wish, but for Romberg, you will have to do something like you did in problem 3.

5) Prove that \(B_i^k(x) \in C^{k-1}(\mathbb{R})\) for \(k \geq 1\). What, you need help? Fine. Let’s break it down. First, since \(B_i^k(x)\) is just a translation of \(B_i^k(x)\), we only need to prove this statement for
So we will ignore the existence of any of the other B-splines, and we will rewrite our recursion formula as:

\[ B^0_k(x) = \frac{x}{k} B^0_{k-1}(x) + \frac{k+1-x}{k} B^0_{k-1}(x-1) \]

And now for a new notation: let \( C^i_k(x) \), \( 0 \leq i \leq k \), be the polynomial piece of \( B^0_k(x) \) on the interval \([i, i+1)\). For instance, one of your homework problems boiled down to finding \( C^0_3(x) \), \( C^1_3(x) \), \( C^2_3(x) \) and \( C^3_3(x) \). Ok, ready?

a) Prove the degree of the polynomial \( C^i_k(x) \) is equal to \( k \), and its leading coefficient is

\[ \frac{(-1)^i}{k!} \binom{k}{i} \]

This is an induction proof, and you have my blessing to assume the very well known fact:

\[ \binom{n}{j} + \binom{n}{j+1} = \binom{n+1}{j+1} \]

b) Prove that if \( B^0_k \) is \( k-1 \) differentiable, then \( B^{k+1}_0 \) must be at least \( k-1 \) differentiable. Hence it remains to show that the \( k \)-th derivative of \( B^{k+1}_0 \) is continuous.

c) Prove that \( \frac{d^k}{dx^k}B^{k+1}_0 \) is continuous. Since it is a piecewise polynomial, you only need to prove continuity at the break points, i.e., at the integers. This problem will boil down to showing

\[
\frac{d^k}{dx^k}C^{k+1}_0(0) = 0 \\
\frac{d^k}{dx^k}C^{k+1}_i(i+1) = \frac{d^k}{dx^k}C^{i+1}_{k+1}(i+1) \quad 0 \leq i \leq k \\
\frac{d^k}{dx^k}C^{k+1}_{k+1}(k+2) = 0
\]

6) On the web is a file test2-pts.mws. In it is a collection of the \( x \)-coordinate and the \( y \)-coordinate samples of a function \( f(x) \).

a) Approximate the derivative \( f'(x) \) at each point. Use the five-point formulas.

b) Approximate the integral of \( f(x) \) on \([0, 1] \). Use composite Simpson’s rule.

c) These points were sampled from a function \( f(x) \) such that \( |f^{(n)}(\xi)| \leq 2 \) for all \( n \) and all \( \xi \). Find an error bound for the derivatives and the integral from parts a) and b).

7) Use Taylor series to show that the value \( f'''(x_0) \) can be approximated by

\[
\frac{1}{2h^3}(f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h))
\]

Find the error term for this approximation.