

Transformations of measure on infinite-dimensional vector spaces*

Denis Bell

Department of Mathematics, University of North Florida
4567 St. Johns Bluff Road South, Jacksonville, FL 32224, U. S. A.
email: dbell@unf.edu

*This paper has appeared in the volume *Seminar on Stochastic Processes*, Vancouver, 1990. Birkhauser, Boston, 1991.

1. Introduction

Let E denote a Banach space equipped with a finite Borel measure ν , $T : E \mapsto E$ a measurable transformation of E , and ν_T the image measure of ν under T , defined by $\nu_T(B) = \nu(T^{-1}(B))$ for Borel sets B . A *transformation theorem* for ν is a result that gives conditions on T under which the measures ν_T and ν are mutually absolutely continuous, and a formula for the corresponding Radon-Nikodym derivatives (RND) when these conditions hold. If E is finite-dimensional then the problem of transformation of measure falls within the scope of the change of variables (Jacobi) theorem. In the infinite-dimensional case the problem is more subtle. In particular, there exists no translation-invariant reference measure. In this setting the study of transformation of measure has largely been restricted to cases where the measure is Gaussian.

In this article we discuss a scheme introduced by the author in [B.2, Section 2] for obtaining transformation theorems for arbitrary Borel measures defined on (finite or) infinite-dimensional vector spaces. Although formal, it is hoped that this procedure can be made rigorous in specific cases where additional structure is assumed. In the general setting discussed here, it yields a formula for the RND $d\nu_T/d\nu$ that we believe to be new.

2. Transformation theorems for Gaussian measure

These come in two varieties, classical and abstract. The theory of transformation of the classical Wiener measure was developed by Cameron and Martin, and Girsanov. Girsanov's theorem is as follows.

Theorem 1 (Girsanov's theorem) *Let w denote a standard real-valued Wiener process and ν the law of w (i.e. the Wiener measure). Let h be a bounded measurable process adapted to the filtration of w and consider the process y defined by*

$$y_t = w_t - \int_0^t h_s ds, \quad t \in [0, 1].$$

Then y is a standard Wiener process with respect to the measure $d\mu(w) \equiv G(w)d\nu(w)$, where

$$G(w) \equiv \exp \left\{ \int_0^1 h_s dw_s - \frac{1}{2} \int_0^1 h_s^2 ds \right\}. \quad (1)$$

A series of (increasingly more general) results have been proved concerning the transformation of abstract Gaussian measure. The quintessential paper in this area is due to Ramer [R]. Let (i, H, E) denote an abstract Wiener spaces as defined by Gross [G]. Let ν denote the corresponding Gaussian measure on E , and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, the inner product and norm on H , respectively.

Theorem 2 (Ramer) *Suppose U is an open subset of E and $T : U \mapsto E$ is a map of the form $I + K$, where I is the identity map on E . Suppose*

- (i) T is a homeomorphism from U to $T(U)$.
- (ii) $K:U \mapsto H$ is an $H - C^1$ map such that the map $x \in U \mapsto DK(x)$ is continuous into the space of Hilbert-Schmidt operators on H .
- (iii) $I_H + DK(x) \in GL(H)$, for every $x \in E$.

Then the measure $\nu_{T^{-1}}$ is absolutely continuous with respect to ν and

$$\frac{d\nu_{T^{-1}}}{d\nu}(x) = |\delta(DT(x))| \exp \left\{ \langle K(x), x \rangle - \text{trace}_H DK(x) - |K(x)|^2/2 \right\} \quad (2)$$

where δ denotes the Carleman-Fredholm determinant defined on $L(H)$. (The difference of the two expressions contained in the quote marks in (2) is defined as the limit of a convergent sequence in L^2 of differences. Both of the terms may fail to exist separately).

In [B.1], the author proved the following non-Gaussian version of the Cameron-Martin theorem (cf. [K.2]).

Theorem 3 (Bell) *Let ν denote a finite Borel measure on E satisfying the following condition with respect to a vector $r \in E$*

$$\int_E D_r \phi d\nu = \int_E \phi X_r d\nu.$$

for test functions ϕ defined on E . Suppose the function $t \in \mathbf{R} \mapsto X_r(x + tr)$ is continuous, a. e. x and the following random variables are locally integrable

$$\sup \{ X_r(x + tr)^4, \quad 0 \leq t \leq 1 \},$$

$$\sup \left\{ \exp -4 \int_s^t X_r(x + ur) du \right\}.$$

Define $T(x) = x - r$. Then the measures ν_T and ν are equivalent and

$$\frac{d\nu_T}{d\nu}(x) = \exp - \int_0^1 X_r(x + ur) du. \quad (4)$$

3. Transformations of measure via a homotopic construction

Let ν be a finite Borel measure on a Banach space E and let U denote a distinguished subclass of the class of transformations of E .

Definition A linear operator $L : U \mapsto L^2(\nu)$ is an *integration by parts operator (IPO)* for ν if the following holds for all C^1 functions $\phi : E \mapsto \mathbf{R}$ and all $h \in U$ for which both sides exist

$$\int_E D\phi(x)h(x)d\nu = \int_E \phi(x)Lh(x)d\nu.$$

Remark. The Malliavin calculus provides a method for obtaining IPOs for measures induced by stochastic differential equations (cf. [B.2, Chapters 2 - 4 and Section 7.3]).

Suppose that L is an IPO for the measure ν with domain U . The following result is easily verified (cf. [B.2, Section 5.3])

Lemma Let $h \in U \cap L^2(\nu)$, $\psi : E \mapsto \mathbf{R} \in L^2(\nu) \cap C^1$, $\psi h \in U$. Then

$$L(\psi h)(x) = \psi(x)Lh(x) - D\psi(x)h(x) \quad a.s.\nu.$$

Remark. The above lemma can be shown to hold for a larger set of functions h and ψ by a closure argument.

Let $T : E \mapsto E$ denote a map of the form $T = I + K$ where $K \in U$. Define

$$T_t = I + tk, t[0, 1].$$

Suppose T_t is invertible for all $t \in [0, 1]$. Note that the transformations T_t are absolutely continuous with respect to ν if and only if there exists a family $\{X_t : t \in [0, 1]\}$ of RNDs such that $X_0 = 1$ and for all test functions ϕ on E , we have

$$\int_E \phi(x)d\nu = \int_E \phi \circ T_t^{-1}(x)X_t(x)d\nu. \quad (5)$$

Thus the RHS of (5) is independent of t . Differentiation wrt t under the integral sign gives

$$\int_E \{D\phi(T_t^{-1}(x))(d/dtT_t^{-1}(x))X_t(x) + \phi \circ T_t^{-1}(x)d/dtX_t(x)\}d\nu = 0. \quad (6)$$

We simplify the first term in the integrand in (6) by means of the relation

$$D\phi(T_t^{-1}(x))d/dtT_t^{-1}(x) = -D(\phi \circ T_t^{-1})(x)K \circ T_t^{-1}(x)$$

to get

$$\int_E \{\phi \circ T_t^{-1}d/dtX_t(x) - D(\phi \circ T_t^{-1})(x)K \circ T_t^{-1}(x)X_t(x)\}d\nu = 0. \quad (7)$$

Assume that $(K \circ T_t)X_t \in U$. Then the defining property of L allows us to write (7) in the form

$$\int_E \phi \circ T_t^{-1}(x)\{d/dtX_t(x) - L[(K \circ T_t^{-1})X_t](x)\}d\nu.$$

This will hold for all test functions ϕ if and only if X_t satisfies the differential equation

$$d/dtX_t(x) = L[(K \circ T_t^{-1})X_t](x). \quad (8)$$

Now assume the functions $K \circ T_t^{-1}$ and h satisfy the conditions on h and ψ respectively in the above lemma. Then applying the lemma to the RHS of (8) yields

$$d/dtX_t(x) = X_t(x)L[(K \circ T_t^{-1})(x) - DX_t(x)K \circ T_t^{-1}(x)]. \quad (9)$$

Writing $X(t, x) = X_t(x)$, $X_1 = dX_t/dt$, $X_2 = DX_t$, and substituting $x = T_t(y)$ in (9) gives

$$X_1(t, T_t(y)) = X(t, T_t(y))L[K \circ T_t^{-1}(T_t(y)) - X_2(t, T_t(y))K(y)].$$

However, since $K = DT_t/dt$, this is equivalent to

$$dX_t/dtX(t, T_t(y)) = X(t, T_t(y))L[K \circ T_t^{-1}](T_t(y)). \quad (10)$$

Solving equation (10) with the initial condition $X(0, x) = 1$ gives

$$X(t, T_t(y)) = \exp \left\{ \int_0^t L[K \circ T_s^{-1}](T_s(y))ds \right\}.$$

We thus arrive at the following expression for X

$$X(t, x) = \exp \left\{ \int_0^t L[K \circ T_s^{-1}](T_s \circ T_t^{-1}(x))ds \right\}. \quad (11)$$

In particular

$$\frac{d\nu_T}{d\nu}(x) = X(t, x) = \exp \left\{ \int_0^1 L[K \circ T_s^{-1}](T_s \circ T^{-1}(x))ds \right\}. \quad (12)$$

Suppose one is given a measure ν on E , an IPO L for ν with domain U , and a map T of E of the form $I + K$ with $K \in U$ such that the maps $T_t = I + tK$ are invertible for all $t \in [0, 1]$. Then one could obtain a transformation theorem for ν by defining $X(t, x)$ by the formula (11) and establishing (5) by reversing the steps in the above argument. This will imply the equivalence of the measures ν_{T_t} and ν , with X_t as the corresponding RNDs. This scheme was implemented in [B.2] in the case K is constant to derive Theorem 3 above.

We now give a condition on K which ensures the invertibility of the maps T_t

Definition The map K is said to be a (strong) contraction on E if there exists a constant $c \in [0, 1)$ such that

$$\|K(x) - K(y)\|_E \leq c\|x - y\|_E, \quad \forall x, y \in E.$$

Proposition If K is a contraction on E then the transformation $T_s = I + sK$ is invertible, for all $s \in [0, 1]$.

Proof. It suffices to prove the result for $T = I + K$. The contraction property trivially implies that T is injective. To show T is surjective, suppose $y \in E$ and define $K_y(x) \equiv y - K(x)$. Then K_y is a contraction on E . By the contraction mapping theorem, K_y has a fixed pnt $x_0 \in E$. Then x_0 satisfies $T(x_0) = y$.

4. Transformation formulae

In this section, we use (12) to derive the formulae for the densities that occur in Theorems 1, 2, and 3.

(A) Let ν denote the standard Wiener measure on the space of real-valued paths with initial point 0, defined on the time interval $[0, 1]$. Then the Itô integral operator L ,

$$Lk \equiv \int_0^1 k'_s dw_s \quad (13)$$

is an IPO for ν . The domain U of L consists of adapted paths k lying in the Cameron-Martin space, i.e. such that $k_0 = 0$ and

$$\int_0^1 k'_s{}^2 ds < \infty.$$

This result is due to Gaveau & Trauber[G-T].

Let $h = h(w)$ denote a bounded adapted path as in the statement of Girsanov's theorem and define

$$K(w) \equiv - \int_0^1 h_u du.$$

Substituting the operator L defined in (13) into (12), we have

$$\begin{aligned} \frac{d\nu_T}{d\nu} \circ T(w) &= \exp \left\{ \int_0^1 \int_0^1 -h_u d \left(w_u - s \int_0^u h_v dv \right) ds \right\} \\ &= \exp - \left\{ \int_0^1 \int_0^1 h_u dw_u - s \int_0^1 h_u^2 du \right\} \\ &= \exp - \left\{ \int_0^1 h_u dw_u - \frac{1}{2} \int_0^1 h_u^2 du \right\} \\ &= 1/G(w) \end{aligned}$$

where w is as in (1). Thus we obtain the formula for the density in Girsanov's theorem.

(B) Let (i, H, E) denote an abstract Wiener space with Gaussian measure ν on E . The Gaussian *divergence operator*¹

$$LK(x) \equiv \langle K(x), x \rangle - \text{trace}_H DK(x)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on H , is an IPO for ν (cf., e.g. K.2)]. The domain of L can be chosen to be the set of C^1 functions from E into E^* . where E^* is identified with its image in E under the inclusions

$$E^* \xrightarrow{i^*} H^* \cong H \xrightarrow{i} E.$$

This domain can then be extended to the set U consisting of the class of maps $K : E \mapsto H$ in Theorem 2, using the argument in Ramer's paper [R]. For $K \in U$, one then has

$$LK(x) = \langle K(x), x \rangle - \text{trace}_H DK(x) \tag{14}$$

where the quote marks have the same meaning as in [R].

In order to derive the density formula (2) from (12) and (14), it will be necessary to perform some manipulations on the trace term in (14). In the present generality these manipulations are necessarily of a formal nature, since the trace may fail to exist as a separate entity (this problem could be circumvented by performing the manipulations on the sequence of approximations used to define the random variable on the RHS in (14), in the spirit of Ramer [R], then passing to the limit). In order to avoid this problem, we make the stronger assumption that K is a C^1 map into E^* . In this case, both terms on the RHS in (14) exist and we have

$$\begin{aligned} & \int_0^1 L[K \circ T_s^{-1}](T_s(x)) ds = \\ & \exp \left\{ \langle K(x), x \rangle + |K(x)|^2/2 - \text{trace}_H \int_0^1 D[K \circ T_s^{-1}](T_s(x)) ds \right\} \\ & = |DetDT(x)|^{-1} \exp \left\{ \langle K(x), x \rangle + |K(x)|^2/2 \right\} \end{aligned} \tag{15}$$

where (15) follows from the identity

$$\exp \left\{ \text{trace}_H \int_0^1 D[K \circ T_s^{-1}](T_s(x)) ds \right\} = |DetDT(x)|.$$

We obtain from (15)

$$\frac{d\nu_{T^{-1}}}{d\nu}(x) = |DetDT(x)| \exp - \left\{ \langle K(x), x \rangle + |K(x)|^2/2 \right\}. \tag{16}$$

¹In the case where the abstract Wiener space is the *classical* Wiener space, the operator L can be used to define an extension of the Itô integral with anticipating integrands. This extension is known as the *Skorohod* integral.

This is the formula for the RND in a transformation formula due to Kuo [K.1]. To derive the analogous formula (2) in Ramer's theorem, it is necessary to introduce the (formal) quantity $\text{trace}_H DK(x)$ into the exponential in (16). The corresponding adjustment outside the exponential converts the standard determinant in (16) into the Carlkeman-Fredholm determinant and gives rise to the formula (2).

(C) Suppose ν is a finite Borel measure on E such that (3) holds for some $r \in E$. Define $U \equiv \{\lambda r : \lambda \in \mathbf{R}\}$ and L on U by $L(\lambda r) \equiv \lambda X_r$. Note that in this case $T_s = I - sr$ and $T_s^{-1} = I + sr$. Thus (12) yields

$$\begin{aligned} \frac{d\nu_T}{d\nu}(x) &= \exp - \left\{ \int_0^1 X_r(x + (1-s)r) ds \right\} \\ &= \exp - \left\{ \int_0^1 X_r(x + sr) ds \right\} \end{aligned}$$

and we obtain (4).

References

- [B.1] D. Bell, A quasi-invariance theorem for measures on Banach spaces. *Trans Amer. Math. Soc.* **290** (1985), 841-845.
- [B.2] D. Bell, *The Malliavin Calculus*. Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 34, Wiley, New York/ Longman U.K., 1987.
- [G-T] B. Gaveau and P. Trauber, L'integrale stochastique comme operateur de divergence dans l'espace fonctionnel. *J. Funct. Anal.* **46** (1982), 230-238.
- [G] L. Gross, Abstract Wiener spaces, *Proc. Fifth Berkeley Sympos. Math. Statist. and Prob.* vol. 2, part (i), (1965) 31-42.
- [K.1] H. -H. Kuo, Integration on infinite-dimensional manifolds. *Trans Amer. Math. Soc.* **159** (1971), 57-78.
- [K.2] H. -H. Kuo, *Gaussian Measures in Banach Spaces*, Springer LNM No. 463, Springer-Verlag, Berlin, 1975.
- [7] R. Ramer, On nonlinear transformations of Gaussian measures. *J. Funct. Anal.* **15** (1974), 166-187.