Quasi-invariant measures on the path space of a diffusion

Denis Bell
University of North Florida

Admissible Vector fields and quasi-invariance

Let N be a manifold, equipped with a finite Borel measure γ and let Z be a vector field on N.

Definition. The vector field Z is *admissible* (with respect to γ) if there exists an L^1 function $Div(Z)$ such that for all test functions Φ on N

$$\int_N Z(\Phi) d\gamma = \int_N \Phi Div(Z) d\gamma.$$

Example

γ is Gaussian measure on $\mathbf{R}^n$, $d\gamma/dx = e^{-||x||^2}$ and $Z \in \mathbf{R}^n$ is constant vector. If Φ is a test function, then integration by parts gives

$$\int_{\mathbf{R}^n} Z(\Phi) d\gamma = \int_{\mathbf{R}^n} Z(\Phi) e^{-||x||^2} dx$$

$$= \int_{\mathbf{R}^n} \Phi(x) < Z, x > e^{-||x||^2} dx$$

$$= \int_{\mathbf{R}^n} \Phi(x) < Z, x > d\gamma.$$

Thus Z is admissible and $Div(Z)(x) =< Z, x >$.

Returning to the general case, let Z be a vector field on N.

Consider the flow on N associated to Z, i.e. the map $x \mapsto x^s$, where

$$x^s = Z(x^s).$$

$$x^0 = x.$$

How does the measure γ transform under the flow?

Suppose x is a random variable with law γ and let γ_s denote the law of x^s. How are the measures γ_s related to γ?

Definition. Say γ is *quasi-invariant* under the flow of Z if, for all s, the measures γ_s and γ are equivalent (i.e. γ_s and γ have the same class of null-sets).

This implies there exist positive random variables ρ_s (*Radon-Nikodym derivatives*, denoted $\frac{d\gamma_s}{d\gamma}$) such that

$$\gamma_s(B) = \int_B \rho_s d\gamma.$$

Relationship between admissibilty and quasi-invariance

Suppose the quasi-invariance property holds, i.e. γ_s and γ are equivalent measures for all s and write $\rho_s = d\gamma_s/d\gamma$. Then for a test function Φ on N, we have

$$\int_N \Phi(x_s)d\gamma = \int_N \Phi(x)d\gamma_s = \int_N \Phi\rho_s d\gamma.$$

Differentiating wrt s and setting $s = 0$

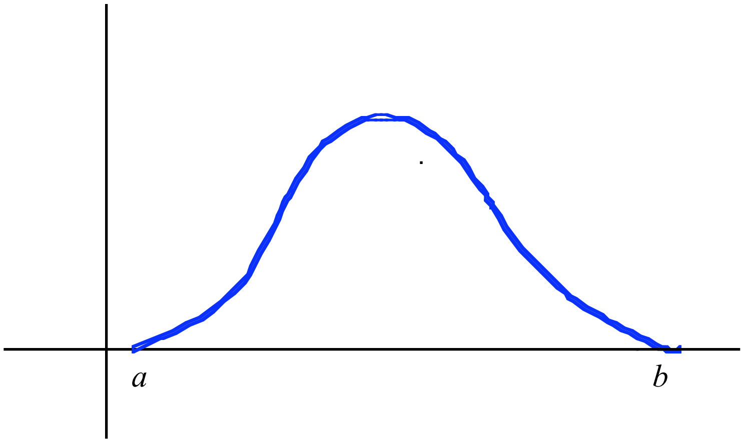
$$\int_N Z(\Phi)d\gamma = \int_N \Phi\Big\{\frac{d\rho_s}{ds}/_{s=0}\Big\}d\gamma.$$

Thus Z is admissible and

$$Div(Z) = \frac{d\rho_s}{ds}/_{s=0}.$$

Does admissibility imply quasi-invariance?

Not necessarily. For example, let γ be a measure on **R**^{n} with a smooth compactly supported density F:

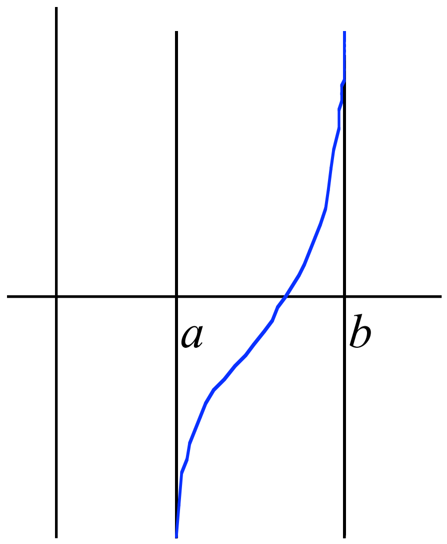


Let $Z \in$ **R**^{n} be constant. Then integration by parts gives

$$\int_{\mathbf{R}^n} Z(\Phi) d\gamma = -\int_{\mathbf{R}^n} \Phi \frac{Z(F)}{F} d\gamma.$$

Z is admissible, however γ is not quasi-invariant under the flow of Z, translations $x \mapsto x + tZ$.

Note that in this case $Div(Z) = -Z(F)/F$ blows up as $x \to a$ and $x \to b$ from within support(F).



In particular for fixed $s > 0$, the function

$$x \mapsto Div(Z)(x + uZ), u \in [0, s]$$

is singular for x in a set of positive γ-measure.

Theorem 1. *Suppose Z be an admissible vector field on N. Let x^s denote the flow generated by Z such that x^0 has law γ and let ν^s denote the law of x^s. Suppose the following conditions are satisfied*

(i) There exists a set $B \subseteq N$ with $\gamma_s(B) = 1$ for all s, such that Div(Z) is defined and Z-differentiable on B.

(ii) The function $s \mapsto Div(Z)(x^s)$ is absolutely continuous for $x^0 \in B$.

Then γ is quasi-invariant under the flow of Z and

$$\frac{d\gamma_s}{d\gamma} = \exp \int_0^s Div(Z)(x^{-u})du.$$

Example (The classical Wiener space). Let C_0 denote the space of continuous paths $\{\sigma : [0,T] \mapsto \mathbf{R} / \sigma(0) = 0\}$ and γ the *Wiener measure* (the law of Brownian motion). Let h be a path in C_0 such that $\int_0^T h'^2(t)dt < \infty$. It can be shown that the constant vector field $Z \equiv h$ is admissible and

$$Div(Z)(w) = \int_0^T h' dw.$$

Note that the flow of Z is the map $w \mapsto w + sh$. Theorem 1 shows that the measures γ and $\gamma_s \equiv \gamma(. + sh)$ are equivalent and

$$\frac{d\gamma_s}{d\gamma}(w) = \exp\left(s \int_0^T h dw - \frac{s^2}{2} \int_0^T h^2 dt\right).$$

This is the *Cameron-Martin* theorem.

Theorem (Dichotomy Theorem for Wiener measure). *Let h be a path in C_0 not in the Cameron-Martin space i.e. a path of infinite-energy*

$$\int_0^T h'^2 dt = \infty.$$

Then the measures γ and $\gamma(.+h)$ are mutually singular, i.e. supported on disjoint sets!

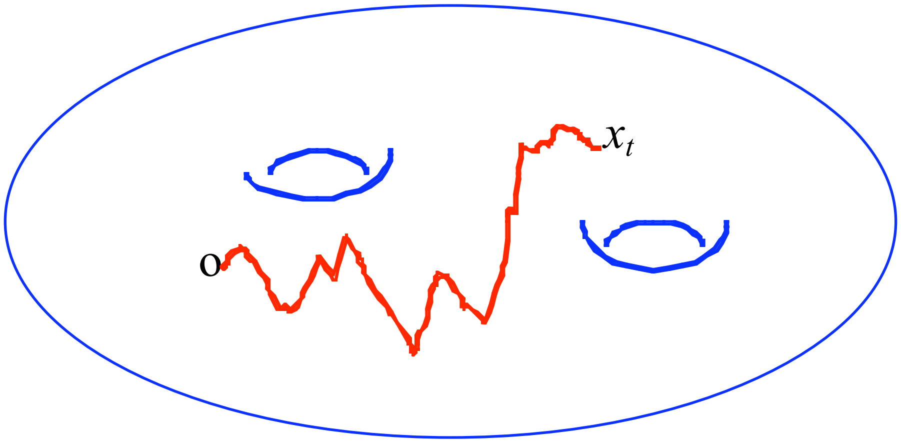
Measures Induced by Stochastic Differential Equations

Let M denote a closed compact manifold and $X_1, \ldots, X_n$ and Y be smooth vector fields on M. Let o be a fixed point in M.

Consider the Stratonovich SDE

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + Y(x_t)dt, \; t \in [0, T]$$

with initial point $x_0 = o$, where $(w_1, \ldots, w_n)$ is a Euclidean Wiener process.



Note: Formally, the above SDE can be thought of as an ODE driven by *white noise*

$$x_t = \sum_{i=1}^{n} X_i(x_t) w_i + Y(x_t).$$

Define N be the space of paths $\{\sigma : [0, T] \mapsto M/\sigma(0) = o\}$ and γ the law of the process x (denote N by $C_o(M)$).

The objective is to construct a class of admissible vector fields on $C_o(M)$ and an associated class of quasi-invariant measures.

We are considering the manifold

$$C_o(M) = \{\sigma : [0, T] \mapsto M/\sigma(0) = o\}$$

equipped with the measure γ, where γ is is the law of the process x defined by the SDE

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + Y(x_t)dt,\ t \in [0, T] \quad (*)$$

Define the *tangent space* $T_x X$ to be the set of paths $V : [0, T] \mapsto TM$ such that $V_0 = 0$ and

$$V_t \in T_{x_t} M, \ \forall t \in [0, T].$$

We will construct vector fields Z on $C_o(M)$ of the form

$$Z(x)_t = \sum_{i=1}^{n} X_i(x_t) h_t^i$$

where $h^1, \dots, h^n$ are real-valued processes, adapted to the filtration of x, i.e. $h_t^i = f(x_u, 0 \leq u \leq t)$.

In order to define the processes h^i, we need to introduce some geometric structure on M associated with the diffusion process (*). This requires the following assumption:

Assume x is *elliptic* i.e. The vector fields $X_1, \dots, X_n$ span TM at each point of M.

Associated Geometric Structure

Denote by $[g_{jk}]_{j,k=1}^{d}$ the Riemannian metric on M defined by $g^{jk} = a_{ij}a_{ik}$ where $X_i = a_{ir}\partial/\partial x_r$ is a local representation of $X_i, 1 \leq i \leq n$ (note: here and elsewhere, we use the summation convention: repeated indices are assumed to be summed on).

Let ∇ denote the corresponding Levi-Civita covariant derivative (this is the unique connection on M that has the following two properties:

(a) *metric compatible*, i.e. for any vector fields X, Y and Z on M, $Z(< X, Y >) =< \nabla_Z X, Y > + < X, \nabla_Z Y >$

(b) *torsion-free*, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$.

The *Riemann curvature tensor* is defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

and the *Ricci tensor*

$$Ric(X) = \sum_{i=1}^{n} R(X, e_i) e_i$$

where $\{\mathrm{e}_i\}_{i=1}^{n}$ is a (locally defined) orthonormal frame on M.

Define a set of 1-forms $\omega^{jk}, 1 \leq j, k \leq n$ on M by

$$\omega^{jk}(.) = < \nabla_{X_j} X_k, . > - < \nabla_{.} X_j, X_k >$$

and functions $B^{jk}, 1 \leq j, k \leq n$ on M by

$$B^{jk} = \frac{1}{2}\Big(< L_{ji} X_i, X_k > - < L_{ij} \nabla_{X_k} X_i >$$

$$- < \nabla_{X_j} X_k, \nabla_{X_i} X_i > + < \nabla_{X_p} X_i, X_k >< \nabla_{X_j} X_p, X_i > \Big)$$

where L_{ij} denotes the differential operator

$$\nabla_{X_i} \nabla_{X_j} - \nabla_{\nabla_{X_i} X_j}.$$

(Recall: summation convention).

Define also the *Cameron-Martin* space $H(\mathbf{R}^n)$ of paths

$$\Big\{ r : [0, T] \mapsto \mathbf{R}^n : r_0 = 0, \int_0^T ||r_t||^2 dt < \infty \Big\}.$$

Statements of Theorems

Theorem 2. *Let* $r = (r^1, \ldots, r^n)$ *be any path in* $H(\mathbf{R}^n)$ *and define* $h^i, 1 \leq i \leq n$ *by the following system of SDE s*

$$dh^i_t = \omega^{ji}(\circ dx_t)h^j_t + \Big[B^{ji} + < \nabla_{X_j} Y, X_i > \Big](x_t) h^j_t dt + r^i_t dt$$

$$h^i_0 = 0.$$

Then the vector field $Z(x)_t \equiv X_i(x_t)h^i_t, t \in [0, T]$ *on* $C_o(M)$ *is admissible and*

$$Div(Z) = \int_O^T \Big(r^i_t + \frac{1}{2} < Ric(Z_t), X_i(x_t) > \Big) dw_i.$$

Theorem 3. *Let T^{ij} and $f^{ij}, 1 \leq j \leq n$ denote, respectively, smooth 1-forms and real-valued functions on M and suppose $g^i \in L^2[0,T], 1 \leq 1 \leq n$ are deterministic functions. Define a vector field V on $C_0(M)$ by $V_t = X_i(x_t)\eta_t^i$, where $\eta_1, \ldots, \eta_n$ satisfy the system of SDE s*

$$d\eta_t^i = T^{ij}(\circ dx_t)\eta_t^j + \Big[f^{ij}(x_t)\eta_t^j + g^i(t)\Big]dt,\ t \in [0,T]$$

with $\eta^i(0) = 0$. Then there exists a solution x^s in $C_o(M)$ to the flow equation

$$\frac{dx^s}{ds} = V(x^s),\ s \in \mathbf{R}$$

where x is as in equation (*). Furthermore, the paths x^s are semimartingales of the form

$$dx_t^s = X_i(s,t) \circ dw_i + X_0(s,t)dt$$

where $X_j(s,.), 0 \leq j \leq n$ are adapted processes (with respect to w) in TM such that, on a set of full Wiener measure, $s \mapsto X_j(s,.)$ is continuous into the space $L^2[0,T]$.

The Main Result

Recall, we are considering the SDE on M

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + Y(x_t)dt,\ t \in [0, T]$$

$$x_0 = o.$$

In Theorem 2, we constructed a class of admissible vector fields Z defined on the solution space $C_o(M)$ of this equation ($Z(x) \in T_x C_o(M)$).

By Theorem 3, Z generates a flow x^s satisfying

$$x^s = Z(x^s)$$

$$x^0 = x.$$

Denote by γ the law of x and by γ_s the law of x^s.

Combining Theorems 1, 2 and 3 we obtain

Theorem 4. *The measures* γ_s *and* γ *are equivalent for all* s *and*

$$\frac{d\gamma_s}{d\gamma}(x) = \exp \int_0^s Div(Z)(x^{-u})du.$$

This result is a generalization of the Cameron-Martin theorem to non-constant vector fields and to the manifold setting.

History of the problem

The problem addressed in this talk (i.e. the construction of quasi-invariant measures on the path space of a diffusion process defined on a manifold) was first successfully treated by Bruce Driver in 1992 (``A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact manifold , *J. Funct. Anal.* 272-376). Partial results had previously been obtained by Bismut and by Shigekawa.

Since 1992, this problem has been studied extensively, e.g. by Driver, Malliavin, Cruzeiro, Hsu, Leandre, Lyons & Qian, Stroock & Enchev, Elworthy, Le Jan & Li, and others. The vector fields studied in Driver s 1992 paper and in these later works are those obtained from Cameron-Martin paths h in T_oM by stochastic parallel transport along the paths of the diffusion process x, i.e. vector fields V on $C_o(M)$ of the form

$$V_t = //_0^t(x)h_t.$$

The stochastic development (*rolling map*) plays a central role in these works.

The work outlined here offers a new approach to the problem. This approach differs from the previous one in the class of admissible vector fields constructed and in the method whereby the quasi-invariance property is established.