Poisson's remarkable calculation - a method or a trick?

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The Gaussian function e^{-x^2} plays a fundamental role in probability and statistics. For this reason, it is important to know the value of the integral

$$I = \int_0^\infty e^{-x^2} dx.$$

Since the integrand does not have an elementary antiderivative, I cannot be evaluated directly by the fundamental theorem of calculus. The familiar computation of the Gaussian integral is via the following remarkable trick, attributed to Poisson. One forms the square of I, interprets it as a double integral in the plane, transforms to polar coordinates and the answer magically pops out. The calculation is as follows

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

$$= \int_0^{\pi/2} \int_0^\infty re^{-r^2} drd\theta = \frac{\pi}{2} \int_0^\infty re^{-r^2} dr$$

$$= \frac{-\pi e^{-r^2}}{4} \Big/_0^\infty = \frac{\pi}{4}.$$

Hence $I = \sqrt{\pi}/2$.

Can this method be used to compute other seemingly intractable integrals? In this article we explore this question. Consider an improper integral

$$J \equiv \int_0^\infty f(x)dx$$

and suppose that f satisfies the functional equation

$$f(x)f(y) = g(x^2 + y^2)h(y/x), \;\; x, y > 0. \tag{1}$$

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Then Poisson's argument yields the relation

$$J^2 = \frac{1}{2}\Big(\int_0^\infty g(x)dx\Big)\Big(\int_0^{\pi/2} h(\tan\theta)d\theta\Big). \tag{2}$$

The determination of J is thereby reduced to the evaluation of two, possibly solvable, integrals. We are therefore led to ask *which functions f satisfy an equation of the form (1), where the functions g and h are such that the integrals in (2) can be evaluated?*

In a previous work [Be], the author characterized the solutions f to equation (1) under the somewhat restrictive assumption that f is asmptotic to a power function at zero. R. Dawson [Da] addressed the problem for the less general equation $f(x)f(y) = g(x^2+y^2)$. In the present article, we establish the following generalization of these results.

Theorem. *Suppose $f : (0,\infty) \mapsto \mathbf{R}$ satisfies equation (1), f is non-zero on a set of positive Lebesgue measure, and the discontinuity set of f is not dense in $(0,\infty)$. Then f has the form*

$$f(x) = Ax^p e^{cx^2}.$$

Furthermore, the corresponding functions g and h are unique up to multiplicative constants and are given by

$$g(x) = A_1 x^p e^{cx}, \tag{3}$$

$$h(x) = A_2\Big(\frac{x}{1+x^2}\Big)^p \tag{4}$$

where $A_1A_2 = A^2$.

Denote

$$J = \int_0^\infty x^p e^{cx^2} dx$$

and assume that $p > -1$ and $c < 0$ to ensure the existence of the integral. The decomposition into (3) and (4) implied by the Theorem results in the identity

$$J^2 = \frac{\Gamma(p+1)}{(-2c)^{p+1}} \int_0^{\pi/2} \sin^p t dt$$

where Γ denotes the gamma function. Furthermore, the evaluation of the above integral in closed form requires that p be an integer, thus $p \in \{0,1,2,\dots\}$. But in this case, J can be reduced to the Gaussian integral I by a scaling substitution and successive integration by parts!

We conclude that *Poisson's argument has no wider applicability as an integration method*. This answers the question posed in the title of the article. As Dawson observes, it is remarkable that Poisson's method turns out to have essentially only one application and that this single application is such a significant one!

The proof of the Theorem differs substantially from the argument in [Be] in focussing on the function g in (1) rather than on h. The proof will require three preliminary results.

Lemma 1. *Suppose f satisfies (1) and f is non-zero on a set of positive Lebesgue measure. Then f never vanishes.*

Proof. Note first that $h(1) \neq 0$ otherwise taking $y = x$ in (1) gives $f \equiv 0$, contradicting the hypothesis. We suppose throughout, without loss of generality, that $h(1) = 1$. Setting $y = x$ in (1) gives

$$f^2(x) = g(2x^2). \tag{5}$$

Substituting for the function g in (1), we obtain

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2 + y^2}{2}}\Big)h\big(\frac{y}{x}\big). \tag{6}$$

Define

$$r(x) = f(\sqrt{x}), \ \ k(x) = h(\sqrt{x}).$$

Then (6) yields

$$r(x)r(tx) = r^2\Big(\frac{x(1+t)}{2}\Big)k(t), \ \ t, x > 0. \tag{7}$$

We now prove the *claim: there exists $\delta > 0$ such that $k(x) \neq 0$ for all x in the interval* $(1 - \delta, 1 + \delta)$. We argue by contradiction. Since the map $x \mapsto x^2$ is strictly monotone on $(0, \infty)$, the non-zero set of r has positive Lebesgue measure λ. Hence there exists an integer N such that $a \equiv \lambda(T) > 0$ where T denotes the set

$$\{x/\ \ r(x) \neq 0\} \cap [N, N + 1].$$

Suppose the claim not hold. Then there exists a sequence $t_n \to 1$ such that $k(t_n) = 0$ for all n. Equation (7) yields

$$r(x)r(t_n x) = 0, \ \ \forall n, x. \tag{8}$$

Define

$$T_n \equiv \{t_n x/\ \ x \in T\}$$

and

$$V \equiv \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} T_k.$$

Then (8) implies T_n and T are disjoint, for all n. Hence

$$V \cap T = \phi. \tag{9}$$

Furthermore

$$\lambda(V) = \lim_n \lambda\Big(\bigcup_{k=n}^{\infty} T_k\Big) \geq \lim_n \lambda(T_n) = \lim_n t_n \lambda(T) = a. \tag{10}$$

If $x \in V$, then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and $\{x_k\} \subset T$ such that $x = t_{n_k} x_k$. Since $t_{n_k} \to 1$, this implies $x \in \bar{T}$, i.e. $V \subset \bar{T}$.

Let U be an arbitrary open set such that $T \subset U$. Then

$$T \cup V \subset \bar{U}. \tag{11}$$

We conclude from (9) - (11) that

$$\lambda(U) = \lambda(\bar{U}) \geq \lambda(T \cup V) = \lambda(T) + \lambda(V) \geq 2a.$$

Thus

$$a = \lambda(T) = \inf \Big\{ \lambda(U) / \ U \text{ open}, T \subset U \Big\} \geq 2a.$$

This implies $a = 0$, a contradiction. The claim follows.

Now suppose f vanishes, so $r(x_0) = 0$ for some $x_0 > 0$. Setting $x = x_0$ in (7) and using the claim, we deduce that $r \equiv 0$ on the interval $x_0(1 - \delta/2, 1 + \delta/2)$. Extrapolating this property results in the conclusion $r \equiv 0$, hence $f \equiv 0$ which contradicts the hypothesis of the Lemma. □

Before proceeding further, we give some examples of functions f satisfying equation (1) that *do not have the form* specified in the Theorem.

Example 1. Let m be a *discontinuous* function on $(0, \infty)$ with the multiplicative property

$$m(x)m(y) = m(xy), \ \ x, y > 0. \tag{12}$$

(The existence of a large class of such functions is well-established, see e.g. [A]). Then equation (1) is satisfied with $f = g = m$ and $h(t) = m(\frac{t}{1+t^2})$. Note that m is *never zero* (otherwise (12) implies $m \equiv 0$).

Example 2. Let $a > 0$ and define $f = I_{\{a\}}$, where $I_{\{a\}}$ denotes the indicator function of the singleton set $\{a\}$, $g = I_{\{2a^2\}}$, and $h = I_{\{1\}}$. Then it is clear that these functions satisfy (1).

Example 3. Let $\mathbf{A}$ denote the set of *algebraic* numbers in $(0, \infty)$. Define $f = g = h = I_{\mathbf{A}}$. Then (1) follows from the fact that $\mathbf{A}$ is closed under the positivity-preserving arithmetic operations and the extraction of square roots. Indeed, this immediately implies that if $I_{\mathbf{A}}(x)I_{\mathbf{A}}(y) = 1$, then $I_{\mathbf{A}}(x^2 + y^2)I_{\mathbf{A}}(y/x) = 1$. Conversely, suppose $I_{\mathbf{A}}(x^2 + y^2)I_{\mathbf{A}}(y/x) = 1$. Write $x^2 + y^2 = \alpha$ and $y/x = \beta$ where $\alpha, \beta \in \mathbf{A}$. Solving for x and y, we have

$$x = \sqrt{\frac{\alpha}{1 + \beta^2}}, \ \ y = \beta \sqrt{\frac{\alpha}{1 + \beta^2}}.$$

Thus $x, y \in \mathbf{A}$ and so $I_{\mathbf{A}}(x)I_{\mathbf{A}}(y) = 1$.

This example in conjunction with Lemma 1, provides a new proof of the well-known fact that (assuming at least one transcendental number exists) *the set of algebraic numbers has zero Lebesgue measure.* In fact, replacing $\mathbf{A}$ in this argument by an arbitrary set, we obtain the following result.

Proposition. *Let E be a measurable proper subset of $(0, \infty)$ closed under addition, multiplication, division, and the extraction of square roots. Then E has zero Lebesgue measure.*

The above examples show that neither of the additional hypotheses in the Theorem is redundant.

Lemma 2. *Suppose f satsifes the hypotheses of the Theorem. Then the function r is continuous everywhere.*

Proof. In view of Lemma 1, we may write (7) in the form

$$r(x) = \frac{r^2\left(\frac{x(1+t)}{2}\right)k(t)}{r(tx)}, \quad t, x > 0. \tag{13}$$

By hypothesis, there exists an interval (a, b) on which r is continuous. Define $c = \frac{a+b}{2}$. *Suppose r is discontinuous at some point $x_0 > 0$.* Choose t such that $c = tx_o$ and define $x_1 = \frac{x_0+c}{2}$. Assume r is continuous at x_1. Then (13) yields

$$\lim_{x\to x_0} r(x) = \frac{r^2(x_1)k(t)}{r(c)} = r(x_0),$$

i.e. r is continuous at x_0, contrary to assumption. We conclude that r is *discontinuous* at x_1. Iteration of this argument shows that r is discontinuous on the sequence of points $\{x_n\}$ defined inductively by

$$x_n = \frac{x_{n-1} + c}{2}, \quad n \geq 1.$$

Since x_n eventually lies in (a, b) this gives a contradiction, thereby proving the Lemma. □

Remark. Lemmas 1 and 2 imply that f and hence r has constant sign, which we may suppose without loss of generality, is positive. We deduce from (7) that k is then strictly positive and everywhere continuous.

Lemma 3. *Suppose f satisfies the hypotheses of the Theorem. Then the function $\log r$ is integrable at 0.*

Proof. The argument is a *quantitative* version of the iterative step in the proof of Lemma 2. We make repeated use of (13), which we write as

$$r(tx) = \frac{r^2\left(\frac{x(1+t)}{2}\right)k(t)}{r(x)}, \quad t, x > 0. \tag{14}$$

First, choose $\delta < .125$ and l, L, m, M such that $0 < l, m < 1$, $M, L > 1$ and

$$l < r(x) < L, \quad x \in [1, 2],$$

$$m < k(t) < M, \quad t \in [\delta, 1].$$

Taking $x = 2$ and letting t vary in the range $[\delta, 1]$ in (14), we have

$$\frac{l^2 m}{L} < r(x) < \frac{L^2 M}{l}, \quad x \in [2\delta, 1]. \tag{15}$$

Now setting $x = 4\delta$ in (14) and using (15) yields

$$\frac{l^5 m^3}{L^2 M} < r(x) < \frac{L^5 M^3}{l^2 m}, x \in [4\delta^2, 2\delta]. \tag{16}$$

Setting $x = 8\delta^2$ and using (16) in (14) we have

$$\frac{l^{12} m^8}{L^7 M^5} < r(x) < \frac{L^{12} M^8}{l^7 m^5}, x \in [8\delta^3, 4\delta^2]$$

Note that the powers of l, m, L, M in these estimates are increasing (roughly) by a factor of 3 each time. Iterating this process, we see that there exist constants $D < 1$ and $E > 1$ such that

$$D^{4^n} < r(x) < E^{4^n}, \quad \epsilon^n < x < \epsilon^{n-1} \tag{17}$$

where $\epsilon = 2\delta$. Let

$$q = 4^{\frac{1}{\log \epsilon}}$$

and note that $q > 1/e$ by choice of δ. Substituting $t = \epsilon^n$ in (17) gives

$$|\log r(x)| < q^{\log t} \max(-\log D, \log E), \;\; x \in [t, t^{\frac{n-1}{n}}].$$

Since

$$\int_0^1 q^{\log t} dt = \int_{-\infty}^0 (eq)^x dx < \infty$$

this implies

$$\int_0^1 |\log r(x)| dx < \infty$$

and we are done. □

Proof of the Theorem. Define

$$G(x) = \log r(x) - \frac{1}{x}\int_0^x \log r(u) du \tag{18}$$

$$= \log r(x) - \int_0^1 \log r(xu) du, \;\; x > 0.$$

(Note that Lemma 3 implies that the integrals exist.) Taking logarithms in (13), we have

$$\log r(x) + \log r(tx) - 2 \log r\Big(\frac{x(1+t)}{2}\Big) = \log k(t), \;\; t, x > 0.$$

Thus

$$G(x) + G(tx) - 2G\Big(\frac{x(1+t)}{2}\Big)$$

$$= \log k(t) - \int_0^1 \Big\{ \log r(xu) + \log r(txu) - 2\log r\Big(\frac{x(1+t)u}{2}\Big)\Big\} du$$

$$= \log k(t) - \int_0^1 \log k(t) du = 0.$$

Setting $y = tx$ gives

$$G(x) + G(y) = 2G\big(\frac{x+y}{2}\big), \quad x, y > 0. \tag{19}$$

Equation (19) is a variant of the Cauchy functional equation. It is well-known (and easy to show) that the only continuous functions G satisfying (19) are *linear* functions $x \mapsto ax + b$. We can therefore write

$$\log r(x) - \frac{1}{x}\int_0^x \log r(u) du = \frac{cx + p}{2}$$

for constants c and p. Multiplying by x and differentiating yields

$$\frac{xr'(x)}{r(x)} = cx + \frac{p}{2}.$$

Solving this ODE for r gives

$$r(x) = Ax^{p/2}e^{cx}.$$

Hence

$$f(x) = Ax^p e^{cx^2}$$

as claimed. Substituting this expression into (5) and (6), we obtain the functions g and h. □

We conclude with the following remarks.

The methods of this paper can be used to treat the more general functional equation

$$f(x)g(y) = F(x^2 + y^2)G(y/x), \quad x, y > 0.$$

Assuming the discontinuity sets of f and g are non-dense and f and g are non-zero on sets of positive Lebesgue measure, we can show that (up to multiplicative constants) the functions necessarily have the form

$$f(x) = x^{p_1}e^{cx^2}, \quad g(x) = x^{p_2}e^{cx^2},$$

$$F(x) = x^p e^{cx}, \quad G(x) = \frac{x^{p_2}}{(1+x^2)^p},$$

where $p_1 + p_2 = 2p$.

The subject of functional equations, which originated with Cauchy and Abel, has spawned an extensive body of advanced techniques (see, e.g. [A]). These techniques have been used to prove far more general results than those presented here (cf. [Ba], [L] and [M]). The advantage of the present approach is that it provides a complete analysis of equation (1) by direct and elementary means.

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