ARBITRAGE-FREE OPTION PRICING MODELS

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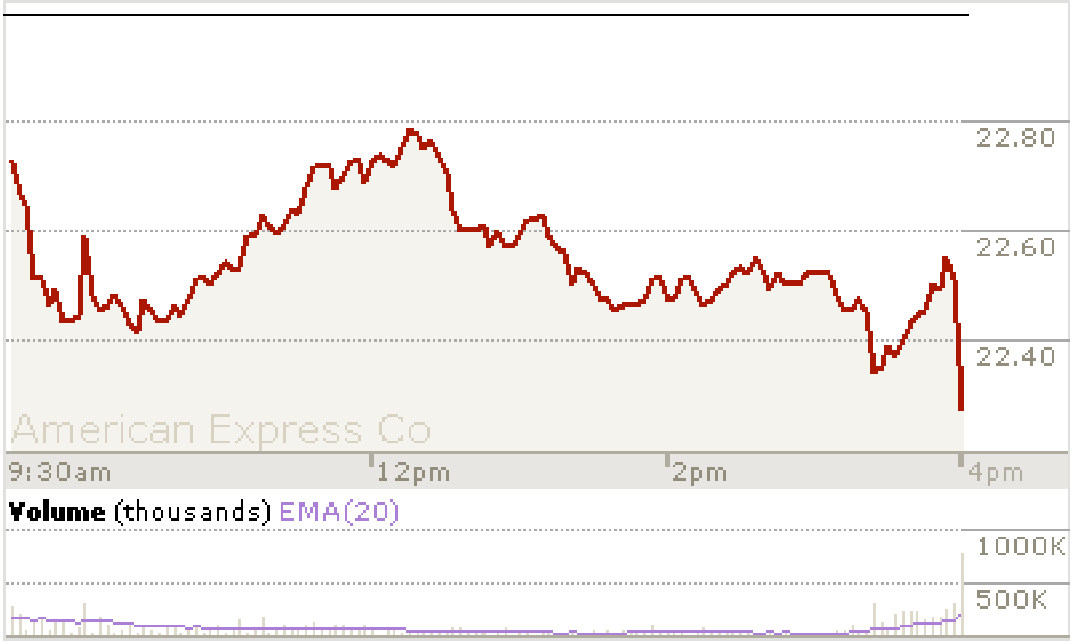
Modelling Stock Prices

Example American Express



In mathematical finance, it is customary to model a stock price by an (Ito) stochatic differential equation (SDE)

$$dS_t = a(S_t)dt + b(S_t)dw_t$$

where w is a Wiener process.

The coefficient function b is called the *volatility* of the stock.

The model is required to be *arbitrage-free*, i.e.

$$E[S_t] = e^{rt} S_0 \tag{1}$$

where r is the riskless interest rate.

Writing (1) as

$$E[e^{-rt} S_t] = S_0 \tag{2}$$

we see that condition (1) says that the process $e^{-rt} S_t$ is a *martingale*.

The Black-Scholes Model

$$dS_t = rS_t dt + \sigma S_t dw_t \qquad (3)$$

The arbitrage-free properly (2) is immediate from the fact that Ito integrals are martingales.

Eq. (3) can be explicitly solved to give

$$S_t = S_0 \exp\Big[(r - \frac{\sigma^2}{2})t + \sigma w_t\Big]. \qquad (4)$$

The present value of a *European call option* with strike price K and maturity time T is

$$V_T = e^{-rT} E\Big[(S_T - K)^+\Big]. \qquad (5)$$

Using (4), we can compute (5). This results in the famous *Black-Scholes formula* for option pricing.

The Cox-Ross Model

Cox & Ross introduced the following fractional version of the B-S model

$$dS_t = rS_t dt + \sigma\sqrt{S_t}dw_t. \tag{6}$$

This model arises naturally in financial situations as a limit of jump processes.

In contrast to the B-S model, *it is impossible to solve (6) in closed form* and hence find an elementary option pricing formula for the C-R model analogous to the Black-Scholes formula.

Cox & Ross computed the option price V_T as an infinite series.

A more general version of this problem was recently studied by Delbaen and coworkers using Bessel processes. They found a formula for V_T in terms of theta functions.

An approach to constructing solvable models (Bell-Stelljes)

Assume in the remainder of the talk that if $p \neq 1$ then $p = m/n$ *is a rational with* m *and* n *coprime and of different parity (one even and one odd).*

Consider the *Stratonovich* equation

$$dS_t = \mu S_t dt + \sigma S_t^p \circ dw_t, \ 1/2 \leq p \leq 1. \tag{7}$$

Since Stratonovich differentials transform in the classical way, Eq. (7) *can* be solved by elementary ODE techniques to obtain

$$S_t = S_0 \exp(\mu t + \sigma w_t), \ p = 1 \tag{8}$$

$$S_t = e^{\mu t}\left[(1-p)\sigma \int_0^t e^{\mu(p-1)s} dw_s + S_0^{1-p}\right]^{\frac{1}{1-p}}, \ p \neq 1. \tag{9}$$

We note that:

The assumptions on p ensure that $\frac{1}{1-p}$ is a fraction with odd denominator so S_t in (9) exists for all time t.

The existence/uniqueness theorem for solutions to SDEs *does not apply* to (7) for p in the range $[1/2, 1)$. However, the validity of the solution (9) can be checked by the Ito formula.

Writing (7) in Ito form we obtain

$$dS_t = \left(\mu S_t + \frac{p\sigma^2}{2} S_t^{2p-1}\right)dt + \sigma S_t^p dw_t.$$

The uniqueness question for SDEs of this type has been studied by several authors.

Uniqueness Theorems

Consider the SDE

$$dx_t = b(x_t)dt + \sigma x_t^p dw_t,\ x_0 > 0. \tag{10}$$

where b is locally Lipschitz away from 0. Since both the drift and diffusion coefficients are locally Lipschitz away 0, there will be a unique solution up until the time τ when x_t hits 0. The uniqueness issue is therefore closely related to the question of whether this will happen in finite time.

Theorem (Delbaen-Shirakawa). For $0 < p < 1$, if $b = 0$ then with probability 1, x_t reaches 0 in finite time.

Theorem (Yamabe-Watanabe). If $1/2 \leq p \leq 1$ and b is *Lipschitz at 0* then (10) has a (pathwise) unique solution.

Theorem (Lions-Musiela). If $1/2 \leq p \leq 1$ and

$$\frac{\sigma^2}{2} x^{2p-2} - \frac{b(x)}{x} \tag{11}$$

is bounded near $x > 0$. Then the solution to (10) never reaches 0 in finite time and is thus unique.

None of the foregoing hypotheses apply to the equation under consideration

$$dS_t = \left(\mu S_t + \frac{p\sigma^2}{2} S_t^{2p-1}\right) dt + \sigma S_t^p dw_t \tag{12}$$

for p in the range $[1/2, 1)$. (Note that in this case (11) becomes

$$\frac{\sigma^2}{2}(1-p)x^{2p-2} - \mu.$$

We show that the conclusions of the theorems *do not hold* for (12) if $1/2 < p < 1$.

Define

$$Y_t \equiv \int_0^t e^{\mu(p-1)s} dw_s.$$

By the time-change theorem for Ito integrals, we can write $Y_t = B(\tau_t)$ where B is a *Brownian motion* and

$$\tau_t = \frac{1}{2\mu(p-1)}\Big[e^{2\mu(p-1)t} - 1\Big].$$

Note that if $\mu < 0$ then $\tau_t \to \infty$ as $t \to \infty$ while if $\mu > 0$ then τ_t is bounded as $t \to \infty$. Recall the explicit solution to (12)

$$S_t = e^{\mu t}\Big[(1-p)\sigma Y_t + S_0^{1-p}\Big]^{\frac{1}{1-p}}.$$

Define $\tau = \inf\{t/S_t = 0\}$.

Since B hits every given level in any given time interval $[0, T], (T > 0)$ with *positive probability*, we will have $P(\tau < \infty) > 0$.

Set $V_t =$

$$S_t,\ t \leq \tau$$

$$0,\ t > \tau.$$

Writing (12) in integral form

$$x_t = S_0 + \int_0^t \left(\mu x_u + \frac{p\sigma^2}{2} x_u^{2p-1}\right) du + \sigma \int_0^t x_u^p$$

we see that S and V provide two different solutions to (12).

Note that this argument requires the integrand in the Riemann integral above to vanish when $x_u = 0$. This excludes the case $p = 1/2$ from the discussion.

Bell-Stelljes Model (continued)

We would like to use the Stratonovich equation (7)

$$dS_t = \mu S_t dt + \sigma S_t^p \circ dw_t$$

as a model for stock price. However, unlike Ito integrals, Stratonovich integrals are not martingales. Hence this model *has arbitrage*. We therefore seek a transformation that converts S_t to an arbitrage-free model.

Theorem 1. *Suppose* $G(s,t) : [0,\infty)\times[0,\infty] \mapsto \mathbf{R}$ *is* C^2 *in* s *and* C^1 *in* t *and satisfies the PDE*

$$G_t + \left(\mu s + \frac{p\sigma^2}{2} s^{2p-1}\right) G_s + \frac{\sigma^2}{2} s^{2p} G_{ss} = rG \quad (13)$$

Suppose that for all $T > 0$ *there exists* $n \in \mathbf{N}$ *such that*

$$\sup_{0\le t\le T} |G_s(s,t)| \le s^n \text{ for large s.} \quad (14)$$

Let $R_t = G(S_t, t)$. *Then* $e^{-rt} R_t$ *is a martingale, i.e.* R_t *is arbitrage-free.*

The main purpose of this talk is to explore the class of models that this theorem produces.

This involves studying the set of solutions to (13) satisfying (14).

A standard method for solving PDEs is *separation of variables*. Set $G(s,t) = F(s)H(t)$. Substituting this into (10) and separating gives

$$H(t) = e^{ct}$$

(for an arbitrary constant c), and the following differential equation for F

$$\frac{\sigma^2 s^{2p}}{2} F'' + \left(\mu s + \frac{p\sigma^2}{2} s^{2p-1}\right) F' = (r-c)F. \quad (15)$$

According to the theory of linear ODEs there are two linearly independent solutions to (15) valid in the range $s \in (0, \infty)$, for every c. The question is *which of these solutions are smooth at $s = 0$?*

We look at some special cases of (15) where we can find explicit solutions.

Note that there are two values of p where the equation has a particularly simple form, $p = 1$ and $p = 1/2$.

The case $p = 1$

Here (15) becomes

$$\frac{\sigma^2 s^2}{2} F'' + \Big(\mu + \frac{\sigma^2}{2}\Big) s F' = (r - c) F.$$

This is a *Cauchy-Euler* equation. The solutions are $F(s) = s^k$ where k is a root of the quadratic equation

$$\sigma^2 k^2/2 + \mu k + (c - r) = 0.$$

Solving for c and evaluating

$$R_t = e^{ct} F(S_t)$$

where S_t is given by (8), we get

$$R_t = R_0 \exp\Big[(r - \sigma^2 k^2/2)t + \sigma k w_t\Big].$$

This is the Black-Scholes model with the volatility σ replaced by $k\sigma$ (if σ is an a-priori *unknown* parameter, it is essentially just the B-S model).

The case $p = 1/2$

In this case (15) becomes

$$\frac{\sigma^2 s}{2} F'' + \Big(\mu s + \frac{\sigma^2}{4}\Big) F' = (r - c) F.$$

This equation can be solved by Laplace transforms.

Denoting the Laplace transform of F by L we have

$$L(t) = \Big(t + \frac{2\mu}{\sigma^2}\Big) t^{-(1+\alpha)} \times$$

$$\Big[A \int_0^t u^\alpha \Big(u + \frac{2\mu}{\sigma^2}\Big)^{-(\alpha+1/2)} du + B\Big] \quad (16)$$

where A and B are an arbitrary constants and

$$\alpha = \frac{r - c}{\mu}$$

If α is a *positive integer* then the integral in (16) can be evaluated by integration by parts.

Furthermore, the inverse transform can be computed and shows that F is as a *polynomial* of degree α.

e.g., if $\alpha = 1$ then there exists a solution

$$F(s) = s + \sigma^2/4\mu.$$

This gives rise to the following explicit analogue of the Cox-Ross model

$$R_t = e^{rt}\left[\frac{\sigma}{2}\int_0^t e^{-\mu s/2} dw_s + \sqrt{R_0 - \sigma^2/4\mu}\right]^2 + \frac{\sigma^2}{4\mu}e^{(r-\mu)t}.$$

An option pricing formula based on this model was derived by Bell & Stelljes.

Irregular solutions for $1/2 \leq p < 1$

If $r-c$ has an appropriate value, then Eq. (15):

$$\frac{\sigma^2 s^{2p}}{2} F^{''} + \left(\mu s + \frac{p\sigma^2}{2} s^{2p-1}\right) F' = (r-c)F$$

has a solution $F(s) = s^{1-p}$. Indeed, substituting $F(s) = s^k$ and equating the coefficients of like powers of s gives $k = 1-p$ and $r - c = (1-p)\mu$.

Note that $F'(s)$ blows up as $s \to 0$ so the hypothesis of Theorem 1 is not satisfied by the corresponding function G.

Substituting S_t from (9) into $R_t = e^{ct}F(S_t)$ we get

$$R_t = e^{rt}\left|(1-p)\sigma \int_0^t e^{\mu(p-1)s} dw_s + R_o\right|.$$

Because of the presence of the absolute value, $e^{-rt}S_t$ is *not a martingale*.

This shows the importance of the regularity assumptions on G at $s = 0$ in Theorem 1.

We can use the method of *reduction of order* to find a second solution $F_2(s)$ of equation (15) corresonding to $F_1(s) = s^{1-p}$.

We obtain $F_2 = vF_1$ where

$$v'(s) = s^{p-2} \exp\left[\frac{\mu s^{2(1-p)}}{\sigma^2(1-p)}\right].$$

This implies

$$F_2'(s) \sim s^{-1}, \;\; s \downarrow 0$$

so F_2' also has singular behavior at $s = 0$.

These examples motivate the following result:

A Non-existence Theorem

Theorem 2. *If $1/2 < p < 1$, $\mu < 0$, and $c \neq r$ then (other than the zero function) there is no C^2 solution on $[0, \infty)$ to the ODE*

$$\frac{\sigma^2 s^{2p}}{2} F^{''} + \Big(\mu s + \frac{p\sigma^2}{2} s^{2p-1}\Big) F' = (r - c)F,$$

satisfying: there exists $n \in \mathbf{N}$ such that

$$|F'(s)| \leq s^n, \text{ for large s.}$$

Recall we have seen that such solutions exist for $p = 1/2$ and $p = 1$.

Proof. Consider again the SDE

$$dS_t = \mu S_t dt + \sigma S_t^p \circ dw$$

(The value of S_0 will be chosen later). We write this equation in Ito form

$$dS_t = (\mu S_t + \frac{p\sigma^2}{2} S_t^{2p-1})dt + \sigma S_t^p dw. \tag{17}$$

As we previously observed, the process

$$S_t = e^{\mu t} X_t^{\frac{1}{1-p}} \tag{18}$$

satisfies (17) where

$$X_t = (1-p)\sigma \int_0^t e^{\mu(p-1)s} dw_s + \ S_0^{1-p}.$$

We modify (18) as before to get another solution to (17). Let

$$\tau = \text{inf}\{t / X_t = 0\}.$$

Recall that since $\mu < 0$ we have $\tau < \infty$ a.s. Define $V_t = S_t$ if $t \leq \tau$ and $V_t = 0$ if $t > \tau$.

Suppose that F is a non-zero function with the properties in the statement of Theorem 2.

Choose the initial point $S_0 = V_0$ in Eq. (17) such that $F(V_0) \neq 0$.

Then the function $G(s,t) = e^{ct}F(s)$ satisfies the hypotheses of Theorem 1, so according to the theorem the process

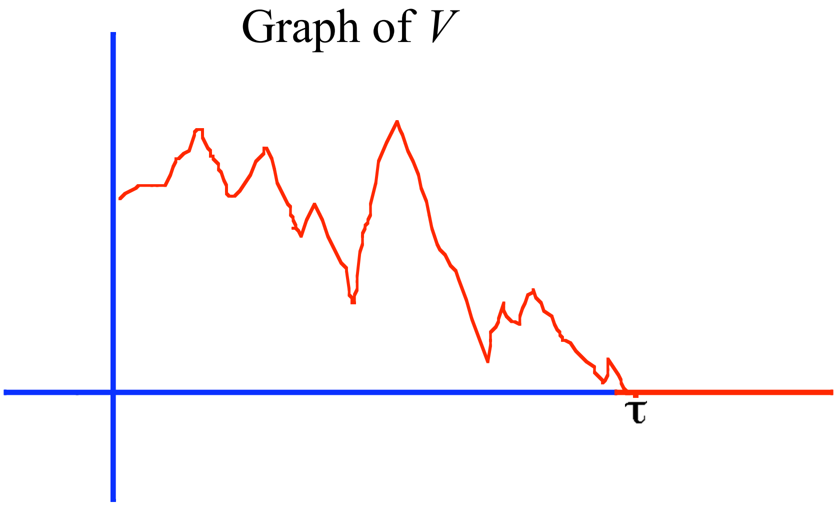
$$e^{(c-r)t}F(V_t)$$

is a *martingale*. In particular

$$e^{(c-r)t}E[F(V_t)] = F(V_0). \tag{19}$$

Hence

$$\lim_{t\to\infty} e^{(c-r)t}E[F(V_t)] = F(V_0) \neq 0. \qquad (20)$$



If $F(0) \neq 0$ then this implies the limit in (20) is 0 (if $c < r$) or ∞ (if $c > r$). In either case we have a contradiction.

If $F(0) = 0$ then, since $F(V_t) \equiv 0$ for $t > \tau$,

$$\lim_{t\to\infty} e^{(c-r)t}E[F(V_t)] = \lim_{t\to\infty} E[e^{(c-r)t}F(V_t)] = 0,$$

again resulting in a contradiction. (The case $c > r$ requires some estimates to justify the passage of the limit inside the expectation).

Ongoing Work

1, Characterize the solution set to the PDE

$$G_t + \left(\mu s + \frac{p\sigma^2}{2} s^{2p-1}\right) G_s + \frac{\sigma^2}{2} s^{2p} G_{ss} = rG.$$

In particular, identify the solutions that are regular at $s = 0$ (these are the solutions that can be expected to give rise to arbitrage-free option pricing models via the approach of B-Stelljes).

2. Find a natural financial interpretation for the resulting arbtrage-free models, or at least relate these models to concrete financial situations.

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