Poissons remarkable calculation - a method or a trick?

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Integrating the Gaussian function

The integral

$$I \equiv \int_0^\infty e^{-x^2}\, dx$$

plays a fundamental role in probability and statistics. It is threfore extremely important to know its value.

It is well-known that the integrand does not have an elementary antiderivative, so the integral cannot be computed directly by the FTC. However it can be evaluated by the following remarkable calculation attributed to Poisson (and/or Liouville).

Squaring I, we have

$$I^2 = \Big(\int_0^\infty e^{-x^2}\, dx \Big) \Big(\int_0^\infty e^{-y^2}\, dy \Big)$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx dy$$

Transforming to polar coordinates

$$x = r\cos\theta,\ \ y = r\sin\theta$$

gives

$$I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr d\theta$$

$$= \int_0^{\pi/2} -\frac{e^{-r^2}}{2}\Big/_0^{\infty}\,d\theta$$

$$\int_0^{\pi/2} 1/2\,d\theta$$

$$= \frac{\pi}{4}$$

Thus

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Can other functions can be integrated by this method?

Let $f : (0, \infty) \mapsto \mathbf{R}$ and consider the integral

$$J = \int_0^\infty f(x)\, dx.$$

Proceeding as before

$$J^2 = \left(\int_0^\infty f(x)\, dx \right) \left(\int_0^\infty f(y)\, dy \right)$$

$$= \int_0^\infty \int_0^\infty f(x) f(y)\, dx dy$$

In order to continue the method further there must exist functions g and h such that

$$f(x)f(y) = g(x^2 + y^2)h(y/x).$$

We will then have

$$J^2 = \int_0^{\pi/2} \int_0^{\infty} g(r^2) h(\tan\theta) r \, dr d\theta$$

$$= \Big(\int_0^{\infty} g(r^2) r \, dr \Big) \Big(\int_0^{\pi/2} h(\tan\theta) d\theta \Big)$$

$$= \frac{1}{2} \Big(\int_0^{\infty} g(x) dx \Big) \Big(\int_0^{\pi/2} h(\tan\theta) d\theta \Big).$$

We now need to compute these integrals. This will not be possible in general.

This leads to the following question:

Which functions f admit a decomposition

$$f(x)f(y) = g(x^2 + y^2) h(y/x)$$

where the functions g and $h \circ tan$ have elementary antiderivatives?

The following theorem answers the first part of this question.

Theorem *Suppose $f : (0, \infty) \mapsto \mathbf{R}$ satisfies an equation of the form*

$$f(x)f(y) = g(x^2 + y^2)h(y/x), \quad x, y > 0.$$

Assume f is continuous on an open interval and non-zero on a set of positive Lebesgue measure. Then there exist constants A, p and c such that

$$f(x) = Ax^p e^{cx^2}.$$

Furthermore, the functions g and h are unique up to scalar multiplication and are given by

$$g(x) = A_1 x^p e^{cx}, \tag{1}$$

$$h(x) = A_2 \Big(\frac{x}{1 + x^2} \Big)^p \tag{2}$$

where $A_1 A_2 = A$.

Define

$$J = \int_0^\infty x^p e^{cx^2} dx.$$

In view of (1) and (2), Poisson s method leads to the identity

$$J^2 = \frac{1}{2^{p+1}} \int_0^\infty x^p e^{cx}\, dx \times \int_0^{\pi/2} \sin^p t\, dt.$$

The existence of the first integral requires that $p > -1$ and $c < 0$. The evaluation of the integrals in closed form requires that p be an integer. But if p is a *non-negative integer* and c is *negative* then J can be evaluated in terms of the original Gaussian integral

$$\int_0^\infty e^{-x^2}\, dx$$

by the substitution $u = x\sqrt{-c}$ and repeated integration by parts. We conclude

Poisson s trick to integrate e^{-x^2} *is of no further use as an integration method*!

Proof of the theorem

Recall we are assuming f is continuous and non-vanishing on some intervals and satisfies

$$f(x)f(y) = g(x^2 + y^2)h(y/x). \qquad (1)$$

We can assume f is *non-negative*, otherwise apply the following argument to $|f|$ (which also satisfies the hypotheses of the theorem). We need two pleliminary results:

Lemma 1 *The function f is strictly positive and continuous everywhere.*

Proof (Assuming f is non-zero on an interval). Note first that $h(1) \neq 0$ otherwise taking $y = x$ in (1) gives $f \equiv 0$. We suppose without loss of generality, that $h(1) = 1$.

Setting $y = x$ in (1) gives

$$f^2(x) = g(2x^2). \qquad (2)$$

Substituting for the function g in (1), we obtain

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)h\Big(\frac{y}{x}\Big). \qquad (3)$$

Define $r(x) = f(\sqrt{x})$ and $k(x) = h(\sqrt{x})$. Then (3) yields

$$r(x)r(tx) = r^2\Big(\frac{x(1+t)}{2}\Big)k(t),\ t, x > 0. \qquad (4)$$

Let (a, b) be an interval where r is non-zero. Taking $x = (a + b)/2$ in (4), we deduce that $k(t) \neq 0$ for

$$\frac{2a}{a+b} < t < \frac{2b}{a+b},$$

an interval that contains the point $t = 1$.

Suppose $r(x_0) = 0$ for some $x_0 > 0$. Setting $x = x_0$ in (4) gives $r(x) = 0$ for x in an interval of the form $x_0(1 - \epsilon, 1 + \epsilon)$. It follows that $r \equiv 0$, a contradiction. Thus r is *non-zero everywhere*.

The proof of the continuity property is similar.

Lemma 1 illustrates a typical feature of equation (1): *properties of f assumed on an arbitrary open interval propagate to* $(0, \infty)$*. No extra information about h is required*.

Lemma 2. *The function* $\ln r$ *is integrable at zero*, i.e. the following improper integral exists for all $\delta > 0$

$$\int_0^\delta |\ln r(x)| dx < \infty.$$

With these lemmas we can prove the theorem.

Proof of the theorem. Define

$$G(x) = \ln r(x) - \frac{1}{x} \int_0^x \ln r(u) du, \; x > 0$$

$$= \ln r(x) - \int_0^1 \ln r(xu) du. \qquad (5)$$

(Note that these integrals exist by Lemma 2.)

Taking logs in (4) gives

$$\ln r(x)+\ln r(tx)-2\ln r\Big(\frac{x(1+t)}{2}\Big)=\ln k(t),\ t,x>0.$$

By (4) and (5)

$$G(x)+G(tx)-2G\Big(\frac{x(1+t)}{2}\Big)$$

$$=\ln k(t)-\int_0^1\Big\{\ln r(xu)+\ln r(txu)$$

$$-2\ln r\Big(\frac{xu(1+t)}{2}\Big)\Big\}du$$

$$=\ln k(t)-\int_0^1\ln k(t)du=0.$$

Setting $y=tx$ gives

$$G(x)+G(y)=2G\Big(\frac{x+y}{2}\Big),\ x,y>0.$$

This equation implies G is *linear*. To see this, fix $\lambda > 0$ and define

$$G_\lambda(x) = G(x+\lambda) - G(\lambda),\ x \geq 0.$$

Then G_λ satisfies

$$G_\lambda(x-\lambda) + G_\lambda(y-\lambda) = 2G_\lambda\Big(\frac{x+y}{2} - \lambda\Big). \quad (6)$$

Since $G_\lambda(0) = 0$, setting $y = \lambda$ gives

$$G_\lambda(x-\lambda) = 2G_\lambda\Big(\frac{x-\lambda}{2}\Big).$$

Substituting this into (6), we arrive at the *Cauchy equation*

$$G_\lambda(x) + G_\lambda(y) = G_\lambda(x+y),\ x,y \geq 0. \qquad (7)$$

Equation (7) and the continuity of G_λ imply

$$G_\lambda(x) = xG_\lambda(1).$$

Thus

$$G(x) = G_\lambda(x-\lambda) + G(\lambda)$$

$$= xG_\lambda(1) + [G(\lambda) - \lambda G_\lambda(1)]\,, x \geq \lambda. \qquad (8)$$

Since the coefficients on the RHS in (8) are obviously independent of λ, we have

$$G(x) = ax + b,\ x > 0.$$

We write

$$G(x) = \ln r(x) - \frac{1}{x}\int_0^x \ln r(u)du$$

$$= \frac{cx + p}{2}$$

for some constants c and p.

Multiplying each side by x and differentiating yields

$$\frac{xr'(x)}{r(x)} = cx + \frac{p}{2}. \tag{9}$$

Dividing each side of (9) by x and integrating

$$\ln r(x) = cx + \frac{p \ln x}{2} + d.$$

Exponentiating

$$r(x) = Ax^{p/2}e^{cx}.$$

Since $f(x) = r(x^2)$,

$$f(x) = Ax^p e^{cx^2}.$$

Substituting

$$r(x) = Ax^{p/2}e^{cx}.$$

into the equations

$$g(x) = r^2\left(\frac{x}{2}\right)$$

$$k(t) = \frac{r(x)r(tx)}{r^2\left(\frac{x(1+t)}{2}\right)}$$

gives

$$g(x) = (A/2^p)x^p e^{cx}$$

and

$$k(t) = \left(\frac{2\sqrt{t}}{1+t}\right)^p$$

Thus

$$h(x) = k(x^2) = \left(\frac{2x}{1 + x^2}\right)^p$$

as claimed.

A preprint containing complete details of the proof can be found on my website.

A class of pathological solutions

The functional equation

$$f(x)f(y) = g(x^2 + y^2)h(y/x) \qquad (10)$$

admits a class of highly irregular solutions of the following type. Let m be a *discontinuous* function on $(0, \infty)$ with the *multiplicative* property

$$m(xy) = m(x)m(y). \qquad (11)$$

It is known that such functions exsit and are discontinuous *everywhere*. Note that m never vanishes otherwise (11) would imply $m \equiv 0$.

It is clear that (10) is satisfied with $f = g = m$ and

$$h(x) = m\Big(\frac{x}{1 + x^2}\Big).$$

This example shows that some continuity assumption on f is *necessary* to ensure the conclusion of the theorem.

A variation on the Poisson theme

Suppose g is a function such that the following integral is *known* and *non-zero*

$$K = \int_0^\infty g(x)dx.$$

Defining

$$J = \int_0^\infty f(x)dx$$

we have

$$JK = \int_0^\infty \int_0^\infty f(x)g(y)dxdy. \qquad (12)$$

We can determine J provided this double integral can be computed.

The prospect of evaluating (12) along the lines of the Poisson calculation leads to the study of the functional equation

$$f(x)g(y) = F(x^2 + y^2)G(y/x),\ x, y > 0. \quad (13)$$

Theorem *Suppose (13) is satisfied and there exist open intervals on which f and g are continuous and sets of positive Lebesgue measure on which they are non-zero. Then (up to multiplicative constants) the functions necessarily have the form*

$$f(x) = x^{p_1} e^{cx^2}$$

$$g(x) = x^{p_2} e^{cx^2}$$

$$F(x) = x^{p} e^{cx}$$

$$G(x) = \frac{x^{p_2}}{(1+x^2)^p}$$

where $p_1 + p_2 = 2p$.

We can calculate the required integrals

$$\int_0^\infty g(x)dx,\ \int_0^\infty F(x)dx$$

only in cases where p_2 and p are integers and this implies p_1 is an integer!

In conclusion...

There is a more general formulation of the problem studied here.

Consider an *arbitrary* change of coordinates $(x, y) \mapsto (u, v)$ such that each of u and v depend on both x and y.

Which functions f satisfy a relation of the type

$$f(x)f(y)dxdy = g(u)h(v)dudv?$$

Conjecture: *Up to scaling, there is generally a two parameter family of functions with this property.*