THE GAUSS-BONNET THEOREM
FOR VECTOR BUNDLES

Denis Bell[1]

Department of Mathematics, University of North Florida

4567 St. Johns Bluff Road South,Jacksonville, FL 32224, U. S. A.

email: dbell@unf.edu

This paper is dedicated to the memory of Philip Bell.

Abstract. We give a short proof of the Gauss-Bonnet theorem for a real oriented Riemannian vector bundle E of even rank over a closed compact orientable manifold M. This theorem reduces to the classical Gauss-Bonnet-Chern theorem in the special case when M is a Riemannian manifold and E is the tangent bundle of M endowed with the Levi-Civita connection. The proof is based on an explicit geometric construction of the Thom class for 2-plane bundles.

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1. Introduction

The classical Gauss-Bonnet theorem gives a remarkable relationship between the topology and the geometry of a compact orientable surface in $\mathbf{R}^3$. In 1944, Chern generalized this theorem to all even-dimensional compact orientable manifolds, proving what is now known as the Gauss-Bonnet-Chern theorem. The Gauss-Bonnet-Chern theorem occupies a central place in differential geometry, opening the way to such important developments as the theory of characteristic classes and index theory. It continues to be a subject of current research, and new proofs of it have appeared within the last decade, e.g. Hsu [4] and Rosenberg [6]. The most recent proofs are based upon the heat kernel method pioneered by Mckean-Singer and Patodi, an approach that was reinvigorated by Bismut's stochastic proof [1] of the index theorem for the Dirac operator on spin bundles.

In this paper we give a short proof of a more general theorem. In contrast to the heat kernel approach, the argument presented here avoids hard analytical estimates and provides direct insight into the structure of the integrand in the Gauss-Bonnet formula. We prove the following:

Theorem 1 *Let E denote a real orientable Riemannian vector bundle of even rank $n = 2k$ over a closed compact manifold M of dimension n. Then*

$$\left(\frac{-1}{2\pi}\right)^k \int_M Pf(\Omega) = \chi(E) \tag{1}$$

where Pf denotes the Pfaffian function acting on square skew-symmetric matrices, Ω the curvature 2-form with respect to any metric connection on E, and $\chi(E)$ the Euler characteristic of E.

This result reduces to the Gauss-Bonnet-Chern theorem in the case when M is a Riemannian manifold and E is the tangent bundle of M endowed with the Levi-Civita connection.

Section 2 contains background information. In particular, the curvature of the vector bundle E is defined and two definitions, geometric and topological, of the Euler class of E are given. The proof of Theorem 1 follows by identifying the geometric and topological Euler classes (Theorem 4). Theorem 4 is proved in Section 3. We first prove the theorem for 2-plane bundles E via an explicit geometric construction of the Thom class of E. This is extended to direct sums of plane bundles by using elementary properties of the Euler classes. We note that Mathai and Quillen [5] have given a geometric construction of the Thom class for a vector bundle of arbitrary even rank. The construction in [5] is based upon equivariant differential forms and the Chern-Weil homomorphism.

2. Euler Classes

Let E denote a real oriented Riemannian vector bundle of even rank $p = 2q$ over a compact connected smooth manifold M (note that M itself does not need to be Riemannian here). In this section, we associate two De Rham cohomology classes to E.

Definition A *connection* on E is a globally defined map $\nabla : \Gamma(E) \mapsto \Gamma(E \otimes T^*(M))$ such that for $f \in C^\infty(M)$ and $X \in \Gamma(E)$,

$$\nabla(fX) = df \otimes X + f\nabla X. \tag{2}$$

We say that ∇ is *metric* if, for all X and $Y \in \Gamma(E)$, we have

$$d < X, Y > = < \nabla X, Y > + < X, \nabla Y > . \tag{3}$$

Suppose that ∇ is a metric connection on E. For every (locally defined) orthonormal frame $e = \{e_1, \ldots, e_p\}^t$ of E, we define a $p \times p$ matrix of *connection 1-forms* $\omega = [\omega_{ij}]$ on M by the relations

$$\nabla e_i = \sum_j \omega_{ij} e_j \qquad (\nabla e = \omega e)$$

and a corresponding $p \times p$ matrix of *curvature 2-forms* Ω by

$$\Omega = d\omega - \omega^2$$

where d denotes exterior derivative and multiplication of matrices of 1-forms is defined in the usual way, using wedge product to multiply the entries. It is clear that the metric compatability condition (3) implies that both ω and Ω are skew-symmetric mtrices. Suppose now that e and f are two positively oriented orthonormal frames of E defined over intersecting neighborhoods U and V of M with connection 1-forms and curvature 2-forms ω_e, Ω_e and ω_f, Ω_f respectively. Then there exists an SO(p)-valued function A on $U \cap V$ such that $f = Ae$. Condition (2) implies the following transformation laws for ω and Ω:

$$\omega_f = (dA)A^{-1} + A\omega_e A^{-1} \tag{4}$$

$$\Omega_f = A\Omega_e A^{-1}. \tag{5}$$

The Pfaffian: There exists a map $Pf : so(p) \mapsto \mathbf{R}$ (where $so(p)$ denotes the set of real $p \times p$ skew-symmetric matrices) such that $Pf(M)$ is a homogeneous polynomial of degree $q = p/2$ in the entries of M, characterized by the following properties:

(i) $Pf \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = \lambda.$

(ii) $Pf \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = Pf(A)Pf(B)$, if A and B are skew-symmetric square blocks.

(iii) $Pf(AMA^{-1}) = Pf(M), A \in SO(p)$.

Since the set of even-degree differential form a commutative ring with $\wedge$ as multiplication we can define $Pf(\Omega_e)$ where, as before, Ω_e is the curvature matrix corresponding to a positively oriented orthonormal frame e of E defined over a neighborhood U of M. By (5) and (iii) above, it follows that if $Pf(\Omega_f)$ is another such expression defined over V with $U \cap V \neq \phi$, then on $U \cap V$ we have $Pf(\Omega_e) = Pf(\Omega_f)$. Thus $Pf(\Omega_e)$ extends to a *globally* defined p-form on M. We denote this by $Pf(\Omega)$.

Theorem 2 $Pf(\Omega)$ *is closed. Furthermore,* $[Pf(\Omega)] \in H^p(M)$ *is independent of the particular choice of connection on E used in its construction.*

A proof of this result can be found in Rosenberg [R].

Define the *geometric Euler class e_g* of E by

$$e_g \equiv [(-1/2\pi)^q Pf(\Omega)] \in H^p(M).$$

Now consider E as a (non-compact) manifold of dimension $n + p$ and denote by $H_c^p(E)$ the p-th cohomology group of E defined by compactly supported forms.

Theorem 3 (Thom isomorphism Theorem) *There exists a unique element $u \in H_c^p(E)$, known as the Thom class of E, such that for each fiber E_x of E*

$$\int_{E_x} u = 1. \tag{6}$$

See Bott and Tu [2] for a proof.

Let $i : M \mapsto E$ denote embedding as the 0-section. The *topological Euler class e_t of E* is defined by

$$e_t \equiv i^*(u) \in H^p(M).$$

Uniqueness of the Thom class implies that e_t is well-defined on the isomorphism class of E, i.e. it is a *topological invariant*. In the case when $p = n$, i.e. the rank of E and the dimension of M coincide, the *Euler characteristic* of E is defined by

$$\chi(E) \equiv \int_M e_t.$$

It is an easy exercise to show that e_g and e_t (both of which we denote for now by $e(E)$) share the following two fundamental properties:

(i) *Whitney duality*: Let $E = E_1 \oplus \ldots \oplus E_r$ be a direct sum of oriented even-dimensional bundles. Then

$$e(E) = e(E_1) \wedge \ldots \wedge (E_r)$$

(ii) *Naturality*: Let $f : N \mapsto M$ be a C^∞ map and consider the pull-back bundle $f^*(E)$ over N

$$
\begin{array}{ccc}
f^*(E) & & E \\
\downarrow & & \downarrow \\
N & \longrightarrow & M \\
& f &
\end{array}
$$

Then $e(f^*(E)) = f^*(e(E))$.

Our main result is

Theorem 4 *Let E denote a real orientable Riemannian vector bundle of even rank $p = 2q$ over a compact orientable manifold M of dimension n. Then*

$$e_t = e_g \in H^p(M). \tag{7}$$

Note that if $p = n = 2k$, then integrating each side of (7) over M gives Theorem 1. The Gauss-Bonnet-Chern Theorem is obtained from Theorem 1 by taking E to be the tangent bundle of an orientable Riemannian manifold M, endowed with the Levi-Civita connection.

3. Proof of Theorem 4

We first prove the theorem for the case where E is a bundle of rank 2, equipped with a metric connection ∇. The idea is to give an explicit construction of the Thom class u of E in terms of ∇.

Let $\{e_1, e_2\}$ be a (locally defined) positively oriented orthonormal frame for E and ω_e the upper off-diagonal entry of the corresponding connection 1-form with respect to this frame. Let v_1, v_2 denote the components of E-vectors with respect to the frame $\{e_1, e_2\}$ and r the radial distance in any fiber E_x. Finally, ρ and γ will denote smooth real-valued functions defined on $[0, \infty)$ of compact support and c a constant, all to be determined in the course of the proof. Consider the following

locally defined 2-form on E, which we introduce as a "template" for constructing the Thom class

$$u \equiv c\{\rho(r^2)dv_1 dv_2 + \rho(r^2)r dr \pi^*(\omega_e) + \pi^*(d\omega)\gamma(r)\}, \tag{8}$$

where $\pi : E \mapsto M$ is the projection map. Note that r and $d\omega$ are intrinsic objects. Changing to another orthonormal frame $\tilde{e}$ with the same orientation as e, related to e by a conterclockwise rotation of θ, yields

$$d\tilde{v}_1 d\tilde{v}_2 = dv_1 dv_2 - r dr \pi^*(d\theta).$$

However, the transformation law (4) implies

$$\tilde{\omega}_e = \omega_e + d\theta.$$

Substituting into (8), we obtain

$$u = c\{\rho(r^2)d\tilde{v}_1 d\tilde{v}_2 + \rho(r^2)r dr \pi^*(\tilde{\omega}_e) + \pi^*(d\omega)\gamma(r)\},$$

i.e. u is *invariantly defined* (it was this realization that motivated us to choose (8) as a general form for u). We shall define ρ, γ, and c so that u is a representative of the Thom class of E. First choose γ so that $\gamma(0) = 1$. Since $r dr = v_1 dv_1 + v_2 dv_2$ and $\pi \circ i = id$, applying i^* to (8) yields

$$i^*(u) = c d\omega \tag{9}$$

Taking the exterior derivative in (8) gives

$$du = c\pi^*(d\omega)dr\{(\gamma'(r) - \rho(r^2)r\}.$$

Thus a sufficient condition for u to be closed is $\gamma'(r) = \rho(r^2)r$. Together with the condition $\gamma(0) = 1$, this implies

$$\gamma(r) = 1 + \int_0^r \rho(s^2)s ds.$$

We now choose ρ to be any smooth function with support contained in $(0, 1)$ such that

$$\int_0^1 \rho(s^2)s ds = -1$$

and extend ρ and γ by defining them to be 0 on $[1, \infty)$. It follows from (8) that

$$\int_{E_x} u = c \int_0^\infty \int_0^\infty \rho(r^2)dv_1 dv_2$$

$$= c \int_0^{2\pi} d\phi \int_0^\infty \rho(r^2)r dr$$

$$= c \int_0^{2\pi} d\phi \int_0^1 \rho(r^2) r dr = -2\pi c.$$

Thus chosing $c = -1/2\pi$ ensures that u represents the Thom class. With this choice of c, (9) implies

$$e_t = i^*(u) = e_g$$

and the theorem is proved for the plane bundle case.

Suppose now that $E = E_1 \oplus \ldots \oplus E_q$ is a sum of oriented plane bundles. Let $E_1, \ldots E_q$ have curvature 2-forms with upper off-diagonal entries $d\omega_1, \ldots, d\omega_q$ with respect to (any) metric connections. Extend these connections (by direct sum) to a connection on E. With the "block multiplicative" property of the Pfaffian we have that

$$pf(\Omega) = d\omega_1 \wedge \ldots \wedge d\omega_q. \tag{10}$$

Let $E_1, \ldots E_q$ have topological Euler classes $e_t^1, \ldots, e_t^q$. Using (10), the theorem for 2-plane bundles, and Whitney duality, we obtain

$$e_g(E) \equiv (-2\pi)^{-q} Pf(\Omega)$$

$$= (-d\omega_1/2\pi) \ldots (-d\omega_q/2\pi)$$

$$\equiv e_g^1 e_g^2 \ldots e_g^q$$

$$= e_t^1 e_t^2 \ldots e_t^q$$

$$= e_t(E).$$

Thus the theorem holds in this case.

We complete the proof by appealing to the following result (a proof of which can be found in [7, Page 196]).

Theorem 5 (Splitting Principle) *Let E denote a real, orientable, even dimensional vector bundle over a manifold M. Then there exists a manifold N and a map $g : N \mapsto M$ such that*
(i) $g^* : H^*(M) \mapsto H^*(N)$ is a monomorphism (i.e. is injective).
(ii) $g^*(E)$ is a sum of orientable plane bundles.

Suppose now E is an arbitrary orientable vector bundle over M. Applying the above result with E and using the already established coincidence of e_g and e_t on sums of orientable plane bundles and the naturality of e_g and e_t, we have

$$g^*(e_g(E)) = e_g(g^*(E)) = e_t(g^*(E)) = g^*(e_t(E)).$$

The result now follows from the injectivity property of the map g^*.

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