

Divergence Theorems in Path Space

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1. Motivation

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$\operatorname{Div}(Z)$ (the divergence of Z) is given by

$$\frac{-1}{\sqrt{\det g}} \sum_{i=1}^d \frac{\partial}{\partial x_i} (a_i \sqrt{\det g}).$$

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The *Laplacian* (Laplace-Beltrami) operator

$$\Delta \equiv -\operatorname{Div} \nabla$$

\implies harmonic functions, harmonic forms, spectral theory, Hodge theory, heat kernel asymptotics and index theory, . . .

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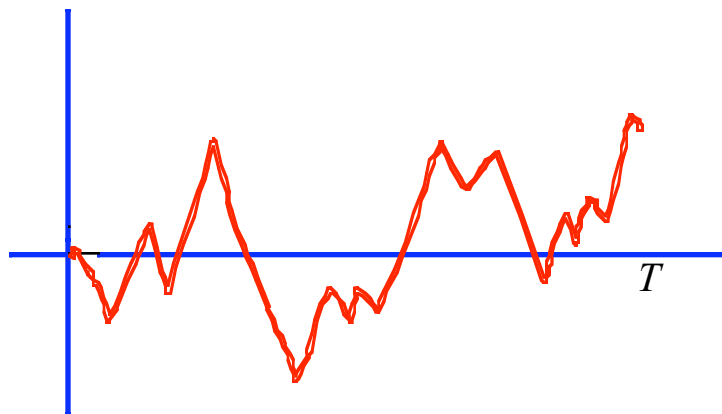
$$\int_X Z(\Phi)d\gamma = \int_X \Phi Div(Z)d\gamma.$$

We say a vector field Z on X is *admissible* if there exists an L^1 random variable $Div(Z)$ such that this relation holds for a dense class of C^1 functions Φ on X .

Example: The *Wiener space*

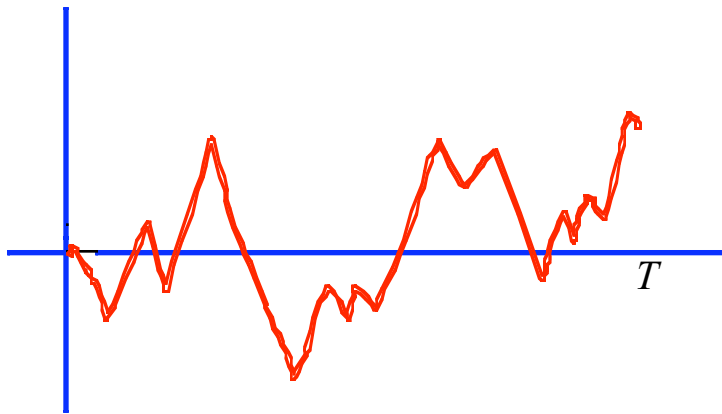
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furnished with the law of Brownian motion (*Wiener measure*).

Let h denote a path with finite *energy*, i.e.

$$\int_0^T |h'(t)|^2 dt < \infty.$$

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$$E[\Phi(w + \epsilon h)] = E[\Phi(w)G_\epsilon(w)]$$

where

$$G_\epsilon(w) = \exp\left(\epsilon \int_0^T h' dw - \frac{\epsilon^2}{2} \int_0^T |h'|^2 dt\right).$$

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Hence the divergence of h exists and is given by

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Problem Given an infinite-dimensional manifold X equipped with a measure γ , characterize the class of admissible vector fields on X . Compute the divergence of these vector fields.

2. Divergence theorems for Wiener space

Suppose now X is the space of continuous paths

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Let H denote the subspace of X consisting of the paths of finite energy (*Cameron-Martin space*).

Theorem. *Suppose $h : X \mapsto H$ is a bounded random adapted path, i.e.*

$$h(s) = f(w_u/u \leq s).$$

Then h is an admissible vector field on X and

$$\text{Div}(h) = \int_0^T h' \cdot dw.$$

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$$E[\Phi(w^\epsilon)] = E[\Phi(w)].$$

$$E[Z(\Phi)(w)] = \frac{d}{d\epsilon} /_{\epsilon=0} E[\Phi(w^\epsilon)] = 0.$$

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Combining the previous two theorems, we arrive at the most general

Divergence theorem for Wiener space. *The process*

$$Z = \int_0^\cdot a dw + \int_0^\cdot b dt$$

where a is a continuous adapted $so(n)$ -valued process and b is a continuous adapted \mathbf{R}^n -valued process, is admissible. The divergence of Z is given by

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The space of such processes Z will be called the *Cameron-Martin-Driver space*.

3. The space of paths over a manifold

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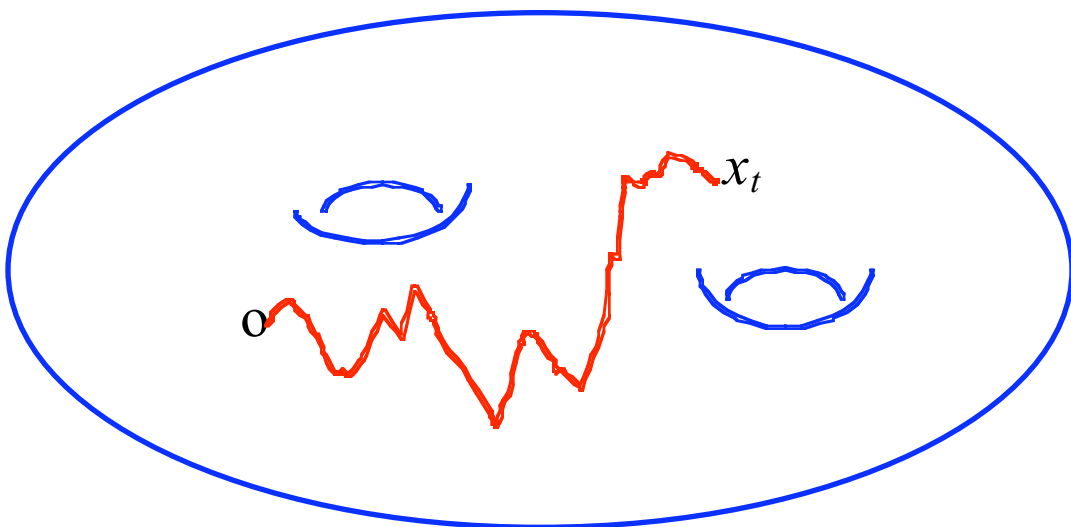
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The idea is to lift the problem from M to the Wiener space by the map $w \mapsto x_T$. This approach works under very weak non-degeneracy conditions on A_1, \dots, A_n (Hormander condition and weaker).

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but requires the *ellipticity* condition: A_1, \dots, A_n span TM at every point in M (a much stronger assumption than the Hormander condition).

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Both these goals have been realized. The first will be described today.

4. The main result

We construct a class of vector fields Z on X in the form

$$Z_t = \sum_{i=1}^n h_i(t) A_i(x_t)$$

where $h_i : [0, T] \mapsto \mathbf{R}$ are suitably defined continuous adapted processes.

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Note that Z lies in $T_x X$. Under the ellipticity assumption *all* continuous adapted vector fields on X have this form.

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Let ∇ denote the Levi-Civita covariant derivative wrt this metric.

Define a set of 1-forms $\omega^{jk}, 1 \leq j, k \leq n$ on M by

$$\omega^{jk}(\cdot) = \langle \nabla_{A_j} A_k, \cdot \rangle - \langle \nabla_{\cdot} A_j, A_k \rangle$$

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and functions $b^{jk}, 1 \leq j, k \leq n$ by

$$b^{jk} = \frac{1}{2} \left(\langle L_{ji} A_i, A_k \rangle - \langle L_{ij} \nabla_{A_k} A_i \rangle - \langle \nabla_{A_j} A_k, \nabla_{A_i} A_i \rangle + \langle \nabla_{A_p} A_i, A_k \rangle \langle \nabla_{A_j} A_p, A_i \rangle \right)$$

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Remark. The ω^{jk} and L_{ij} are not standard geometric objects. However

$$\Delta \equiv \sum_{i=1}^n L_{ii}$$

is the *covariant (Bochner) Laplacian*. This is an operator that frequently appears in the work of stochastic geometers.

Theorem. Let $r = (r_1, \dots, r_n)$ be a path in H . Define $h_i, 1 \leq i \leq n$ by the following system of SDEs

$$dh_i = \omega^{ji}(\circ dx_t)h_j(t) + [b^{ji}(x_t)h_j(t) + r'_i]dt$$

$$h_i(0) = 0.$$

Then the vector field Z on X defined by

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is admissible and

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Remark. The formula for the divergence is identical to that of Driver but the construction of Z is different. In Driver's work admissible vector fields are obtained by parallel transport along x of Cameron-Martin paths in T_oM .

5. Outline of the proof

Let g denote the map $w \mapsto x$ (*Ito map*) defined by the diffusion process

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By *lift* we mean that the following diagram commutes

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Recall that admissibility of r requires

$$r = \int_0^\cdot a dw + \int_0^\cdot b dt$$

where a is an adapted process with values in $so(n)$ and b is an adapted process in \mathbf{R}^n .

Now...

Suppose Φ is a test function in X . By the lifting property we have

$$E[Z(\Phi)(x)] = E[r(\Phi \circ g)(w)].$$

By DTWS, the rhs can be expressed as

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and conclude that Z is admissible with

$$Div(Z) = E\left[\int_0^T b \cdot dw / x\right].$$

The substance of the proof lies in the construction of Z and r satisfying

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Digression: Malliavin's construction for the endpoint problem ($M = \mathbf{R}^d$)

Denote $g_T : w \mapsto x_T$ and let $Z \in \mathbf{R}^d$.

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Write

$$Q \equiv Dg_T(w) \in L(H, \mathbf{R}^d).$$

One can show that if A_1, \dots, A_n satisfy the Hormander condition, then Q is a.s. surjective (i.e. g_T is a *submersion*). Choose

$$r = Q^*(QQ^*)^{-1}Z.$$

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One can show that if A_1, \dots, A_n satisfy the Hormander condition, then Q is a.s. surjective (i.e. g_T is a *submersion*). Choose

$$r = Q^*(QQ^*)^{-1}Z.$$

The operator $QQ^* \in GL(d)$ is known as the *Malliavin covariance matrix*.

Digression: Malliavin's construction for the end-point problem ($M = \mathbf{R}^d$)

Denote $g_T : w \mapsto x_T$ and let $Z \in \mathbf{R}^d$.

Then r is lift of Z (by the map g_T) if

$$Dg_T(w)r = Z.$$

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This construction will not work on the path space level because the surjectivity property does not hold.

Lifting Theorem. *The process $r : \Omega \times [0, T] \mapsto \mathbf{R}^n$ is a lift of the vector field*

$$Z_t \equiv \sum_{i=1}^n h_i(t) A_i(x_t) \quad (1)$$

if and only if r and h are related by the SDE

$$h_k = r_k + \int_0^\cdot \langle [A_j, A_i], A_k \rangle (x_t) h_j \circ dw_i. \quad (2)$$

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The answer is construct (h, r) as a *pair*.

Observe that the problem is that the integrand in

$$h_k = r_k + \int_0^\cdot \langle [A_j, A_i], A_k \rangle (x_t) h_j \circ dw_i \quad (2)$$

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$$\begin{aligned} \langle [A_j, A_i], A_k \rangle &= \langle \nabla_{A_j} A_i, A_k \rangle - \langle \nabla_{A_i} A_j, A_k \rangle \\ &= \langle \nabla_{A_j} A_i, A_k \rangle - \langle \nabla_{A_j} A_k, A_i \rangle \\ &\quad + \langle \nabla_{A_j} A_k, A_i \rangle - \langle \nabla_{A_i} A_j, A_k \rangle . \end{aligned}$$

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Write

$$G_j^{ik}(t) = \left(\langle \nabla_{A_j} A_i, A_k \rangle - \langle \nabla_{A_j} A_k, A_i \rangle \right) (x_t)$$

and

$$T^{jk} = \langle \nabla_{A_j} A_k, \cdot \rangle - \langle \nabla_{\cdot} A_j, A_k \rangle .$$

Equation (2)

$$h_k = r_k + \int_0^\cdot \langle [A_j, A_i], A_k \rangle (x_t) h_j \circ dw_i$$

can now be written

$$\begin{aligned} h_k &= r_k + \int_0^\cdot G_j^{ik} h_j \circ dw_i + \int_0^\cdot T^{jk}(A_i) h_j \circ dw_i \\ &= r_k + \int_0^\cdot G_j^{ik} h_j \circ dw_i + \int_0^\cdot T^{jk}(\circ dx) h_j \quad (3) \end{aligned}$$

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Then we have

$$h_k = \rho_k + \int_0^\cdot G_j^{ik} h_j \circ dw_i + \int_0^\cdot T^{jk}(\circ dx) h_j$$

and (3) holds with r replaced by ρ .

We have established the following conditions:

(i) ρ is a lift of $Z_t \equiv \sum_i h_i(t)A_i(x_t)$ to $C_0(\mathbf{R}^n)$.

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Computation of the divergence of Z :

Recall that

$$Div(Z) = E[Div(\rho)/x].$$

In order to compute $Div(\rho)$, it is necessary to convert the Stratonovich integral in

$$\rho_k = r_k - \int_0^\cdot G_j^{ik} h_j \circ dw_i.$$

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$$\int_0^t G_j^{ik} h_j \circ dw_i = \int_0^t G_j^{ik} h_j dw_i + \frac{1}{2} [G_j^{ik} h_j, w_i](t).$$

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The divergence of the quadratic variation gives rise to the Ricci curvature term in the statement of the theorem.

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